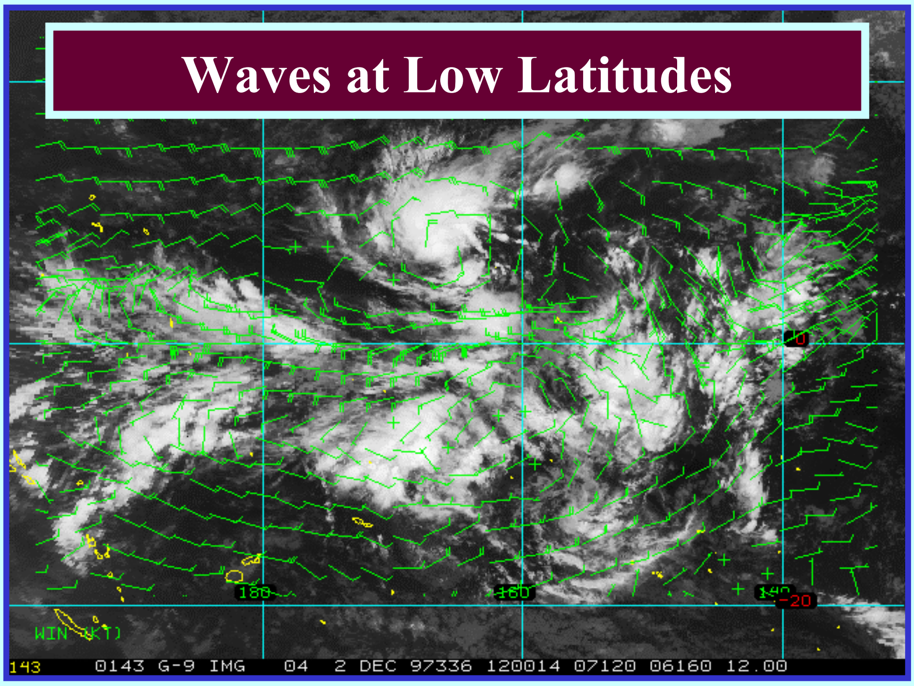
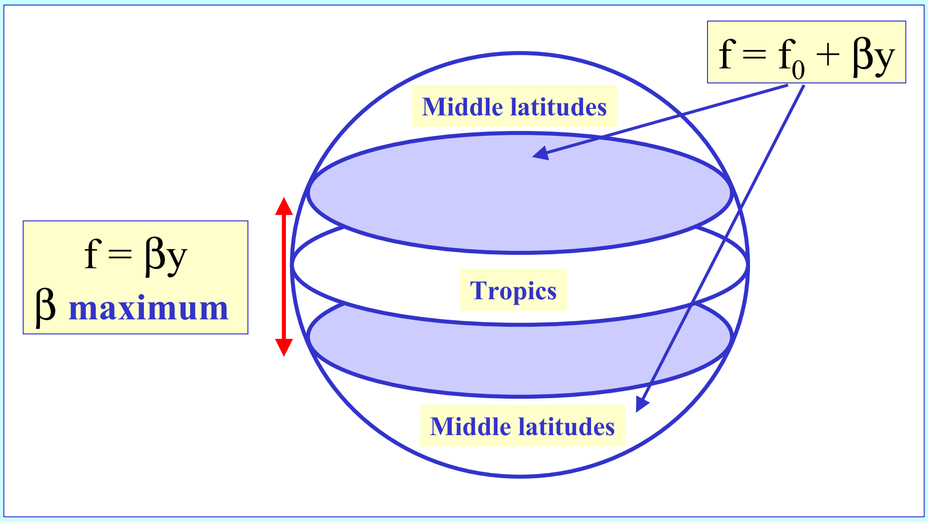
Waves at Low Latitudes



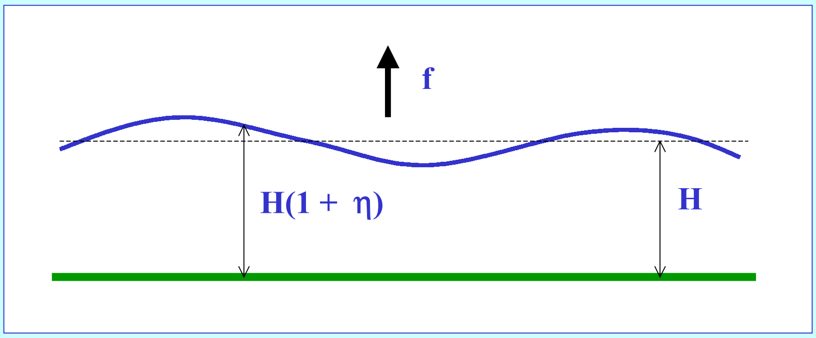
References

- The classical papers on this subject are:
  - **Matsuno** (1966), *J. Meteor. Soc. Japan*
  - **Longuet-Higgins** (1968) *Phil. Trans. Roy. Soc.*
  - **Webster** (1972) *Mon. Wea. Rev.*
  - **Gill** (1980) *Quart. J. Roy. Meteor. Soc.*
- A relatively recent review is given by Lim and Chang (1987) In **Monsoon Meteorology**, Ed. C. P. **Chang** and T. N. **Krishnamurti**, Oxford Univ. Press

Tropics versus Middle Latitudes



Theory of wave motions in a divergent barotropic fluid on an f-plane



Linearized "shallow-water" equations

$$\partial_t u - f v = -c^2 \partial_x \eta$$

$$\partial_t v + f u = -c^2 \partial_y \eta$$

$$\partial_t \eta + \partial_x u + \partial_y v = 0$$

On an f-plane put

$$v = \hat{v} \sin(kx - \omega t)$$

$$(u, \eta) = (\hat{u}, \hat{\eta}) \cos(kx - \omega t)$$

$$\Rightarrow \quad \omega\left(\omega^2 - f^2 - c^2 k^2\right) = 0$$

Dispersion relation

$$\omega\left(\omega^2 - f^2 - c^2 k^2\right) = 0$$

$$\Rightarrow \quad \omega = 0 \quad \text{or} \quad \omega^2 = f^2 + c^2 k^2$$

Steady (geostrophic) flow ($\omega = 0$)

Inertia gravity waves ($\omega^2 = f^2 + c^2k^2$)

Inertia-gravity waves

$$\omega^2 = f^2 + c^2k^2$$

$$c_p = \frac{\omega}{k} = \pm\sqrt{\left(c^2 + \frac{f^2}{k^2}\right)} = \pm c\sqrt{\left(1 + \frac{1}{L_R^2\,k^2}\right)}$$

$L_R = c/f$ **is the Rossby radius of deformation**

The importance of inertial effects compared with gravitational effects is characterized by the size of the parameter $L_R^2\,k^2$, i.e. by the wavelength of waves compared with the Rossby radius of deformation.

Wave motions in a divergent barotropic fluid on a mid-latitude β-plane

Now f **is a function of** y **(specifically, where** $f = f_0 + \beta y,\ f_0 \neq 0$**)**

~~$\partial_t u - f(y)v = -c^2\,\partial_x \eta$~~ $\qquad \partial_t u - f_0 v = -c^2\,\partial_x \eta$

~~$\partial_t v + f(y)u = -c^2\,\partial_y \eta$~~ $\qquad \partial_t \zeta + \beta v = f_0\,\partial_t\,\eta$

$$\partial_t \eta + \partial_x u + \partial_y v = 0$$

$$\zeta = v_x - u_y$$

$$\left(\omega^2 - c^2\,k^2\right)\left(\omega k + \beta\right) - f_0^2\,\omega k = 0$$

Dispersion relation

$$\left(\omega^2 - c^2 k^2\right)\left(\omega k + \beta\right) - f_0^2 \omega k = 0$$

Write $\omega = f_0 \nu$ **and** $k = \mu / L_R$

$$\Rightarrow \quad (\nu^2 - \mu^2)(\nu\mu + \varepsilon) - \nu\mu = 0$$

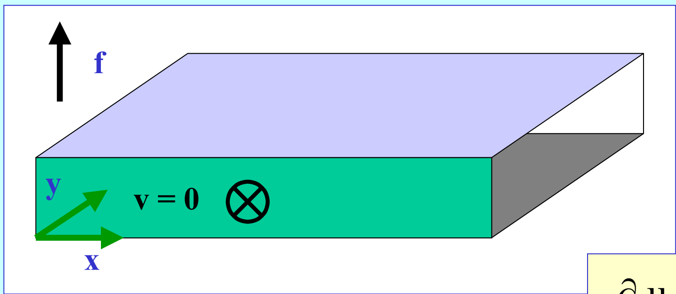
Solution $\omega = 0$ **now becomes** $\nu = -\dfrac{\varepsilon\mu}{1+\mu^2}$

Divergent Rossby waves $\omega = -\dfrac{\beta k}{k^2 + 1/L_R^2}$

Inertia-gravity waves $\omega^2 = f^2 + c^2k^2$

Kelvin wave

A wave that owes its existence to the presence of a boundary



$v \equiv 0$ $\Rightarrow$

$$\partial_t u = -c^2 \partial_x \eta$$
$$f u = -c^2 \partial_y \eta$$
$$\partial_t \eta + \partial_x u = 0$$

$$\partial_t u = -c^2 \partial_x \eta$$
$$fu = -c^2 \partial_y \eta$$
$$\partial_t \eta + \partial_x u = 0$$

$$\partial_{tt} \eta = c^2 \partial_{xx} \eta$$

$$\eta = F(x - ct, y) + G(x + ct, y)$$

Put $X = x - ct,\ Y = x + ct$

$$\partial_x u = -\partial_t \eta = c(\partial_X F - \partial_X G) = c(\partial_x F - \partial_x G)$$

Integrate $\partial_x u = c(\partial_x F - \partial_x G)$

$$u = c(F - G)$$

$$\eta = F(x - ct, y) + G(x + ct, y)$$

$$fu = -c^2 \partial_y \eta$$

$$fc[F - G] = -c^2 [\partial_y F + \partial_y G]$$

Since F and G are arbitrary functions

$\partial_y F + (f/c) F = 0$ **and** $\partial_y G - (f/c) G = 0$

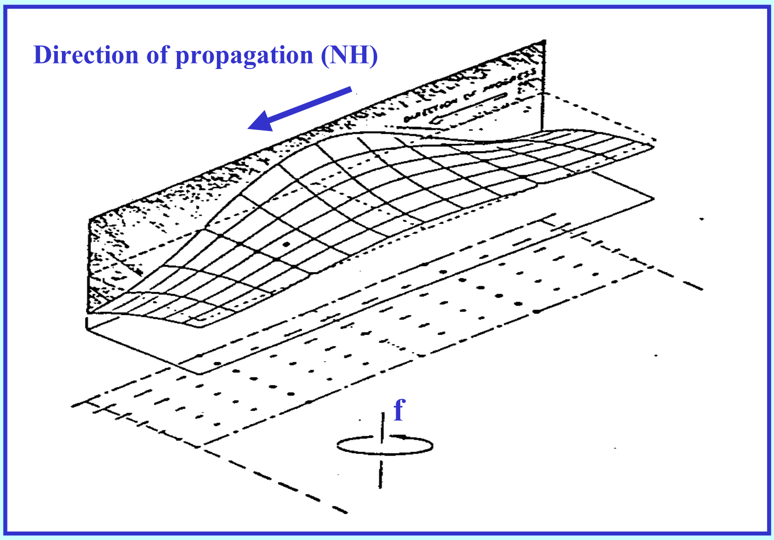
$$\partial_y F + (f/c)F = 0 \quad \Rightarrow \quad F = F_0(X)e^{-fy/c}$$

$$\partial_y G - (f/c)G = 0 \quad \Rightarrow \quad$$ ~~$G = G_0(Y)e^{fy/c}$~~ **if** $y \to \infty$

$$\Rightarrow \quad \eta = F_0(x - ct)e^{-fy/c}$$

$$u = cF_0(x - ct)e^{-fy/c}$$

- **The Kelvin wave is essentially a gravity wave that is "trapped" along the boundary by the rotation.**
- **The velocity perturbation u is always such that geostrophic balance occurs in the y-direction**



Flow configuration in a Kelvin wave

The solution

$$\eta = G_0(x + ct)\, e^{fy/c}$$

represents a trapped wave moving at speed c with the boundary on the right (left) in the Northern (Southern) Hemisphere when $f > 0$ ($f < 0$).

The equatorial beta-plane approximation

At the equator, $f_0 = 0$, but β is a maximum.

⇨ Near the equator, set $f = \beta y$.

This equatorial beta-plane approximation that may be derived from the equations for motion on a sphere (see e.g. Gill, 1982).

$$\partial_t u - \beta y v = -c^2 \partial_x \eta \qquad \partial_t \eta + \partial_x u + \partial_y v = 0$$

$$\partial_t v + \beta y u = -c^2 \partial_y \eta \qquad \partial_t (\zeta - f\eta) + \beta v = 0$$

$$\partial_t[(\partial_{tt} v + f^2 v)/c^2 - (\partial_{xx} v + \partial_{yy} v)] - \beta \partial_x v = 0$$

Put $v = \hat{v}(y)\exp\left[i(kx - \omega t)\right]$

$$\Rightarrow \quad \frac{d^2\hat{v}}{dy^2} + \left[\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right]\hat{v} = 0$$

Scale the independent variables t, x, y, **using the time scale** $(\beta c)^{-1/2}$ **and length scale** $(c/\beta)^{1/2}$**.**

the equatorial Rossby radius L_E

Scale $\omega = (c\beta)^{1/2}\nu$ **and** $k = \mu\,(c/\beta)^{-1/2}$ **.**

Schrödinger's Wave Equation

$$\frac{d^2\hat{v}}{dy^2} + \left[\nu^2 - \mu^2 - \frac{\mu}{\nu} - y^2\right]\hat{v} = 0$$

Schrödinger´s equation - arises in quantum mechanics

Solutions that are bounded as $y \to \pm\infty$ **are possible only if**

$$\nu^2 - \mu^2 - \mu\nu^{-1} = 2n + 1, (n = 0, 1, 2, \ldots)$$

These solutions have the form of parabolic cylinder functions

In dimensional terms

$$v(x,y,t) = H_n\left((\beta/c)^{1/2} y\right) \exp\left(-\beta y^2/2c\right) \cos(kx - \omega t)$$

Multiply by $2^{-n/2}$

$$\Rightarrow \quad v(x,y,t) = D_n\left((\beta/c)^{1/2} y\right) \cos(kx - \omega t)$$

D_n **is the parabolic cylinder function of order** n **and**
H_n **is the Hermite polynomial of order** n**.**

Dispersion relation

$$\omega^2/c^2 - k^2 - \beta k/\omega = (2n+1)\beta/c$$

A cubic equation for ω **for each value of** n

- **A whole range of wave modes is possible:**
  - **The equatorial Kelvin wave**
  - **Equatorial gravity waves**
  - **Rossby waves**
  - **The mixed gravity-Rossby wave**

The equatorial Kelvin Wave

$$v = 0$$

$$\partial_t u = -c^2 \partial_x \eta$$
$$\beta y u = -c^2 \partial_y \eta$$
$$\partial_t \eta + \partial_x u = 0$$

cf

$$\partial_t u = -c^2 \partial_x \eta$$
$$f u = -c^2 \partial_y \eta$$
$$\partial_t \eta + \partial_x u = 0$$

→ $$\partial_{tt} \eta = c^2 \partial_{xx} \eta$$

→ $$\eta = F(x - ct, y) + G(x + ct, y)$$

As before

$$\partial_y F + (\beta y / c) F = 0$$
$$\partial_y G - (\beta y / c) G = 0$$

→ $$F = F_0(X) \exp(-\beta y^2 / 2c)$$

Note that $G \to \infty$ **as** $y \to \pm \infty$

The equatorial Kelvin wave

$$\eta(x,y,t) = F_0(x-ct)\exp\left(-\beta y^2/2c\right)$$
$$u(x;y,t) = c\,F_0(x-ct)\exp\left(-\beta y^2/2c\right.$$
$$v(x,y,t) = 0$$

- **The wave is an eastward propagating gravity wave that is trapped in the equatorial waveguide by Coriolis forces.**
- **Note that it is nondispersive and has a meridional scale on the order of $L_E = (c/\beta)^{1/2}$.**

Equatorial Gravity Waves

Dispersion relation

$$\omega^2/c^2 - k^2 - \beta k/\omega = (2n+1)\,\beta/c$$

For $n \geq 1$, the waves subdivide into two classes of solutions:

Case $\beta k/\omega$ small:

$$\omega^2 \approx (2n+1)\beta c + k^2c^2$$

These waves are equatorially-trapped gravity waves, or equatorially-trapped Poincaré waves.

Equatorial Rossby Waves

Case ω^2/c^2 **small:**

Dispension relation $\omega^2/c^2 - k^2 - \beta k/\omega = (2n+1)\beta/c$

$$\Rightarrow \quad \omega \approx \frac{-\beta k}{k^2 + (2n+1)\beta/c}$$

These waves are equatorially-trapped planetary waves, or equatorially-trapped Rossby waves.

The mixed Rossby - Gravity Wave

When $n = 0$

$$\nu^2 - \mu^2 - \mu\nu^{-1} = 2n+1, \quad (n=0, 1, 2, \ldots)$$

$$\Rightarrow \quad (\nu+\mu)(\nu-\mu-1/\nu) = 0$$

$$\Rightarrow \quad \nu = -\mu \quad \text{or} \quad \nu^2 - \nu\mu - 1 = 0$$

Indeterminate solution for u

$$\omega_+ = \tfrac{1}{2}kc + [\tfrac{1}{4}k^2c^2 + c\beta]^{1/2}$$

- an eastward propagating gravity wave

$$\omega_- = \tfrac{1}{2}kc - [\tfrac{1}{4}k^2c^2 + c\beta]^{1/2}$$

- a gravity wave if k is small
- a Rossby wave if k is large

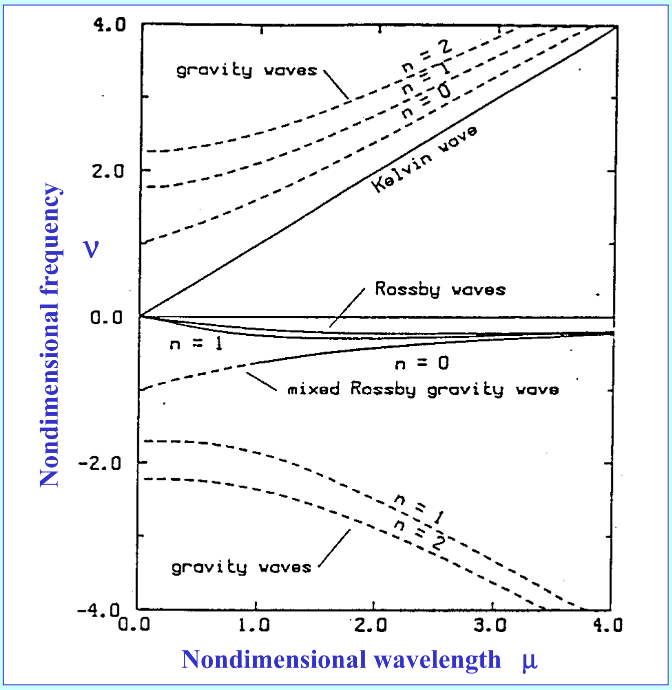
- as $k \to 0$, $\omega \to -(c\beta)^{1/2}$, which agrees with the limit of the gravity wave solution.
- as $k \to \infty$, $\omega \approx -\beta/k$, which agrees with the limit of the Rossby wave solution.
- The solution $n = 0$ is called a **mixed Rossby-gravity wave.**
- The **phase velocity** of this mode can be either eastward or westward, but the **group velocity** is always eastward.

- The **Kelvin wave** solution is sometimes called the $n = -1$ wave because

$$\omega^2/c^2 - k^2 - \beta k/\omega = (2n+1)\,\beta/c$$

is satisfied by the Kelvin-wave dispersion relation (i.e. $\omega = kc$) when $n = -1$ (last term is then $-\beta k/\omega$).

Dispersion relation diagram



Structure of wave modes

Substitute $v = \hat{v}\sin(kx - \omega t)$

$$(u, \eta) = (\hat{u}, \hat{\eta})\cos(kx - \omega t)$$

in $\partial_t u - \beta y v = -c^2 \partial_x \eta$

$$\Rightarrow \quad \omega\hat{u} - \beta y\hat{v} = kc^2\hat{\eta}$$

$$\partial_t \eta + \partial_x u + \partial_y v = 0$$

$$\Rightarrow \quad \omega\hat{\eta} - k\hat{u} + \frac{d\hat{v}}{dy} = 0$$

Solve for $\hat{u}\,,\hat{\eta}$ →

Scale and put $y = L_E Y$

$$\begin{cases} \left(\nu^2 - \mu^2\right)\hat{u} = \nu Y \hat{v} - \mu \dfrac{d\hat{v}}{dY} \\ \left(\nu^2 - \mu^2\right)\hat{\eta} = \mu Y \hat{v} - \nu \dfrac{d\hat{v}}{dY} \end{cases}$$

Now

$$\hat{v}(Y) = \hat{v}_n = \exp(-\tfrac{1}{2} Y^2) H_n(Y)$$

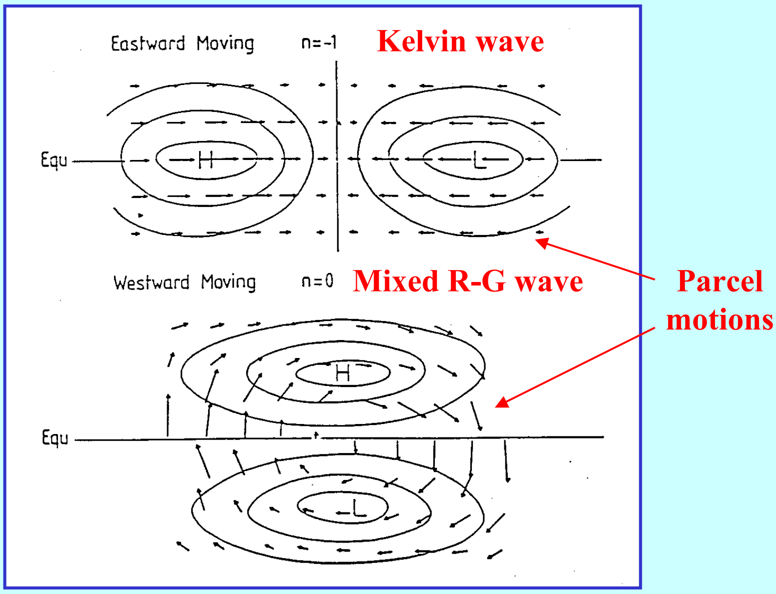
and

$$\frac{d\hat{v}}{dY} = -Y \hat{v}_n + \exp\left(-\tfrac{1}{2} Y^2\right) \frac{dH_n}{dY}$$

Properties of the Hermite polynomials

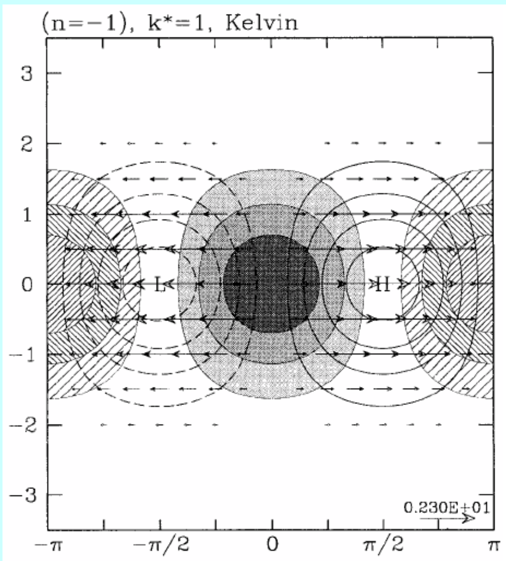
$$\begin{cases} \dfrac{dH_n}{dY} = 2n H_{n-1}(Y) \\ H_{n+1}(Y) = 2Y H_n(Y) - 2n H_{n-1}(Y) \end{cases}$$

$$\Rightarrow \begin{cases} (\nu^2 - \mu^2)\hat{u}_n = \tfrac{1}{2}(\nu + \mu)\hat{v}_{n+1} + n(\nu - \mu)\hat{v}_{n-1} \\ (\nu^2 - \mu^2)\hat{\eta}_n = \tfrac{1}{2}(\nu + \mu)\hat{v}_{n+1} - n(\nu - \mu)\hat{v}_{n-1} \end{cases}$$

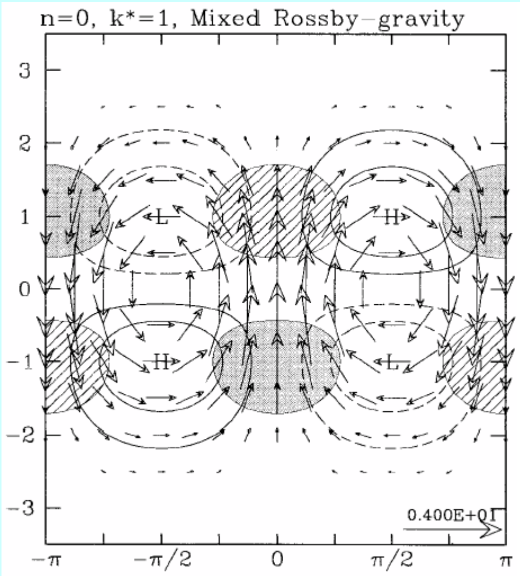


Horizontal structure of the Kelvin wave and of a westward propagating Mixed Rossby-gravity wave.

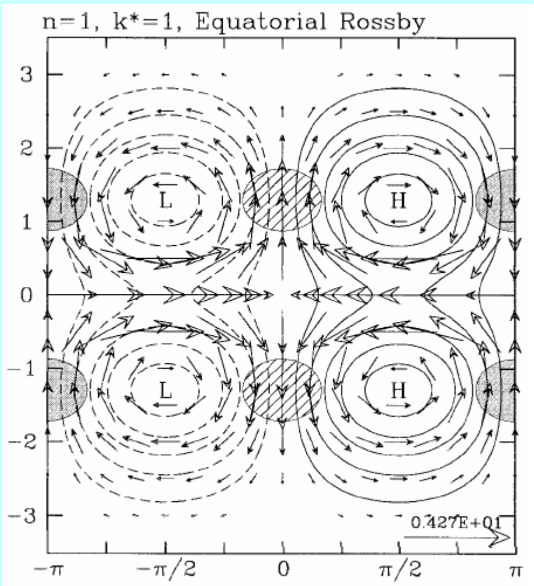
The equatorial Kelvin Wave



The mixed Rossby-gravity wave



The equatorial Rossby wave



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