1 Axisymmetric equations of motion in cylindrical (r, λ, z)-coordinates

The primitive equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \lambda} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \lambda} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = -\frac{1}{\rho r}\frac{\partial p}{\partial \lambda} + F_\lambda, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \lambda} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho r u}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \lambda} + \frac{\partial \rho w}{\partial z} = 0, \tag{4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + \frac{v}{r}\frac{\partial \theta}{\partial \lambda} + w\frac{\partial \theta}{\partial z} = \dot{\theta} \tag{5}$$

$$\rho = p_* \pi^{\frac{1}{\kappa}-1}/(R_d \theta) \tag{6}$$

where u, v, w are the Eulerian velocity components in the three orthogonal coordinate directions, (F_r, F_λ, F_z) is the divergence of the sub-grid-scale turbulent momentum fluxes and/or the frictional force per unit mass, $\dot{\theta}$ is the diabatic heating rate. Temperature is given by $T = \pi\theta$, where θ is the potential temperature and π is the *Exner function*.

The diabatic heating rate, $\dot{\theta}$, in units of Kelvin per second is given by

$$\dot{\theta} = \frac{\dot{Q}}{c_{pd}\pi}, \tag{7}$$

where $\dot{Q}$ is the heating rate per unit mass in units of Joules per second per kilogram.

2 The balance equations

Assuming gradient wind balance and hydrostatic balance, Eqs. (1) and (3) give,

$$\frac{\partial p}{\partial r} = \rho\left(\frac{v^2}{r} + fv\right), \tag{8}$$

and

$$\frac{\partial p}{\partial z} = -\rho g. \tag{9}$$

Taking $(\partial/\partial z)$[Eq. (8)] and $(\partial/\partial r)$[Eq. (9)] to eliminate the pressure we obtain

the *thermal wind equation*

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}, \tag{10}$$

where

$$C = \frac{v^2}{r} + fv \tag{11}$$

denotes the sum of the centrifugal and Coriolis forces per unit mass. Equation (10) is a linear first-order partial differential equation for $\ln \rho$.

The characteristics of the equation satisfy

$$\frac{dz}{dr} = \frac{C}{g}. \tag{12}$$

The characteristics coincide with the isobaric surfaces because a small displacement (dr, dz) along an isobaric surface satisfies $(\partial p/\partial r)dr + (\partial p/\partial z)dz = 0$. Then, using the equations for hydrostatic balance $(\partial p/\partial z = -\rho g)$ and gradient wind balance $(\partial p/\partial r = \rho C)$ gives the equation for the characteristics (Eq. (12)). The density variation along a characteristic is governed by the equation

$$\frac{d}{dr} \ln \rho = -\frac{1}{g}\frac{\partial C}{\partial z}, \tag{13}$$

which is another form of Eq. (10).

3 Stability of a balanced vortex

The local axisymmetric stability of a baroclinic vortex depends on the three spatially-varying parameters characterizing:

the *static stability*

$$N^2 = -\frac{g}{\chi}\frac{\partial \chi}{\partial z}; \tag{14}$$

the *generalized inertial stability*

$$I_g^2 = I^2 + \frac{C}{\chi}\frac{\partial \chi}{\partial r}; \tag{15}$$

the *baroclinicity*

$$B = \frac{1}{\chi}\frac{\partial}{\partial z}(C\chi) = \xi\frac{\partial v}{\partial z} + \frac{C}{\chi}\frac{\partial \chi}{\partial z}. \tag{16}$$

The condition for a baroclinic vortex to be symmetrically stable is

$$\Delta = N^2 I_g^2 - B^2 > 0. \tag{17}$$

4 The secondary circulation: derivation of the dry Eliassen equation

If the vortex is axisymmetric, evolves strictly axisymmetrically and adheres to strict gradient wind and hydrostatic balance, we can derive an equation for the streamfunction, ψ, of the secondary circulation, i.e. the circulation in a vertical plane. This streamfunction is such that

$$u = -\frac{1}{r\rho}\frac{\partial \psi}{\partial z}, \qquad w = \frac{1}{r\rho}\frac{\partial \psi}{\partial r}. \tag{18}$$

which ensures that the quasi-steady form of equation (4) is satisfied exactly. The equation for ψ follows by differentiating the thermal wind equation in the form $g\partial\chi/\partial r = -\partial(C\chi)/\partial z$ with respect to time t and using the azimuthal momentum equation and thermodynamic equation to eliminate the time derivatives. It is convenient to write the last two equations in the following form

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = -\dot{V} \tag{19}$$

and

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta} \tag{20}$$

where $\zeta = (1/r)(\partial(rv)/\partial r)$ is the relative vorticity and $S = \partial v/\partial z$ is the vertical shear of the tangential wind. Note that we have added a momentum sink term $F_\lambda = -\dot{V}$ in the former equation to represent the effect of surface friction on the tangential wind component of the vortex in the balance formulation. Here the term $\dot{V}$ represents a distributed body force and is confined to a thin layer adjacent to the lower boundary.

For general forcing terms $\dot{V}$ and $\dot{\theta}$, v and χ given by Eqs. (19) and (20) will change at rates that will destroy thermal wind balance. To ensure that the vortex remains in balance as time proceeds, the time derivative of the thermal wind equation, Eq. (10) expressed in terms of[1] χ, must be satisfied, i.e., $\partial v/\partial t$ and $\partial \chi/\partial t$ must satisfy

$$g\frac{\partial}{\partial r}\left(\frac{\partial \chi}{\partial t}\right) + \frac{\partial}{\partial z}\left(C\frac{\partial \chi}{\partial t} + \chi\frac{\partial C}{\partial t}\right) = 0. \tag{21}$$

Substituting the time derivatives from Eqs. (19) and (20) in Eq. (21) gives

$$-g\frac{\partial}{\partial r}\left(u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} + \dot{\Theta}\right) - \frac{\partial}{\partial z}\left[C\left(u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} + \dot{\Theta}\right) + \chi\xi\left(u(\zeta+f) + wS + \dot{V}\right)\right] = 0,$$

where $\chi = 1/\theta$ and $\dot{\Theta} = \chi^2\dot{\theta}$. We will refer to $\dot{\Theta}$ as *the heating function.* Then

$$\frac{\partial}{\partial r}\left[-g\frac{\partial \chi}{\partial z}w - g\frac{\partial \chi}{\partial r}u\right] - \frac{\partial}{\partial z}\left[\left(\chi\xi(\zeta+f) + C\frac{\partial \chi}{\partial r}\right)u + \frac{\partial}{\partial z}(\chi C)w\right] = g\frac{\partial \dot{\Theta}}{\partial r} + \frac{\partial}{\partial z}(C\dot{\Theta}) + \frac{\partial}{\partial z}(\chi\xi\dot{V})$$

or, using Eq. (10),

$$\frac{\partial}{\partial r}\left[-g\frac{\partial \chi}{\partial z}w + \frac{\partial}{\partial z}(\chi C)u\right] - \frac{\partial}{\partial z}\left[(\chi\xi(\zeta+f) + C\frac{\partial \chi}{\partial r})u + \frac{\partial}{\partial z}(\chi C)w\right]$$
$$= g\frac{\partial \dot{\Theta}}{\partial r} + \frac{\partial}{\partial z}(C\dot{\Theta}) + \frac{\partial}{\partial z}(\chi\xi\dot{V}). \tag{22}$$

Then, substitution for u and w from Eqs. (18) into Eq. (22) gives

$$\frac{\partial}{\partial r}\left[-g\frac{\partial \chi}{\partial z}\frac{1}{\rho r}\frac{\partial \psi}{\partial r} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial z}\right] + \frac{\partial}{\partial z}\left[\left(\chi\xi(\zeta+f) + C\frac{\partial \chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial \psi}{\partial z} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right]$$
$$= g\frac{\partial \dot{\Theta}}{\partial r} + \frac{\partial}{\partial z}(C\dot{\Theta}) + \frac{\partial}{\partial z}(\chi\xi\dot{V}). \tag{23}$$

[1] It is easy to verify that Eq. (10) has exactly the same form when expressed in terms of χ.

This linear second-order partial differential equation is called the *Eliassen equation*. In terms of the parameters introduced in Section 3, the equation may be written finally as

The Eliassen equation

$$\frac{\partial}{\partial r}\left[\gamma N^2\frac{\partial\psi}{\partial r} - \gamma B\frac{\partial\psi}{\partial z}\right] + \frac{\partial}{\partial z}\left[\gamma I_g^2\frac{\partial\psi}{\partial z} - \gamma B\frac{\partial\psi}{\partial r}\right] = g\frac{\partial\dot{\Theta}}{\partial r} + \frac{\partial}{\partial z}(C\dot{\Theta}) + \frac{\partial}{\partial z}(\chi\xi\dot{V}), \tag{24}$$

where

$$\gamma = \chi/(\rho r). \tag{25}$$

Equation (24) contains the same three spatially-varying parameters N^2, I_g^2 and B that arise in the local axisymmetric linear stability analysis of a baroclinic vortex (Section 3).

The discriminant of the Eliassen equation (24) is

$$D = 4\gamma^2(I_g^2N^2 - B^2), \tag{26}$$

which is proportional to Δ, where $\Delta = I_g^2N^2 - B^2$ characterizes the local axisymmetric stability of a baroclinic vortex. Recall that a baroclinic vortex is symmetrically stable if $\Delta > 0$. Equation (24) is *elliptic* if $D > 0$, *parabolic* if $D = 0$, and *hyperbolic* if $D < 0$. Thus the symmetric stability of the vortex globally is a requirement that the Eliassen equation is elliptic globally, which, in turn, is a requirement that the equation may be solved as a diagnostic equation for the streamfunction, ψ, subject to suitable boundary conditions on ψ along the horizontal and vertical domain boundaries of the axisymmetric vortex.

The Eliassen equation can be written in the form

$$\bar{A}\frac{\partial^2\psi}{\partial r^2} + 2\bar{B}\frac{\partial^2\psi}{\partial r\partial z} + \bar{C}\frac{\partial^2\psi}{\partial z^2} + \bar{D}\frac{\partial\psi}{\partial r} + \bar{E}\frac{\partial\psi}{\partial z} = g\frac{\partial\dot{\Theta}}{\partial r} + \frac{\partial}{\partial z}(C\dot{\Theta}) + \frac{\partial}{\partial z}(\chi\xi\dot{V}), \tag{27}$$

where

$$\bar{A} = -g\frac{\partial\chi}{\partial z}\frac{1}{\rho r} = \gamma N^2, \tag{28}$$

$$\bar{B} = -\frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r} = -\gamma B, \tag{29}$$

$$\bar{C} = \left(\chi\xi(\zeta + f) + C\frac{\partial\chi}{\partial r}\right)\frac{1}{\rho r} = \gamma I_g^2, \tag{30}$$

$$\bar{D} = \frac{\partial\bar{A}}{\partial r} + \frac{\partial\bar{B}}{\partial z}, \tag{31}$$

and

$$\bar{E} = \frac{\partial\bar{B}}{\partial r} + \frac{\partial\bar{C}}{\partial z}. \tag{32}$$

Again, the discriminant of this equation is $4(\bar{A}\bar{C} - \bar{B}^2)$ (Sneddon 1957, p108 uses $\bar{B}$ as the coefficient of the mixed derivative term instead of $2\bar{B}$ used here and defines the discriminant as $\bar{B}^2 - 4\bar{A}\bar{C}$. We have chosen here to reverse the sign so that positive potential vorticity (Section 5) corresponds with positive discriminant.).

4.1 The moist Eliassen equation

We can write

$$\dot{\Theta} = \chi^2\dot{\theta} = \chi^2\frac{\dot{Q}}{c_p\pi}. \tag{33}$$

Let us approximate the material diabatic heating rate for slantwise ascent by[2]

$$\dot{Q} \approx -L_v\left(u\frac{\partial r_v^*}{\partial r} + w\frac{\partial r_v^*}{\partial z}\right). \tag{34}$$

Then, using the formula (34) for $\dot{Q}$ and formulae (18) for u and w gives,

$$\chi^2\dot{\theta} = -\frac{\gamma L_v}{c_pT}J(\psi, r_v^*), \tag{35}$$

where $\gamma = \chi/(\rho r)$ and $J(\cdot,\cdot)$ denotes the Jacobian operator. In the foregoing, we have used the fact that

$$\frac{\chi^2 L_v}{c_p\pi} = \frac{L_c\chi}{c_pT}, \tag{36}$$

because $\theta = T/\pi$.

It follows from (35) that

$$\frac{\partial}{\partial r}(g\chi^2\dot{\theta}) = -\frac{\partial}{\partial r}\left(\frac{g\gamma L_v}{c_pT}J(\psi, r_v^*)\right), \tag{37}$$

and

$$\frac{\partial}{\partial z}(C\chi^2\dot{\theta}) = -\frac{\partial}{\partial z}\left(\frac{C\gamma L_v}{c_pT}J(\psi, r_v^*)\right). \tag{38}$$

The diabatic forcing terms involving the stream function ψ may then be moved to the left hand side of Eq. (24) to give

$$\frac{\partial}{\partial r}\left[-g\frac{\partial\chi}{\partial z}\frac{1}{\rho r}\frac{\partial\psi}{\partial r} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial z} - g\dot{\Theta}\right] + \\ \frac{\partial}{\partial z}\left[\left(\chi\xi(\zeta+f) + C\frac{\partial\chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial\psi}{\partial z} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial r} - C\dot{\Theta}\right] = \frac{\partial}{\partial z}(\chi\xi\dot{V}). \tag{39}$$

Inserting the foregoing relations for the diabatic forcing terms then yields

$$\frac{\partial}{\partial r}\left[\gamma N^2\frac{\partial\psi}{\partial r} - \gamma B\frac{\partial\psi}{\partial z} + \gamma\frac{gL_v}{c_pT}J(\psi, r_v^*)\right] + \frac{\partial}{\partial z}\left[\gamma I_g^2\frac{\partial\psi}{\partial z} - \gamma B\frac{\partial\psi}{\partial r} + \gamma\frac{CL_v}{c_pT}J(\psi, r_v^*)\right] = \\ \frac{\partial}{\partial z}(\chi\xi\dot{V}). \tag{40}$$

Finally,

the Eliassen equation for *moist slantwise ascent* is

$$\frac{\partial}{\partial r}\left[\gamma\left\{\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right)\frac{\partial\psi}{\partial r} - \left(B + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right)\frac{\partial\psi}{\partial z}\right\}\right] + \\ \frac{\partial}{\partial z}\left[\gamma\left\{\left(I_g^2 - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right)\frac{\partial\psi}{\partial z} - \left(B - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right)\frac{\partial\psi}{\partial r}\right\}\right] = \frac{\partial}{\partial z}\left(\chi\xi\dot{V}\right). \tag{41}$$

[2]It is readily verified that retention of the local time derivative of r_v^* in (34) does not change the discriminant condition (Eq. (45) below) that characterizes the partial differential equation type for ψ in the case of explicit moist adiabatic dynamics. Its retention merely adds a time derivative 'forcing' term to the right hand side of the corresponding Eliassen equation. This forcing term corresponds to a higher order term in the balance formulation, which would be typically neglected for slowly evolving axisymmetric tropical cyclone vortices. A scale analysis confirms that this term is subdominant in comparison to the vertical advection and radial advection advection of r_v^*. In the limit of a slowly evolving vortex, this term is asymptotically small, justifying its neglect herein.

In the form of Eq. (27), the coefficients of the second-order derivatives are now given by:

$$\bar{A} = \gamma \left(N^2 + \frac{g L_v}{c_p T} \frac{\partial r_v^*}{\partial z} \right), \tag{42}$$

$$\bar{B} = -\gamma \left[B + \frac{1}{2} \frac{L_v}{c_p T} \left(g \frac{\partial r_v^*}{\partial r} - C \frac{\partial r_v^*}{\partial z} \right) \right], \tag{43}$$

$$\bar{C} = \gamma \left(I_g^2 - \frac{C L_v}{c_p T} \frac{\partial r_v^*}{\partial r} \right). \tag{44}$$

Then,

the moist discriminant is

$$D_m = 4\gamma^2 \left[\left(I_g^2 - \frac{C L_v}{c_p T} \frac{\partial r_v^*}{\partial r} \right) \left(N^2 + \frac{g L_v}{c_p T} \frac{\partial r_v^*}{\partial z} \right) - \left\{ B + \frac{L_v}{2 c_p T} \left(g \frac{\partial r_v^*}{\partial r} - C \frac{\partial r_v^*}{\partial z} \right) \right\}^2 \right]. \tag{45}$$

Note that, at least up to this stage, we have made no approximations concerning the variation of L_v/T.

5 Dry PV

The *dry Rossby-Ertel PV* is defined as

$$P = \frac{(\boldsymbol{\omega} + \mathbf{f}) \cdot \nabla\theta}{\rho}. \tag{46}$$

For a symmetric vortex with tangential wind speed distribution $v(r, z)$,

$$\boldsymbol{\omega} + \mathbf{f} = -\frac{\partial v}{\partial z}\mathbf{i} + (\zeta + f)\mathbf{k} = -\frac{1}{r}\frac{\partial M}{\partial z}\mathbf{i} + \frac{1}{r}\frac{\partial M}{\partial r}\mathbf{k},$$

and

$$\nabla\theta = \frac{\partial\theta}{\partial r}\mathbf{i} + \frac{\partial\theta}{\partial z}\mathbf{k},$$

where $\mathbf{i}$ and $\mathbf{k}$ as unit vectors in the radial and vertical directions, respectively. Therefore

$$P = \frac{1}{r\rho}\left(-\frac{\partial M}{\partial z}\frac{\partial\theta}{\partial r} + \frac{\partial M}{\partial r}\frac{\partial\theta}{\partial z} \right) = \frac{1}{r\rho} J(M, \theta), \tag{47}$$

where J(.,.) is the Jacobian operator. Now

$$-\frac{\partial M}{\partial z}\frac{\partial\theta}{\partial r} + \frac{\partial M}{\partial r}\frac{\partial\theta}{\partial z} = \mathbf{j} \cdot \nabla\theta \wedge \nabla M,$$

where $\mathbf{j}$ is a unit vector in the tangential direction. It follows that zero dry PV is equivalent to the congruence of the M- and θ-surfaces.

It is straightforward to show that the discriminant of the dry Eliassen equation is proportional to P. Specifically,

$$D = 4g\frac{\chi^3}{\rho r^2}\xi P. \tag{48}$$

6 Saturation moist PV

The *saturation moist PV* is defined by the analogous formula:

$$P_m^* = \frac{(\boldsymbol{\omega} + \mathbf{f}) \cdot \nabla \theta_e^*}{\rho}, \tag{49}$$

where

$$\nabla \theta_e^* = \frac{\partial \theta_e^*}{\partial r}\mathbf{i} + \frac{\partial \theta_e^*}{\partial z}\mathbf{k}.$$

Therefore

$$P_m^* = \frac{1}{r\rho}\left(-\frac{\partial M}{\partial z}\frac{\partial \theta_e^*}{\partial r} + \frac{\partial M}{\partial r}\frac{\partial \theta_e^*}{\partial z}\right) = \frac{1}{r\rho}J(M, \theta_e^*). \tag{50}$$

Now

$$-\frac{\partial M}{\partial z}\frac{\partial \theta_e^*}{\partial r} + \frac{\partial M}{\partial r}\frac{\partial \theta_e^*}{\partial z} = \frac{1}{r\rho}\mathbf{j} \cdot \nabla \theta_e^* \wedge \nabla M.$$

Then

$$P_m^* = \frac{1}{r\rho}\mathbf{j} \cdot \nabla \theta_e^* \wedge \nabla M. \tag{51}$$

Note that this formula does not depend on any approximation for θ_e^*. Thus zero moist PV is equivalent to the congruence of the M- and θ_e^*-surfaces.

7 An approximate expression for P_m^*

We could write

$$P_m^* = \frac{\theta_e^*}{r\rho}\mathbf{j} \cdot \nabla \ln \theta_e^* \wedge \nabla M. \tag{52}$$

Now, $\ln \theta_e^* = \ln \theta + L_v r_v^*/(c_p T) = -\ln \chi + L_v r_v^*/(c_p T)$ and therefore, treating $L_v/(c_p T)$ as a constant to a first approximation in comparison to spatial variations of r_v^*,

$$\frac{1}{\theta_e^*}\nabla \theta_e^* = -\frac{1}{\chi}\nabla \chi + \frac{L_v}{c_p T}\nabla r_v^* = \left(-\frac{1}{\chi}\frac{\partial \chi}{\partial r} + \frac{L_v}{c_p T}\frac{\partial r_v^*}{\partial r}\right)\mathbf{i} + \left(-\frac{1}{\chi}\frac{\partial \chi}{\partial z} + \frac{L_v}{c_p T}\frac{\partial r_v^*}{\partial z}\right)\mathbf{k},$$

so that

$$P_m^* = \frac{\theta_e^*}{\rho}\left[-\frac{\partial v}{\partial z}\left(-\frac{1}{\chi}\frac{\partial \chi}{\partial r} + \frac{L_v}{c_p T}\frac{\partial r_v^*}{\partial r}\right) + (\zeta + f)\left(-\frac{1}{\chi}\frac{\partial \chi}{\partial z} + \frac{L_v}{c_p T}\frac{\partial r_v^*}{\partial z}\right)\right]. \tag{53}$$

Recall that

$$B = \frac{1}{\chi}\frac{\partial}{\partial z}(C\chi) = \xi\frac{\partial v}{\partial z} + \frac{C}{\chi}\frac{\partial \chi}{\partial z}, \tag{54}$$

$$I_g^2 = I^2 + \frac{C}{\chi}\frac{\partial \chi}{\partial r}, \tag{55}$$

$$g\frac{\partial \chi}{\partial r} = -\frac{\partial}{\partial z}(C\chi) = -\chi B, \tag{56}$$

$$N^2 = -\frac{g}{\chi}\frac{\partial \chi}{\partial z}, \tag{57}$$

and note that

$$\frac{\partial C}{\partial z} = B - N^2 C/g. \tag{58}$$

Then, upon multiplying P_m^* by $g\rho\xi/\theta_e^*$ yields

$$\frac{g\rho\xi}{\theta_e^*}P_m^* = \underbrace{\left(\frac{C}{\chi}\frac{\partial \chi}{\partial z} - B\right)}_{-\frac{\partial C}{\partial z}}\left(\underbrace{-\frac{g}{\chi}\frac{\partial \chi}{\partial r}}_{B} + \frac{gL_v}{c_p T}\frac{\partial r_v^*}{\partial r}\right) + \underbrace{\xi(\zeta + f)}_{I_g^2 - \frac{C}{\chi}\frac{\partial \chi}{\partial r}}\left(\underbrace{-\frac{g}{\chi}\frac{\partial \chi}{\partial z}}_{N^2} + \frac{gL_v}{c_p T}\frac{\partial r_v^*}{\partial z}\right),$$

or, interchanging the two terms on the right-hand-side,

$$\frac{g\rho\xi}{\theta_e^*}P_m^* = \underbrace{\left(I_g^2 - \frac{C}{\chi}\frac{\partial\chi}{\partial r}\right)}_{I_g^2+\frac{C}{g}B,\text{ using Eq. (56)}} \left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) - \frac{\partial C}{\partial z}\left(B + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right).$$

Now, using Eqs. (58),

$$B\left(\frac{C}{g}N^2 - \frac{\partial C}{\partial z}\right) = -B^2,$$

whereupon

$$\frac{g\rho\xi}{\theta_e^*}P_m^* = I_g^2\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) + B\frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z} - B^2 - \frac{\partial C}{\partial z}\frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}. \tag{59}$$

Our expectation at first was that P_m^* would be proportional to D_m, where

$$D_m = 4\gamma^2\left[\left(I_g^2 - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right)\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) - \left\{B + \frac{L_v}{2c_pT}\left(g\frac{\partial r_v^*}{\partial r} - C\frac{\partial r_v^*}{\partial z}\right)\right\}^2\right]. \tag{60}$$

However, this turns out not to be the case. Nevertheless, comparison of the Eqs. (59) and (60) suggested adding

$$-\frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) \tag{61}$$

to the right-hand-side of Eq. (59) and then subtracting it. The last three terms on the right-hand-side of (59) with the expression (61) subtracted then sum to give

$$-B\left(B - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) + \frac{L_v}{c_pT}\frac{\partial r_v^*}{\partial r}\underbrace{\left(-g\frac{\partial C}{\partial z} + CN^2\right)}_{-gB,\text{ using Eqs. (54) and (57)}} + gC\left(\frac{L_v}{c_pT}\right)^2\frac{\partial r_v^*}{\partial r}\frac{\partial r_v^*}{\partial z},$$

which factorizes to give

$$-\left(B - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right)\left(B + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right).$$

It follows that

the saturation moist PV is

$$\frac{g\rho\xi}{\theta_e^*}P_m^* = \left(I_g^2 - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right)\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) - \left(B - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right)\left(B + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right). \tag{62}$$

8 Relationship between D_m and P_m^*

From Eq. (60) above,

$$\frac{D_m}{4\gamma^2} = \left(I_g^2 - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right)\left(N^2 + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right) - \left[B + \frac{L_v}{2c_pT}\left(g\frac{\partial r_v^*}{\partial r} - C\frac{\partial r_v^*}{\partial z}\right)\right]^2. \tag{63}$$

Subtracting Eq. (62) from Eq. (63) gives

$$\frac{D_m}{4\gamma^2} - \frac{g\rho\xi}{\theta_e^*}P_m^* = \left(B - \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z}\right)\left(B + \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}\right) - \left[B + \frac{L_v}{2c_pT}\left(g\frac{\partial r_v^*}{\partial r} - C\frac{\partial r_v^*}{\partial z}\right)\right]^2. \tag{64}$$

Writing

$$\alpha = \frac{CL_v}{c_pT}\frac{\partial r_v^*}{\partial z} \text{ and } \beta = \frac{gL_v}{c_pT}\frac{\partial r_v^*}{\partial r}, \tag{65}$$

Eq. (64) becomes

$$\begin{aligned}
\frac{D_m}{4\gamma^2} - \frac{g\rho\xi}{\theta_e^*}P_m^* &= (B-\alpha)(B+\beta) - \left[B + \frac{1}{2}(\beta-\alpha)\right]^2 \\
&= B^2 + B(\beta-\alpha) - \alpha\beta - \left[B^2 + B(\beta-\alpha) + \frac{1}{4}(\beta-\alpha)^2\right] \\
&= -\frac{1}{4}(\alpha+\beta)^2 \\
&= -\frac{1}{4}\left(\frac{L_v}{c_pT}\right)^2\left(g\frac{\partial r_v^*}{\partial r} + C\frac{\partial r_v^*}{\partial z}\right)^2 .
\end{aligned} \tag{66}$$

Now $r_v^* = r_v^*(p,T)$, whereupon application of the chain rule gives

$$g\frac{\partial r_v^*}{\partial r} + C\frac{\partial r_v^*}{\partial z} = g\left(\frac{\partial r_v^*}{\partial p}\frac{\partial p}{\partial r} + \frac{\partial r_v^*}{\partial T}\frac{\partial T}{\partial r}\right) + C\left(\frac{\partial r_v^*}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial r_v^*}{\partial T}\frac{\partial T}{\partial z}\right)$$

$$= \frac{\partial r_v^*}{\partial p}\underbrace{\left(g\frac{\partial p}{\partial r} + C\frac{\partial p}{\partial z}\right)}_{=0 \text{ from Eqs. (8) and (9)}} + \frac{\partial r_v^*}{\partial T}\left(g\frac{\partial T}{\partial r} + C\frac{\partial T}{\partial z}\right) = g\left(\frac{\partial r_v^*}{\partial T}\right)_p\left(\frac{\partial T}{\partial r}\right)_p,$$

where the change of constraint relation

$$\left(\frac{\partial T}{\partial r}\right)_p = \left(\frac{\partial T}{\partial r}\right)_z + \frac{C}{g}\left(\frac{\partial T}{\partial z}\right)_r,$$

has been used. Finally,

the relation between D_m and P_m^* is

$$D_m = 4g\gamma^2\left[\frac{\rho\xi P_m^*}{\theta_e^*} - \underbrace{\frac{g}{4}\left(\frac{L_v}{c_pT}\right)^2\left(\frac{\partial r_v^*}{\partial T}\right)_p^2\left(\frac{\partial T}{\partial r}\right)_p^2}_{>0}\right]. \tag{67}$$

Important conclusions:

- The congruence of the M and θ_e^* surfaces does not imply that $D_m = 0$.
- If M and θ_e^* surfaces are congruent, $D_m < 0$ and the moist Eliassen equation is hyperbolic.