# Chapter 5

# WAVES AT LOW LATITUDES

A characteristic of the atmosphere is its shallow depth; 99% of the mass lies below a height of 30 km whereas the mean earth radius is 6,380 km. Over this 30 km which extends into the middle stratosphere there is a considerable variation in the vertical structure. However much can be learned about low latitude motions by considering the atmosphere to be a uniform layer of fluid with variable depth. Put another way, consideration of the *horizontal structure* of the vertical mean atmosphere yields rich information about the predominant wave modes, especially at low latitudes. The classical papers on this subject are those of Matsuno (1966) and Longuet-Higgins (1968) with important contributions also from Webster (1972) and Gill (1980) amongst others. A recent review is given by Lim and Chang (1987).

Although we begin our study by assuming a uniform vertical structure, we shall find that the effort is not in vain for it turns out that the solutions to the divergent barotropic system are, in fact, the horizontal part of the *baroclinic* modes.

In contrast to Longuet-Higgins (1968) and Webster (1972) who use full-spherical geometry, we follow Matsuno (1966) and Gill (1980) and consider motions on an equatorial beta plane. To begin with, we review the theory of wave motions in a divergent barotropic fluid on an $f$-plane or on a mid-latitude $\beta$-plane as described in DM, Chapter 11. The basic flow configuration is shown in Fig. 5.1. The fluid layer has undisturbed depth $H$.

We consider small amplitude perturbations about a state of rest in which the free surface elevation is $H(1 + \eta)$. As shown in DM, the linearized "shallow-water" equations take the form

$$\frac{\partial u}{\partial t} - fv = -c^2 \frac{\partial \eta}{\partial x}, \tag{5.1}$$

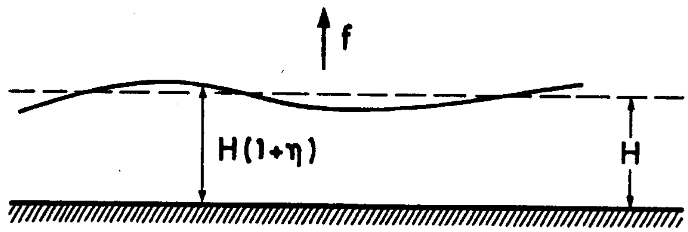


Figure 5.1: Configuration of a one-layer fluid model with a free surface.

$$\frac{\partial v}{\partial t} + fu = -c^2 \frac{\partial \eta}{\partial y}, \tag{5.2}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.3}$$

where $c = \sqrt{(gH)}$ is the phase speed of long waves in the absence of rotation ($f = 0$).

On an $f$-plane, Eqs. (5.1) - (5.3) have sinusoidal travelling wave solutions in the $x$-direction with *wavelength* $2\pi/k$ and *period* $2\pi/\omega$ (*wavenumber* $k$, *frequency* $\omega$) of the form

$$v = \hat{v} \ \sin\,(kx - \omega t) \tag{5.4}$$

$$(u,\, \eta) = (\hat{u},\, \hat{\eta}) \ \cos\,(kx - \omega t)\,, \tag{5.5}$$

where $\hat{u}$, $\hat{v}$, $\hat{\eta}$ are constants, provided

$$\omega\, \left(\omega^2 - f^2 - c^2\, k^2\right) = 0. \tag{5.6}$$

This dispersion relation yields $\omega = 0$, which corresponds with a steady ($\partial/\partial t = 0$) geostrophic flow, or $\omega^2 = f^2 + c^2\, k^2$, corresponding with inertia - gravity waves. The phase speed of these waves, $c_p$, is given by

$$c_p = \frac{\omega}{k} = \pm \sqrt{\left(c^2 + \frac{f^2}{k^2}\right)} = \pm c \sqrt{\left(1 + \frac{1}{L_R^2\, k^2}\right)}, \tag{5.7}$$

where $L_R = c/f$ is the Rossby radius of deformation. Clearly, the importance of inertial effects compared with gravitational effects is characterized by the size of the parameter $L_R^2 k^2$, i.e. by the wavelength of waves compared with the Rossby radius of deformation.

On a mid-latitude $\beta$-plane where $f$ is a function of $y$ (specifically, where $f = f_0 + \beta y$, $f_0 \neq 0$), it is inconsistent to seek a solution of the form (5.4) - (5.5) with $\hat{u}$, $\hat{v}$ *and* $\hat{\eta}$ as constants, unless meridional particle displacements are relatively small. In that case, the effects of variable $f$ can be incorporated by replacing the second equation of (5.1) - (5.3) by the vorticity equation

$$\frac{\partial \zeta}{\partial t} + \beta v = f_0 \frac{\partial v}{\partial t}, \tag{5.8}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},. \tag{5.9}$$

Note that variations of $f$ are included only in so much as they appear in the advection of planetary vorticity by the meridional velocity component. Substituting again (5.4) into (5.1), (5.8) and (5.9) shows that, solutions are possible now only if $\omega$ satisfies the equation

$$\left(\omega^2 - c^2 k^2\right) (\omega k + \beta) - f_0^2 \omega k = 0. \tag{5.10}$$

It is convenient to scale $\omega$ by $f_0$ and $k$ by $1/L_R$, say $\omega = f_0 \nu$, $k = m/L_R$. Then (5.10) reduces to

$$(\nu^2 - \mu^2)(\nu\mu + \varepsilon) - \nu\mu = 0 \tag{5.11}$$

where $\varepsilon = \beta L_R / f_0$. At latitude $45°$, $\beta/f_0 = 1/a$, where a is the earths radius. It follows that for Rossby radii $L_R << a$, then $\varepsilon << 1$.

When $\varepsilon = 0$, implying that $\beta = 0$, Eq. (5.11) has solutions $\nu = 0$ and $\nu = \mu^2 + 1$ as before.

If $\varepsilon << 1$, there is a root of $0(\varepsilon)$ which emerges if we set $\nu = \varepsilon \nu_0$, where $\nu_0$ is $0(1)$ and neglect higher powers of $\varepsilon$. It follows easily that

$$\nu = -\frac{\varepsilon\, \mu}{1 + \mu^2} \tag{5.12}$$

which in dimensional form, $\omega = -\beta\, k / \left[k^2 + 1/L_R^2\right]$, is the familiar dispersion relation for *divergent Rossby waves*. The other two roots for small $\varepsilon$ are the same as when $\varepsilon = 0$, and again correspond with inertia-gravity wave modes. We consider now a rather special wave type that owes its existence to the presence of a boundary - the so-called Kelvin wave. Consider the flow configuration on an $f$-plane sketched in Fig. 5.2. The equation set (5.1) - (5.3) has a solution in which $v \equiv 0$. In that case, they reduce to

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \eta}{\partial x}, \tag{5.13}$$

$$fu = -c^2 \frac{\partial \eta}{\partial y}, \tag{5.14}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{5.15}$$

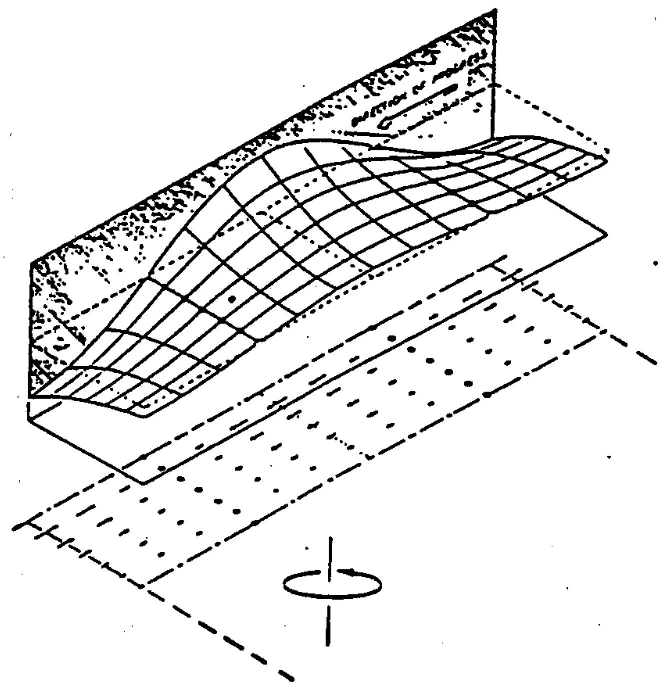

Figure 5.2: Flow configuration of a Kelvin wave.

Cross-differentiating (5.13) and (5.15) to eliminate $u$ gives

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2}, \tag{5.16}$$

which has a general travelling-wave solution of the form

$$\eta = F\ (x - ct,\, y) + \ G\ (x + ct,\, y)\,, \tag{5.17}$$

where $F$ and $G$ are arbitrary functions. Define $X = x - ct$ and $Y = x + ct$. Then using (5.15) we have

$$\frac{\partial u}{\partial x} = c\left(\frac{\partial F}{\partial X} - \frac{\partial G}{\partial Y}\right) = c\left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial x}\right),$$

$$u = c\,(F - G)\,, \tag{5.18}$$

which may be integrated partially with respect to $x$ to give ignoring an arbitrary function of $y$ and $t$. Substitution of (5.18) in (5.14) gives

$$fc\,[F - G] = c^2\left[\frac{\partial F}{\partial y} + \frac{\partial G}{\partial y}\right],$$

and since $F$ and $G$ are arbitary functions we must have

$$\frac{\partial F}{\partial y} + (f/c)\ F = 0, \tag{5.19}$$

and

$$\frac{\partial G}{\partial y} - (f/c)\ G = 0. \tag{5.20}$$

These first order equations in $y$ may be integrated to give the $y$ dependence of $F$ and $G$, i.e.

$$F = F_0\ (X)\ e^{-fy/c},\ G = G_0\ (Y)\ e^{fy/c}.$$

In the configuration shown in Fig. 5.2, we must reject the solution $G$ as this is unbounded as $y \to \infty$. In this case, the solution is

$$\eta = F_0\ (x - ct)\ e^{-fy/c}. \tag{5.21}$$

This represents the surface elevation of a wave that moves in the *positive* $x$-direction with speed c and decays exponentially away from the boundary with decay scale $c/f$ which is simply the Rossby radius of deformation, $L_R$. The solution for $u$ from (5.18) is simply

$$u = cF_0\ (x - ct)\ e^{-fy/c}. \tag{5.22}$$

The Kelvin wave is essentially a gravity wave that is "trapped" along the boundary by the rotation. The velocity perturbation $u$ is always such that geostrophic balance occurs in the $y$ direction, expressed by (5.14). If the fluid occupies the region $y < 0$, then the appropriate solution is the one for which $F_0 \equiv 0$ and then

$$u = cF_0\ (x - ct)\ e^{-fy/c}. \tag{5.23}$$

Again this represents a trapped wave moving at speed $c$ with the boundary on the right (left) in the Northern (Southern) Hemisphere when $f > 0 (f < 0)$.

## 5.1 The equatorial beta-plane approximation

At the equator, $f_0 = 0$, but $\beta$ is a maximum. In the vicinity of the equator, (5.1)-(5.3) must be modified by setting $f = \beta y$. This constitutes the *equatorial beta-plane approximation* that may be derived from the equations for motion on a sphere (see e.g. Gill, 1982, §11.4). The perturbation equations are now

$$\frac{\partial u}{\partial t} - \beta y v = -c^2 \frac{\partial \eta}{\partial x}, \tag{5.24}$$

$$\frac{\partial v}{\partial t} + \beta y u = -c^2 \frac{\partial \eta}{\partial y}, \tag{5.25}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.26}$$

$$\frac{\partial}{\partial t}\left(\zeta - f\eta\right) + \beta v = 0, \tag{5.27}$$

where again

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{5.28}$$

Taking

$$-\left(\frac{\beta y}{c}\right)\frac{\partial}{\partial t}\ (5.24) + \left(\frac{1}{c}\right)\frac{\partial^2}{\partial t^2}\ (5.25) - c\frac{\partial^2}{\partial y \partial t}\ (5.26) - c\frac{\partial}{\partial x}\ (5.27)$$

and using (5.28) and remembering that $f = \beta y$ gives

$$\frac{\partial}{\partial t}\left[\frac{1}{c^2}\left(\frac{\partial^2 v}{\partial t^2} + f^2 v\right) - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right] - \beta\frac{\partial v}{\partial x} = 0, \tag{5.29}$$

which has only the dependent variable $v$. We follow the usual procedure and look for travelling-wave solutions of the form

$$v = \hat{v}\left(y\right) \exp\left[i\ \left(kx - \omega t\right)\right], \tag{5.30}$$

whereupon $\hat{v}\left(y\right)$ has to satisfy the ordinary differential equation obtained by substituting (5.30) into (5.29). Note that, unlike the previous case we cannot assume that $\hat{v}$ is a constant because Eq.(5.29) has a *coefficient* (namely $f^2$) that depends on $y$. The equation for $\hat{v}\left(y\right)$ is

$$\frac{d^2\hat{v}}{dy^2} + \left[\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right]\hat{v} = 0. \tag{5.31}$$

Before attempting to find solutions to this equation we scale the *independent* variables $t, x, y$, using the time scale $(\beta c^{-1/2})$ and length scale $(c/\beta^{-1/2})$, the latter defining the *equatorial Rossby radius* $L_E$. This scaling necessitates scaling $\omega = (c\beta)^{-1/2}\nu$ and $k = \mu(c/\beta)^{-1/2}$ also. Then the equation becomes

$$\frac{d^2\hat{v}}{dy^2} + \left[\nu^2 - \mu^2 - \frac{\mu}{\nu} - y^2\right]\hat{v} = 0 \tag{5.32}$$

which is the same form as Schrödingers equation that arises in the theory of quantum mechanics. The solutions are discussed succinctly by Sneddon (1961, see especially Chapter V). A brief sketch of the main results that we require are given in an appendix to this chapter. There it is shown that solutions for $\hat{v}$ that are bounded as $|y| \to \infty$ are possible only if

$$\nu^2 - \mu^2 - \mu\,\nu^{-1} = 2n + 1\ ,\ (n\ =\ 0,\ 1,\ 2, ...)\,. \tag{5.33}$$

These solutions have the form of parabolic cylinder functions. In dimensional terms,

$$v\,(x,\,y,\,t) = H_n\left((\beta/c)^{\ 1/2}y\right)\,\exp\,\left(-\beta y^2/2c\right)\,\cos\,(kx - \omega t)\,, \tag{5.34}$$

which on multiplication by $2^{-n/2}$ can be written as

$$v\,(x,\,y,\,t) = D_n\left((\beta/c)^{\ 1/2}y\right)\ \cos\,(kx - \omega t)\,, \tag{5.35}$$

where $D_n$ is the parabolic cylinder function of order $n$ and $H_n$ is the Hermite polynominal of order $n$. In dimensional form the corresponding dispersion relation, (5.33), is

$$\omega^2/c^2 - k^2 - \beta k/\omega = (2n + l)\beta/c. \tag{5.36}$$

Like (5.10), this is a cubic equation for $\omega$ for each value of $n$ and evidently a whole range of wave modes is possible. We shall consider the structure of these presently.

## 5.2 The Kelvin Wave

First we note that (5.32) has a trivial solution $\hat{v} = 0$. As in the case of the Kelvin wave discussed earlier, this solution corresponds with a nontrivial wave mode. To see this we substitute $\hat{v} = 0$ into Eqs. (5.24) - (5.26) to obtain

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \eta}{\partial x}, \tag{5.37}$$

$$\beta\, y\, u = -c^2\, \frac{\partial \eta}{\partial y}, \tag{5.38}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{5.39}$$

These equations are identical with (4.12) if we set $f = \beta y$ in (5.14). In particular the solutions for $\eta$ and $u$ are exactly the same as (5.17) and (5.18), respectively. Then (5.38) gives

$$\frac{\partial F}{\partial y} + (\beta y/c)\; F = 0, \tag{5.40}$$

and

$$\frac{\partial G}{\partial y} - (\beta y/c)\; G = 0, \tag{5.41}$$

analogous to (5.19) and (5.20). Now (5.40) has the solution

$$F = F_0\,(X)\;\exp\left(-\beta y^2/2c\right), \tag{5.42}$$

whereas the solution for $G$ is unbounded as $y \to \pm\infty$. Therefore Eqs. (5.37) -(5.39) have a solution

$$\left.\begin{array}{l} \eta\;(x,\,y,\,t) = F_0\,(x - ct)\;\exp\,(-\beta y^2/2c)\,, \\ u\;(x;\,y,\,t) = c\,F_0\;(x - ct)\;\exp\,(-\beta y^2/2c)\,, \\ v(x,y,t) = 0 \end{array}\right\} \tag{5.43}$$

This solution is called an *equatorial Kelvin wave*. It is an eastward propagating gravity wave that is trapped in the *equatorial waveguide* by Coriolis forces. Note that it is nondispersive and has a meridional scale on the order of $L_E = (c/\beta)^{1/2}$.

## 5.3 Equatorial Gravity Waves

We return now to the dispersion relation (5.36). For $n \geq 1$, the waves subdivide into two classes like the solutions of (5.10). There are two solutions for which $\beta k/\omega$ is small, whereupon the dispersion curves are given approximately by

$$\omega^2 \approx (2n + 1)\;\beta c + k^2 c^2. \tag{5.44}$$

The form is similar to that for inertia-gravity waves (sometimes called Poincaré waves also - e.g., in Gill, 1982). These waves are *equatorially-trapped gravity waves*, or *equatorially-trapped Poincaré waves.*

## 5.4 Equatorial Rossby Waves

There are solutions also of (5.36) for which $\omega^2/c^2$ is small. Then the dispersion relation is approximately

$$\omega \approx -\beta k / \left[ k^2 + (2n+1)\ \beta/c \right]. \tag{5.45}$$

These modes are called equatorially trapped planetary waves or equatorially trapped Rossby waves. The various dispersion curves are plotted in Fig. 5.3. This figure includes also the dispersion curves for the case $n = 0$ described below and for the Kelvin wave.

## 5.5 The mixed Rossby-gravity wave

When $n = 0$, Eq. (5.33) may be written

$$(\nu + \mu)\ (\nu - \mu - 1/\nu) = 0. \tag{5.46}$$

The solution $\nu = -\mu$ must be excluded since it leads to an indeterminate solution for $u$ (see later). Therefore the solutions are, in dimensional form,

$$\omega_+ = \tfrac{1}{2}\, kc + \left[ \tfrac{1}{4}\, k^2 c^2 + c\beta \right]^{1/2}, \tag{5.47}$$

which represents an eastward propagating inertia-gravity wave, and

$$\omega_- = \tfrac{1}{2}\, kc - \left[ \tfrac{1}{4}\, k^2 c^2 + c\beta \right]^{1/2}, \tag{5.48}$$

which represents an inertia-gravity wave if $k$ is small and a Rossby wave if $k$ is large (see Ex. 5.2). Note that as $k \to 0$, $\omega_- \approx -(c\beta)^{1/2}$, which agrees with the long wavelength limit of the inertia-gravity wave solution (5.44), while as $k \to \infty$, $\omega_- \approx -\beta/k$, which agrees with the limit of the Rossby wave solution (5.45). The solution $n = 0$ is called therefore a *mixed Rossby-gravity wave.* The phase velocity of this mode can be either eastward or westward, but the group velocity is always eastward (Ex. 5.2). The Kelvin wave solution is sometimes called the $n = -1$ wave because (5.36) is satisfied by the Kelvin-wave dispersion relation (i.e. $\omega = kc$) when $n = -1$.

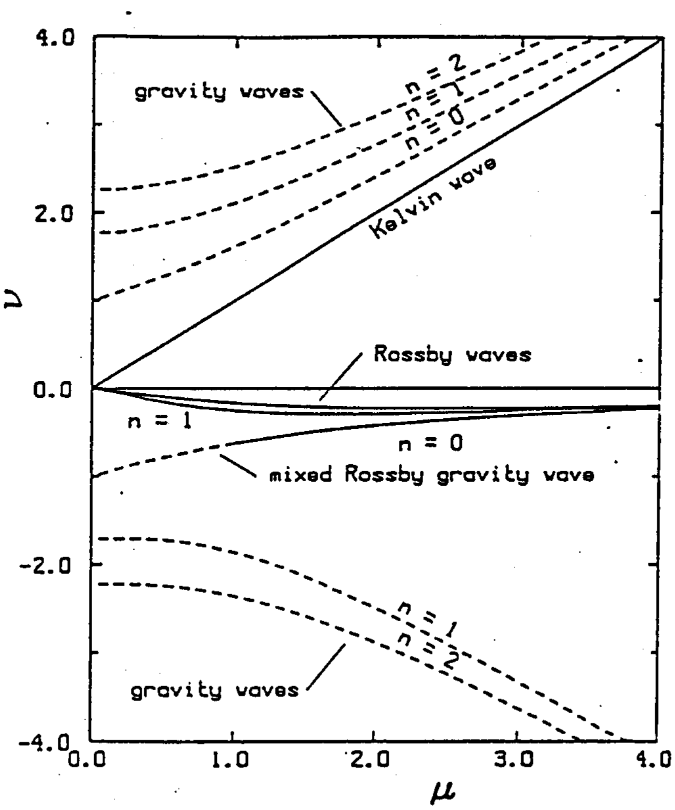


Figure 5.3: Nondimensional frequencies $\nu$ from (5.36) as a function of nondimensional wavenumber $\mu$.

To calculate the complete structure of the various wave modes we need to obtain solutions for $u$ and $\eta$ corresponding to the solution for $v$ in Eq. (5.34).

We return to the linearized equations (5.24)-(5.28). The substitution $v = \hat{v}\ \sin\,(kx - \omega\, t)$, $(u, \eta, = (\hat{u}, \hat{\eta})\cos\,(kx - \omega\, t)$ in (5.24) and (5.26) gives

$$\omega\hat{u} - \beta y\hat{v} = kc^2\hat{\eta}, \tag{5.49}$$

$$\omega\hat{\eta} - k\hat{u} + \frac{d\hat{v}}{dy} = 0, \tag{5.50}$$

which may be solved for $\hat{u}$ and $\hat{\eta}$ in terms of $\hat{v}$ and $d\hat{v}/dy$, i.e.,

$$\left(\omega^2 - k^2\, c^2\right)\ \hat{u} = \omega\,\beta\, y\,\hat{v} - k\, c^2\frac{d\hat{v}}{dy}, \tag{5.51}$$

$$\left(\omega^2 - k^2\, c^2\right)\, \hat{\eta} = k\, \beta\, y\, \hat{v} - \omega \frac{d\hat{v}}{dy}. \tag{5.52}$$

With the previously introduced scaling and $y = L_E Y$, these become

$$\left(\nu^2 - \mu^2\right)\, \hat{u} = \nu\, Y\, \hat{v} - \, \mu \frac{d\hat{v}}{dY}, \tag{5.53}$$

and

$$\left(\nu^2 - \mu^2\right)\, \hat{\eta} = \mu\, Y\, \hat{v} - \nu \frac{d\hat{v}}{dY}. \tag{5.54}$$

Also, from (5.34)

$$\hat{v}\,(Y) = \hat{v}_n = \exp\, \left(-\tfrac{1}{2}\, Y^2\right)\, H_n\,(Y), \tag{5.55}$$

whereupon

$$\frac{d\hat{v}}{dY} = -Y\, \hat{v}_n + \exp\, \left(-\tfrac{1}{2}\, Y^2\right)\, \frac{dH_n}{dY}. \tag{5.56}$$

We use now two well-known properties of the Hermite polynomials:

$$\frac{dH_n}{dY} = 2n\, H_{n-1}\,(Y), \tag{5.57}$$

and

$$H_{n+1}\,(Y) = 2\, Y H_n\,(Y) - 2n\, H_{n-1}\,(Y). \tag{5.58}$$

It follows that

$$\left(\nu^2 - \mu^2\right)\, \hat{u}_n = \tfrac{1}{2}\, (\nu + \mu)\, \hat{v}_{n+1} + n\, (\nu - \mu)\, \hat{v}_{n-1} \tag{5.59}$$

and

$$\left(\nu^2 - \mu^2\right)\, \hat{\eta}_n = \tfrac{1}{2}\, (\nu + \mu)\, \hat{v}_{n+1} - n\, (\nu - \mu)\, \hat{v}_{n-1} \tag{5.60}$$

Figure 5.4 shows the horizontal structure of the Kelvin wave and of a westward propagating Mixed Rossby-gravity wave. Air parcels move parallel to the equator in the case of the Kelvin wave and move clockwise around elliptical orbits in the case of the mixed wave. The equatorial wave-guide equation (5.31) has the form

$$\frac{d^2\hat{v}}{dy^2} + \frac{\beta^2}{c^2}\, \left[y_c^2 - y^2\right]\, \hat{v} = 0, \tag{5.61}$$

where

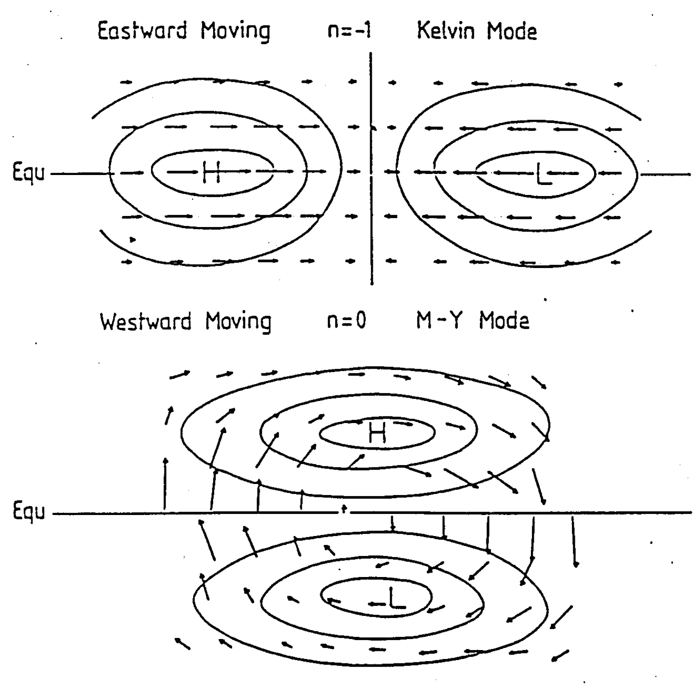


Figure 5.4: Contours of surface elevation and arrows representing for : (a) a Kelvin wave; and (b) a westward propagating mixed Rossby-gravity wave. The latter shows only one half the zonal wavelength (after Matsuno, 1966).

$$\beta^2 y_c^2 = \omega^2 - k^2 c^2 - \beta k c^2/\omega = (2n+1)\ \beta c, \tag{5.62}$$

using (5.36). Solutions thereto have a wave-like structure in the meridional ($y$-) direction if $y < y_c$ and an exponential structure if $y > y_c$. Thus $y_c$ corresponds with a critical latitude for a particular mode, a latitude beyond which wave-like propagation is not possible. If the phase of a particular wave changes rapidly enough with $y$, we can define a local meridional wavenumber $\kappa$ for each value of $y$, the assumption being that $\kappa$ varies only slowly with $y$. One may then use the WKB technique outlined in Gill (1982, §8.12, pp 297-302) to find approximate solutions to (5.61). Such solutions have the form

$$\hat{v} = \kappa^{-1/2} \exp\ \left[ i \left\{ kx + \int \kappa dy - \omega t \right\} \right], \tag{5.63}$$

where

$$\kappa^2 = \frac{\beta^2}{c^2}\left(y_c^2 - y^2\right). \tag{5.64}$$

This approximate solution is valid provided that

$$\delta = \kappa^{-3/2}\frac{d^2}{dy^2}\left(\kappa^{-1/2}\right) << 1. \tag{5.65}$$

At the equator $\delta = 1/\left[2(2n+1)^2\right]$ i.e. the approximate solution is valid provided $n$ is large. The group velocity of waves follows from (5.62), i.e.,

$$\mathbf{c}_g = \frac{(2k + \beta/\omega,\ 2\kappa)}{2\ \omega/c^2 + \beta k/\omega^2} \tag{5.66}$$

whereupon wave packets propagate along rays defined by $d\mathbf{x}/dt = \mathbf{c}_g$, or in this case,

$$\frac{dy}{dx} = \frac{\kappa}{k + \beta/2\omega} \tag{5.67}$$

Using (5.64) it follows readily that ray paths have the form

$$y = y_c\ \sin\left[c^{-1}\beta x/\ (k + \beta/2\omega)\right] \tag{5.68}$$

i.e., they are sinusoidal paths about the equator in which wave energy is reflected at the critical latitudes $y = \pm y_c$. Figure (5.5) shows an example of this type of behaviour for the case of gravity waves with no variation in $x$, i.e. $k = 0$. Then the term $\beta/2\omega$ can be ignored in (5.66) and (5.68). Since $k = 0$ the group velocity is given by

$$\mathbf{c}_g = (0\ ,\ c_{gy})\ =\ \frac{(0,\ 2\kappa)}{2\ \omega/c^2 + \beta k/\omega^2}. \tag{5.69}$$

In the calculation in Fig. (5.5) a uniform wind stress is suddenly applied over the ocean over a small range of latitudes that are remote from the equator. This results in the generation of inertial waves. The path followed by these waves can be calculated from $dy/dt = c_{gy}$. Integration of (5.69) with respect to time shows that the path followed by the waves is sinusoidal in time about the equator. As seen in Fig. (5.5) the waves move backwards and forwards across the equator along ray paths that are described quite well by a sinusoidal function.

Another effect of the waveguide is the discretization of modes $n = 1,\ 2,$ .... in the meridional direction. For long inertia-gravity waves ($k \to 0$) this implies a discrete set of frequencies given by

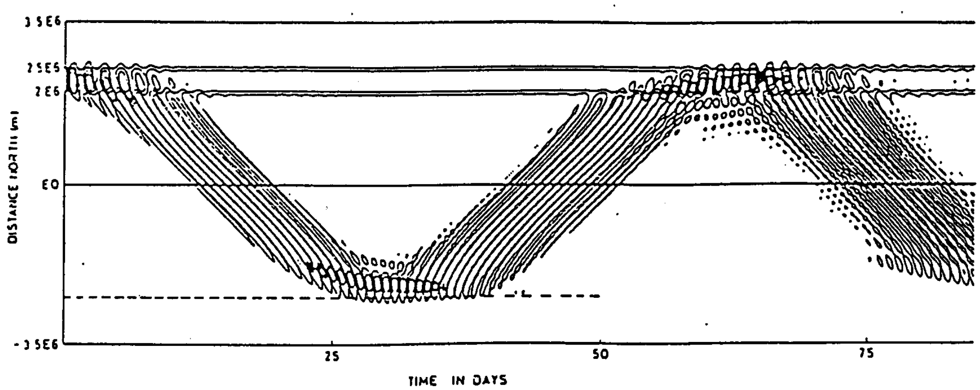


Figure 5.5: An illustration of beta dispersion of gravity waves. An eastward wind stress is applied in the strip 2000 km $< y <$ 2500 km from $t = 0$. At first local inertial waves are generated as on an $f$-plane, but the variation of $f$ with latitude causes the waves to propagate backward and forward across the equator. Contours are of the meridional velocity. (from Gill, 1982)

$$\omega^2 \approx (2n+1)\ \beta c, \tag{5.70}$$

obtained from (5.62). Gill (1982, p 442) notes that this frequency selection shows up in Pacific sea-level records because variations associated with the first baroclinic mode have magnitudes of the order of centimeters which is large enough to be detected. For these modes $c \approx 2.8\, ms^{-1}$ giving periods of $5\frac{1}{2}$, 4 and 3 days for $n$ = 1, 2 and 3, respectively. See Gill (1982, p442) for further details.

## 5.6 The planetary wave motions

Planetary waves have the approximate dispersion relation (5.45) which, in the long-wave *limit* ($k \to 0$), is $\omega \approx -kc/(2n+1)$. The phase speed $\omega/k$ is in the opposite direction to the Kelvin wave (i.e. westward) and the amplitudes are reduced by factors 3, 5, 7 etc. For example, for the first baroclinic mode in the Pacific Ocean, $c = 2.8\ ms^{-1}$, so that the planetary wave with $n = 1$ has speed 0.9 $ms^{-1}$. This mode would require 6 months to cross the Pacific Ocean from east to west. Other modes would be slower. These facts have implications for the coupled atmospheric-ocean response to perturbations in the tropics.

Figure 5.6 shows the dispersion curve (5.45) for the planetary wave modes. Differentiating (5.45) with respect to $k$ gives

$$\frac{1}{\omega}\frac{d\omega}{dk} = \frac{1}{k} - \frac{2k}{\left[k^2 + \ (2n+1)\,\beta/c\right]}.$$

Thus the curve has zero slope where

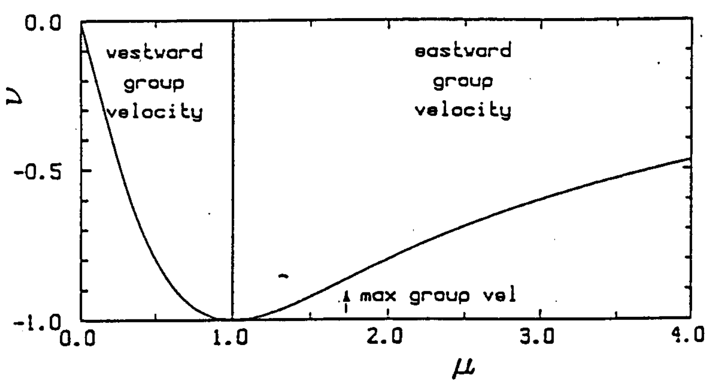


Figure 5.6: Graph of the planetary wave dispersion relation (5.45). The units are defined by the expression (5.70) and (5.71), i.e. $\mu = k/k_\star$ and $\upsilon = \omega/\omega_\star$.

$$k_\star = \left[(2n+1)\,/L_E^2\right]^{1/2} = f_c/c, \tag{5.71}$$

$f_c = \beta y_c$, and $y_c$ is defined by (5.62).

At the point $k = k_\star$ the frequency has a *maximum* absolute value

$$\omega_\star = \frac{1}{2}\beta/[(2n+1)\,\beta/c]^{1/2} = \frac{1}{2}\beta c/f_c. \tag{5.72}$$

For example when $n = 1$, this corresponds to a minimum period of 31 days for a first baroclinic ocean mode with $c = 2.8\ ms^{-1}$, 74 days for a higher mode with $c = 0.5\ ms^{-1}$, and 12 days for an atmospheric mode with $c = 20\ ms^{-1}$.

For waves with wavelength shorter than $2\pi/k$, the group velocity $(\partial\omega/\partial k)$ is positive (i.e. eastward) and therefore in the direction opposite to the phase velocity. The maximum group velocity is $c_g = \frac{1}{8}c/\,(2n+1)$ when $k = \left[3\ (2n+1)\,/L_E^2\right]^{1/2}$. Thus only short waves can carry information eastwards and then at only one eighth of the speed at which long waves can carry information westwards.

## 5.7 Baroclinic motions in low latitudes

The equations for small amplitude perturbations to an incompressible stratified fluid at rest are

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial P}{\partial x}, \tag{5.73}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial P}{\partial y}, \tag{5.74}$$

$$\frac{\partial \sigma}{\partial t} + N^2 \omega = 0, \tag{5.75}$$

$$\frac{\partial P}{\partial z} - \sigma = 0, \tag{5.76}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5.77}$$

where $P = p/\overline{\rho}$. Elimination of $w$ and $\sigma$ from (5.75) - (5.77) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left\{N^{-2}\frac{\partial P}{\partial z}\right\}\right] = 0. \tag{5.78}$$

If we choose P to satisfy the equation (5.79)

$$\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial P}{\partial z}\right) + \frac{P}{c^2} = 0 \tag{5.79}$$

then (5.32) becomes

$$\frac{\partial P}{\partial t} + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{5.80}$$

whereupon Eqs. (5.73), (5.74) and (5.80) have exactly the same form as the shallow-water equations (5.1) - (5.3) if we identify $c^2\eta$ in the latter with $P$ in the former. Equation (5.79) with appropriate boundary conditions leads to an eigenvalue problem for the vertical structure of wave perturbations and the corresponding eigenvalue $c$. Consider the case of an isothermal atmosphere with $N^2$ = constant. Then differentiating (5.79) with respect to $z$ and $t$ and using (5.75) and (5.76) to eliminate $P$ in preference to $w$ gives

$$\frac{\partial^2 w}{\partial z^2} + \frac{N^2}{c^2}w = 0. \tag{5.81}$$

For a liquid layer bounded by rigid horizontal boundaries at $z = 0$ and $z = H$, where $w = 0$, Eq. (5.81) has the solution

$$w = \hat{w}_n(x, y, t)\ \sin\left(\frac{n\pi z}{\mathrm{H}}\right)\ , \quad (n = 1, 2, 3, \ ...)\ , \tag{5.82}$$

and the corresponding eigenvalues are

$$c = c_n = \frac{NH}{n\pi}\ , \quad (n = 1, 2, 3, ...)$$

As usual, the gravest mode (the one with the largest phase speed, the case $n = 1$) has a single vertical velocity maximum in the middle of the layer, $z = 1/2H$.

Taking typical values for $N(= 10^{-2}s^{-1})$ and $H$ (the tropical tropopause $= 16$ km), the phase speed of the gravest mode $c_1 = 51\ ms^{-1}$.

It is easily verified that (5.81) holds even if $N$ is a function of $z$, but the eigenvalue problem will then be more difficult to solve.

In an unbounded vertical domain and when $N$ is a constant, Eq. (5.81) has solutions proportional to $exp(\pm imz)$, where $m^2 = N^2/c^2$. Therefore, the system of equations (5.73) - (5.77) have solutions in which, for example,

$$v = D_n\left[(2\beta/c)^{1/2}\ y\right]\ \exp\left[i\left(kx + mz - \omega t\right)\right], \tag{5.83}$$

where

$$c = N/\left|m\right|. \tag{5.84}$$

Note that $c$ is a property of the mode in question and is equal to the phase speed *only* in special cases such as the Kelvin wave. For an isothermal compressible atmosphere, Eq. (5.81) is a little more complicated, but is still given by (5.84) to a good approximation provided that $1/(4m^2H_s^2) << 1$, where $H_s$ is the scale height. Even for vertical wavelength of 20 km, this number is only about 0.03, so that the incompressible approximation is reasonable.

Now consider the dispersion relation $\omega = \omega(\mathbf{k})$ for the various types of waves with vector wavenumber $\mathbf{k} = (k, m)$. It is convenient to scale the wavenumber components by writing $k = (\beta/\omega)k_\star$ and $m = (\beta N/\omega^2)m_\star$. Then, the dispersion relation for the Kelvin wave, $\omega = kc$ becomes

$$m_\star = k_\star. \tag{5.85}$$

For the mixed Rossby- gravity wave $(n = 0)$, $\omega m/N - k - \beta/\omega = 0$ from (5.46], which becomes,

$$m_\star = k_\star + 1. \tag{5.86}$$

The remaining waves satisfy (5.36)

$$m_*^2 - (2n+1)m_* = k_*^2 + k_*,$$

or

$$m_* = n + \frac{1}{2} + \left[\left(k_* + \tfrac{1}{2}\right)^2 + n\,(n+1)\right]^{1/2}. \tag{5.87}$$

It can be shown that modes corresponding with the positive root in (5.87) are gravity waves while those corresponding with the negative root are planetary waves. The full set of dispersion curves is shown in Fig. 5.7. The gravity wave curves are the hyperbolae in the upper part of the diagram. The planetary wave curves are hyperbolae also and are shown on the expanded plot in the inset. The corresponding curves in the **k**-plane are the curves of constant frequency. The group velocity $\mathbf{c}_g = \nabla_k \omega$ is at right angles to these curves and in the direction of increasing $\omega$. The corresponding directions are shown in Fig. 5.7.

We carry out the calculations for the Kelvin wave and mixed Rossby-gravity waves.

*Kelvin wave*

$$\omega = \frac{Nk}{|m|} = \frac{N}{m} k \text{ sgn } (m)$$

Then

$$c_{g1} = \frac{\partial \omega}{\partial k} = \frac{N}{|m|} = c_{3p} \text{ , say , and } \quad c_{g3} = \frac{\partial \omega}{\partial m} = -\frac{N}{m^2} k \text{ sgn } (m) \text{ ,}$$

whereupon

$$c_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right) = \frac{N}{|m|}\left(1, -\frac{k}{m}\right).$$

Note that $c_{g3} > 0$ if $m < 0$, i.e. the Kelvin wave solution that propagates energy vertically upwards has phase lines that slope upward in the eastward direction (5.8). Figure 5.8 shows the structure of this mode.

*Mixed Rossby-gravity wave*

$\omega$ satisfies

$$\frac{\omega m}{N} \text{ sgn } (m) - k - \frac{\beta}{\omega} = 0, \tag{5.88}$$

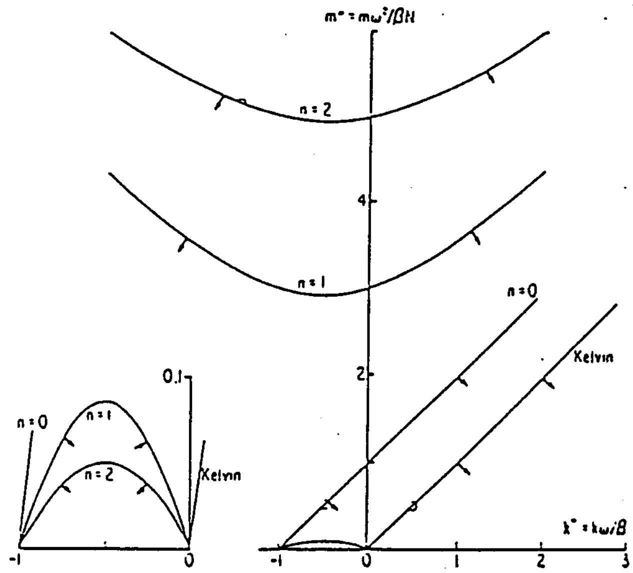


Figure 5.7: Dispersion curves for vertically-propagating equatorially-trapped waves. m is the vertical wavenumber and $\lambda$ the eastward wavenumber. The curves collapse into a single set when scaled with the frequency $\omega$, buoyancy frequency $N$, and beta as indicated. The direction of the group velocity, being the gradient of frequency in wavenumber space, is as indicated. The curves for m negative are obtained by reflection in the $k^*$ axis and have an upward-directed group velocity. The inset at the left is a blowup of the region near the origin to show the planetary waves $n = 1$, 2. The upper $n = 1$, 2 curves are the corresponding gravity waves. The circles represent observed waves.

whereupon

$$\frac{\partial \omega}{\partial k} = 1 / \left[ \frac{|m|}{N} + \frac{\beta}{\omega^2} \right],$$

i.e.

$$\underline{c}_g = \left( 1, \ -\frac{\omega}{N} \operatorname{sgn}(m) \right) / \left[ \frac{|m|}{N} + \frac{\beta}{\omega^2} \right]. \tag{5.89}$$

Now (5.88) gives

$$\omega = \tfrac{1}{2} k \frac{N}{|m|} \pm \left[ \tfrac{1}{4} k^2 \frac{N^2}{|m|^2} + \beta \frac{N}{|m|} \right]^{1/2},$$

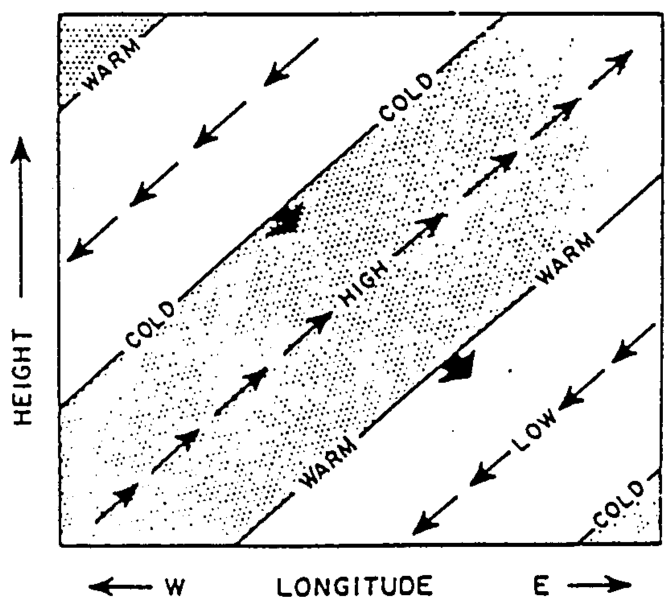


Figure 5.8: Longitudinal-height section along the equator showing pressure, temperature and perturbation wind oscillations in the Kelvin wave. Thick arrows indicate direction of phase propagation. (After Wallace, 1973).

from which it follows that the mixed Rossby-gravity wave, i.e. the solution for the negative square root, has $\omega < 0$. From (5.89) we see that this has an upward-directed group velocity if $m > 0$. Thus the phase lines of the mixed Rossby-gravity wave tilt westward with height (Fig. 5.9). Figure reffig4-9 shows the structure of this mode also. Note that poleward-moving air is correlated with positive temperature perturbations so that the eddy heat flux v'T' averaged over a wave is positive. The mixed Rossby-gravity wave removes heat from the equatorial region.

Both Kelvin wave and mixed Rossby-gravity wave modes have been identified in observational data from the equatorial stratosphere. The observed Kelvin waves have periods in the range 12-20 days and appear to be primarily of zonal wavenumber 1. The corresponding observed phase speeds of these waves relative to the ground are on the order of 30 $ms^{-1}$. In applying our theoretical formulas for the meridional and vertical scales, however, we must use the Doppler-shifted phase speed $c_p - U$, where $U$ is the mean zonal wind speed. Assuming $u = -10\ ms^{-1}$, $c_p - U = 40\ ms^{-1}$, whereupon $L_E = \sqrt{[(c_p - U)/\beta]} \approx 1300\,km$. This corroborates with observational evidence that the Kelvin waves have significant amplitude only within about

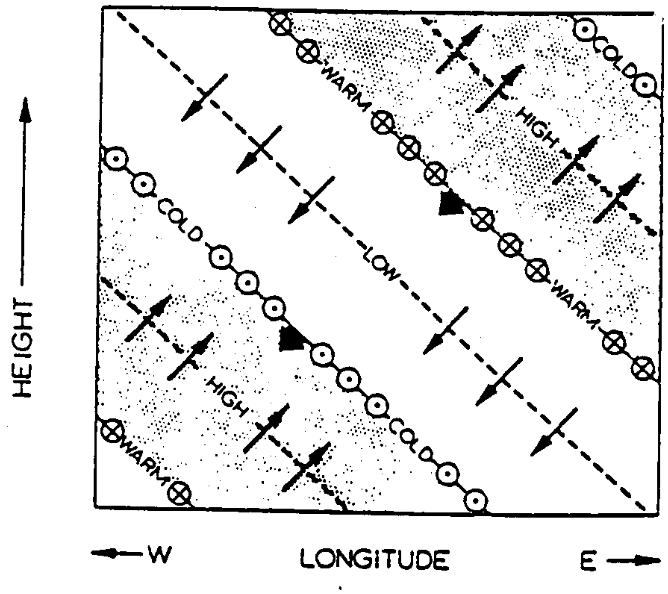


Figure 5.9: Longitudinal-height section at a latitude north of the equator showing pressure, temperature and perturbation wind oscillations in the mixed Rossby-gravity wave. Meridional wind components are indicated by arrows pointed into the page (northward) and out of the page (southward). Thick arrows indicate direction of phase propagation.

20° latitude of the equator. Knowledge of the observed phase speed also allows one to calculate the theoretical vertical wavelength of the Kelvin wave. Assuming that $N = 2 \times 10^{-2}\, s^{-1}$ (a stratospheric value) we find that

$$\frac{2\pi}{|m|} = 2\pi \frac{(c_p - U)}{N} \approx 12\ km,$$

which agrees with the vertical wavelength deduced from observations. (Note that for the Kelvin wave, $c_p = c$).

Figure (5.10) shows an example of zonal wind oscillations associated with the passage of Kelvin waves at a station near the equator. During the observational period shown in the westerly phase of the so-called quasi-biennial oscillation is descending so that at each level there is a general increase of the mean zonal wind with time. Superposed on this trend is a large fluctuating component with a period between speed maxima of about 12 days and a vertical wavelength (computed from the tilt of the oscillations with height) of about 10-12 km. Observations of the temperature field for the same period reveal that the temperature oscillation leads the zonal wind oscillation by one quarter of a cycle (that is, the maximum temperature occurs one-quarter of a period prior to maximum westerlies), which is just the phase relationship required for Kelvin waves (see Fig. 5.8). Furthermore, additional observations from other stations indicate that these oscillations do propagate eastward at about 30 $ms^{-1}$. Therefore there can be little doubt that the observed oscillations are Kelvin waves.

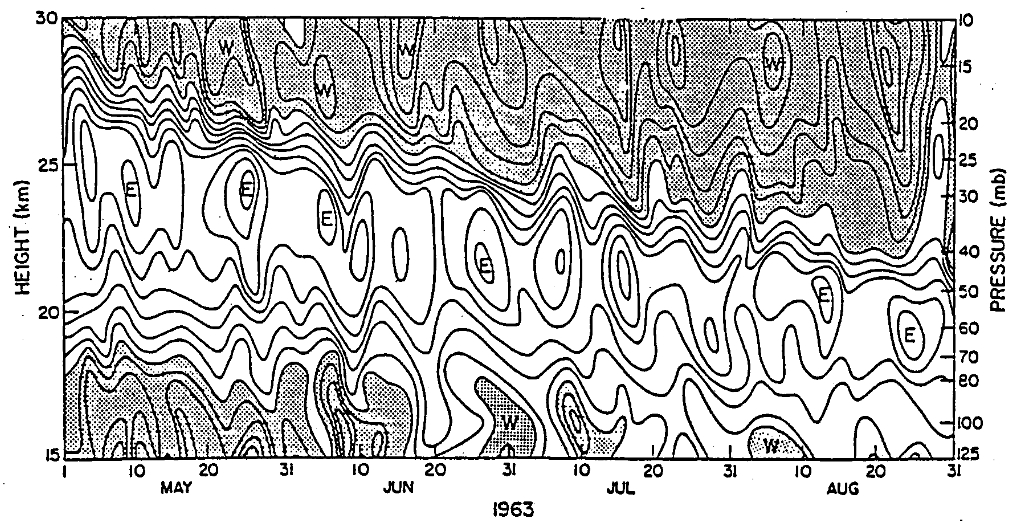


Figure 5.10: Time-height section of zonal wind at Kwajalein (9° latitude). Isotachs at intervals of $ms^{-1}$. Westerlies are shaded. (After Wallace and Kousky, 1968).

The existence of the mixed Rossby-gravity mode has been confirmed also in observational data from the equatorial Pacific. This mode is most easily identified in the meridional wind component, since v is a maximum for it at the equator. The observed waves of this mode have periods in the range of 4-5 days and propagate westward at about 20 $ms^{-1}$. The horizontal wavelength appears to be about 10,000 km, corresponding to zonal wavenumber-4. The observed vertical wavelength is about 6 km, which agrees with the theoretically derived wavelength within the uncertainties of the observations. These waves appear to have significant amplitudes only within about 20° latitude of the equator also, which is consistent with the e-folding width $\sqrt{2L_E} = 2300$ km. Note that in this case, $c = c_p$, but using (5.88) and (5.84), for the mixed Rossby-gravity wave mode. At present it appears that both the Kelvin waves and the mixed Rossby-gravity waves are excited by oscillations in the large-scale convective heating pattern in the equatorial troposphere. Although these waves do not contain much energy compared with typical tropospheric disturbances, they are the predominant disturbances of the equatorial stratosphere. Through their vertical energy and momentum transport they play a crucial role in the general circulation of the stratosphere.

# Exercises

5.1 The linearized momentum equations for a Boussinesq fluid on an equatorial $\beta$-plane are (see Gill, 1982, p449):

$$\frac{\partial u}{\partial t} + 2\Omega w - \beta\, y\, v = -\,(1/\rho_o)\,\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + \beta y u = -\,(1/\rho_o)\,\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} - 2\Omega u = -\,(1/\rho_o)\,\frac{\partial p}{\partial z} + \sigma,$$

Consider the following scaling: horizontal length scale $L = (c/\beta)^{1/2}$, time scale $T = (\beta c)^{-1/2}$, horizontal velocity scale U, pressure scale $P = \rho_o c U$, vertical length scale $H = c/N$, vertical velocity scale $w = (\omega/N)U$, where $\omega$ is a frequency. Show that the Coriolis acceleration associated with the horizontal component of rotation $2\Omega w$ can be neglected if $2\Omega < N$. Show that in this case the vorticity equation reduces to

$$\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0,$$

where $f = \beta y$. Give an interpretation of this equation in the case of steady motions. Note that the foregoing scaling is suggested by a linearized wave analysis of the approximated equation set (see Gill *op. cit.*).

5.2 The dispersion relation for the mixed Rossby-gravity wave is

$$\omega = \tfrac{1}{2}\,kc - \left[\tfrac{1}{4}\,k^2c^2 + c\beta\right]^{1/2}.$$

Show that $\omega \to -\beta/k$ as $k \to 0$ and $\omega \to \frac{1}{2}kc$ as $k \to \infty$.

Show also that although the phase velocity is westward for all wavenumbers, the group velocity is eastward.

# Chapter 6

# Forced Equatorial Waves

## 6.1 Response to steady forcing

Consider homogeneous ocean layer of mean depth H forced by a surface wind stress $\mathbf{X} = (X, Y)$ per unit area. We assume that through the process of turbulent mixing in the vertical, this wind stress is distributed uniformly with depth as body force $\mathbf{X}/(\rho H)$ per unit mass. Suppose that there is also a drag per unit mass acting on the water, modelled by the linear friction law $-r\mathbf{u}$ per unit mass. Then the equations analogous to (3.21) and (3.22) are

$$-\beta y v = -gH\frac{\partial \eta}{\partial x} + X/(\rho H) - ru, \tag{6.1}$$

$$\beta y u = -gH\frac{\partial \eta}{\partial y} + Y/(\rho H) - rv. \tag{6.2}$$

The continuity equation analogous to (3.23) takes the form

$$c^2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -gE/\rho - c^2 r\eta, \tag{6.3}$$

where $E$ may be interpreted as an evaporation rate (i.e. rate of mass removal) and $r\eta$ with $r > 0$ represents a linear damping of the free surface displacement. In the atmospheric situation, a positive/negative evaporation rate is equivalent to the effect of convective heating/cooling (see Chapter 5) and the damping term represents Newtonian cooling due, for example, to infra-red radiation space. Although formally obtained for a shallow homogeneous layer, we have shown that these equations apply for each normal mode, but with a value of appropriate to that mode. Also, the magnitude of the forcing is then determined by expanding the forcing function in normal modes.