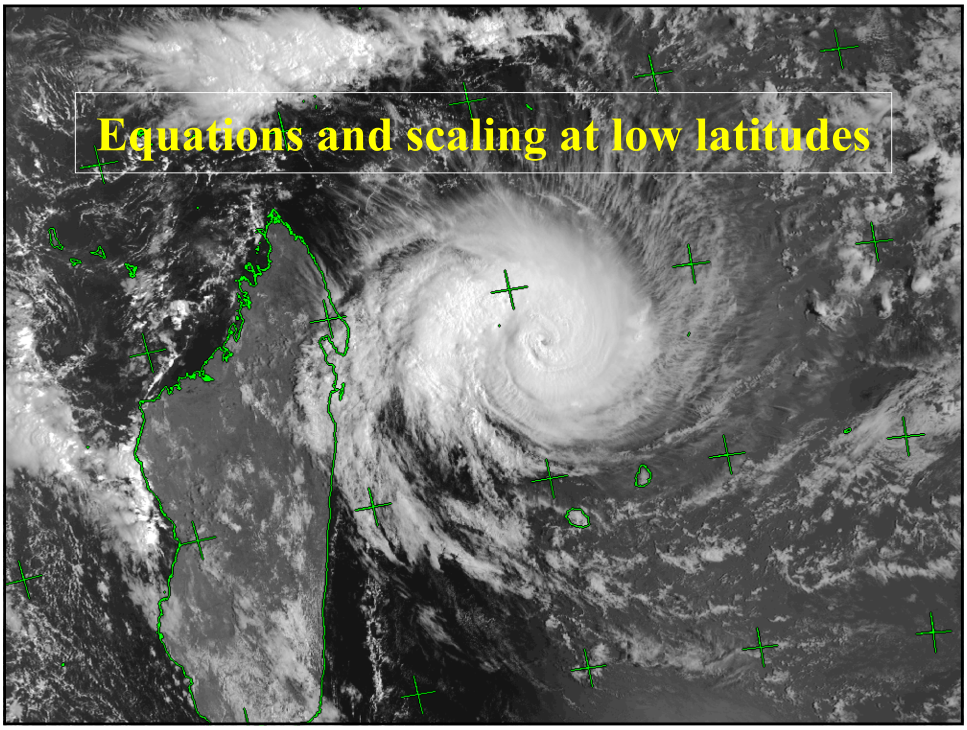


The governing equations on a sphere

Equations of motion for a dry atmosphere:

Momentum $$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} - \Omega \wedge \mathbf{u} + \frac{1}{\rho}\mathbf{F}$$

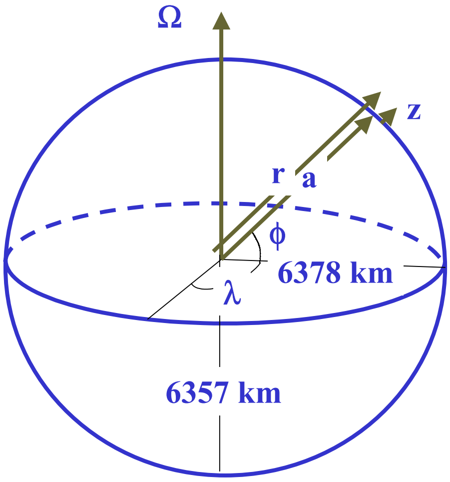
Continuity $$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

Thermodynamic $$\frac{D}{Dt}\ln\theta = \frac{Q}{c_p T}$$

State $$p = \rho R T$$

Definition of θ $$\theta = T\left(\frac{p^*}{p}\right)^{\kappa}$$

The (λ, ϕ, z) coordinate system



$r = a + z$

$z \ll a \Rightarrow r \approx a$

The equations in the (λ, ϕ, z) coordinate system

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \underline{\frac{uw}{a}} = -\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} + 2\Omega\, v\sin\phi - \underline{2\Omega\, w\cos\phi}$$

$$\frac{Dv}{Dt} + \frac{Dw}{Dt} - \underline{\frac{u^2 + v^2}{a}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \underline{2\Omega u\cos\phi}$$

$$\frac{Dw}{Dt} - \frac{u^2\tan\phi}{a} + \underline{\frac{vw}{a}} = -\frac{1}{\rho a}\frac{\partial p}{\partial\phi} - 2\Omega\, u\sin\phi$$

$$\frac{D\rho}{Dt} = -\frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right] - \rho\frac{\partial w}{\partial z} - 2\rho\underline{\frac{w}{a}}$$

$$\mathbf{u} = a\cos\phi\frac{d\lambda}{dt}\mathbf{i} + r\frac{d\phi}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

The hydrostatic equation at low latitudes

Introduce the buoyancy force σ:

$$\frac{Dw}{Dt}+\frac{1}{\rho}\frac{\partial p}{\partial z}-\sigma=\cancel{\frac{u^2+v^2}{a}}+\cancel{2\Omega u\cos\phi}$$

Scales $\quad \frac{W}{\tau} \quad \frac{\delta p}{\rho D} \quad \Sigma \quad \frac{U^2}{a} \quad 2\Omega U$

$\delta p \approx 1$ mb $= 10^2$ Pa **over the troposphere depth** ≈ 20 km

$\Rightarrow$ $\delta p/(\rho D) \approx 10^2 \text{ Pa} \div (1.0 \times 2.0 \times 10^4) = 0.5 \times 10^{-2} \text{ ms}^{-2}$

$U \approx 10 \text{ ms}^{-1}$, $\Omega \approx 10^{-5} \text{ s}^{-1}$, $a \approx 6 \times 10^6$ m

$\Rightarrow$ $U^2/a \approx 10^{-4} \text{ ms}^{-2}$, $2\Omega U \approx 10^{-4} \text{ ms}^{-2}$

Hydrostatic equation (continued)

The question remains, how large is the ratio:

$$\left|\frac{Dw}{Dt}\Big/\left(\frac{1}{\rho}\frac{\delta p}{\delta z}\right)\right| \approx \frac{W}{\tau}\Big/\left(\frac{1}{\rho}\frac{\delta p}{D}\right)$$

The horizontal equation of motion gives two possible scales for δp :

High frequency limit:

$1/\tau >> f, \qquad \delta p \approx P_1 = \rho L U/\tau \qquad \frac{W}{\tau}\Big/\left(\frac{1}{\rho}\frac{P_1}{D}\right) = \frac{W}{U}\frac{D}{L}$

Hydrostatic balance if $W << U$ **and/or** $D << L$**, provided the other ratio is no more than** $O(1)$**.**

Hydrostatic equation (continued)

Low frequency limit:

$$1/\tau << f, \qquad \delta p \approx P_2 = \rho LUf \qquad \frac{W}{\tau} / \left(\frac{1}{\rho} \frac{P_2}{D} \right) = \frac{W}{U} \frac{D}{L} \frac{1}{\tau f}$$

Now even if $W \approx U$ and $D \approx L$, the hydrostatic equation is valid (note that P_2 was derived on the assumption that $1/\tau << f$).

For synoptic-scale motions ($L \approx 10^6$ m) and planetary-scale motions ($L \approx a$), $L >> D$ and the hydrostatic approximation is valid even if ($\tau \approx f$), and therefore as f decreases towards the equator.

⇒ **We are well justified in treating planetary-scale motions as hydrostatic**

Some caution is required:

The unapproximated kinetic energy equation is:

$$\frac{D}{Dt}\left[\frac{1}{2}\left(u^2+v^2+w^2\right)\right] = -\frac{1}{\rho}\left[\frac{u}{a\cos\phi}\frac{\partial p}{\partial \lambda} + \frac{v}{a}\frac{\partial p}{\partial \phi} + w\frac{\partial p}{\partial z}\right] - gw$$

It contains no geometric terms or Coriolis terms!

But if we simply make the system hydrostatic and assume that $|w| << |u|, |v|$, the approximate form becomes:

$$\frac{1}{2}\frac{D}{Dt}\left(u^2+v^2\right) = -\frac{1}{\rho}\left[\frac{u}{a\cos\phi}\frac{\partial p}{\partial \lambda} + \frac{v}{a}\frac{\partial p}{\partial \phi}\right] - \left[2\Omega uw\cos\phi - \left(\frac{u^2+v^2}{a}\right)w\right]$$

A spurious energy source!

Energetic consistency

$$\frac{1}{2}\frac{D}{Dt}\left(u^2+v^2\right)=-\frac{1}{\rho}\left[\frac{u}{a\cos\phi}\frac{\partial p}{\partial\lambda}+\frac{v}{a}\frac{\partial p}{\partial\phi}\right]-\left[2\Omega uw\cos\phi-\left(\frac{u^2+v^2}{a}\right)w\right]$$

- **The last term in square brackets represents a fictitious or spurious energy source and arises from the lack of consistency in scaling the system of equations.**
- **That is, each equation is interrelated to the others and it is incorrect to scale one without consideration of the others.**
- **If the hydrostatic equation is used, energetic consistency requires that certain curvature and Coriolis terms must be omitted also.**
- **These are the terms underlined earlier by a red line.**
- **Similar considerations to these are necessary when "sound - proofing" the equations (see e.g. ADM, Ch. 2).**

The equations in the (λ, φ, z) coordinate system

$$\frac{Du}{Dt}-\frac{uv\tan\phi}{a}+\underline{\frac{uw}{a}}=-\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda}+2\Omega\, v\sin\phi-\underline{2\Omega\, w\cos\phi}$$

$$\frac{Dv}{Dt}+\frac{Dw}{Dt}-\underline{\frac{u^2+v^2}{a}}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+\underline{2\Omega u\cos\phi}$$

$$\frac{Dw}{Dt}-\frac{u^2\tan\phi}{a}+\underline{\frac{vw}{a}}=-\frac{1}{\rho a}\frac{\partial p}{\partial\phi}-2\Omega\, u\sin\phi$$

$$\frac{D\rho}{Dt}=-\frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda}+\frac{\partial}{\partial\phi}\left(v\cos\phi\right)\right]-\rho\frac{\partial w}{\partial z}-2\rho\underline{\frac{w}{a}}$$

$$\mathbf{u}=a\cos\phi\frac{d\lambda}{dt}\mathbf{i}+r\frac{d\phi}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$$

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla$$

Hydrostatic formulation of the momentum equations with friction terms included

The energetically-consistent hydrostatic formulation of the momentum equations is:

$$\frac{Du}{Dt} = -\frac{1}{\rho a \cos\phi}\frac{\partial p}{\partial \lambda} + \left(2\Omega + \frac{u}{a\cos\phi}\right) v \sin\phi + F_\lambda$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} - \left(2\Omega + \frac{u}{a\cos\phi}\right) u \sin\phi + F_\phi$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Friction terms

Scaling at low latitudes

$$(\partial_t + \mathbf{V}\cdot\nabla_h)\mathbf{V} + w\partial_z \mathbf{V} + f\mathbf{k}\wedge\mathbf{V} = -(1/\rho)\nabla_h p$$

$$0 = -(1/\rho)\partial_z p - g$$

$$(\partial_t + \mathbf{V}\cdot\nabla_h)\rho + \rho\nabla_h\cdot\mathbf{V} + \partial_z(\rho w) = 0$$

$$(\partial_t + \mathbf{V}\cdot\nabla_h)\ln\theta + w\partial_z \ln\theta = Q/(c_p T)$$

Pressure scale height: $\frac{1}{H_s} = -\frac{1}{p_0}\frac{dp_0}{dz}$

Hydrostatic ⇒ $H_s = p_0 / g\rho_0$

Middle-latitude quasi-geostrophic scaling

Then $\delta p = \rho_o LUf$

$$\frac{\delta p}{p_0} \approx \frac{\delta p}{\rho_o g H_s} = \frac{fUL}{gH_s} = \frac{F^2}{Ro}\Bigg|_{Ro<<1}$$

$H_s = p_0 / g\rho_0$

$Ro = \frac{U}{fL}$ **is the Rossby number**

$F = \frac{U}{(gH_s)^{1/2}}$ **is the Froude number**

Low-latitude scaling

$$\frac{\delta p}{p_0} \approx \frac{\delta p}{\rho_o g H_s} = \frac{fUL}{gH_s} = \frac{F^2}{Ro}\Bigg|_{Ro<<1}$$

This is still valid for $Ro \approx 1$ **because** $\delta p = \rho LUf$ **provides the same scale as the inertial scale** $\delta p = \rho U^2$.

The hydrostatic equation $\delta_z p' = -g\rho' \Rightarrow \delta p/D = g\delta\rho$

$$\frac{\delta\rho}{\rho_0} \approx \frac{\delta p}{gD\rho_0} \approx \frac{\delta p}{p_0}\left(\frac{H_s}{D}\right) \approx \frac{\delta p}{p_0} = \frac{F^2}{Ro}\Bigg|_{Ro<<1}$$

Assume $D \approx H_s$

Low-latitude scaling (continued)

From the definition of θ, $(1-\kappa)\ln p = \ln \rho + \ln \theta + \text{constant}$

$$\Rightarrow \frac{\delta\theta}{\theta_0} \approx -\kappa \frac{\delta p}{p_0} \approx \frac{F^2}{Ro}\Big|_{Ro<<1}$$

Typically: $g \approx 10\ ms^{-2}$, $H_s \approx 10^4$ m.

For $U \approx 10\ ms^{-1}$, $f \approx 10^{-4}\ s^{-1}$, $Ro = 0.1$ **and** $F^2 = 10^{-3}$, $\Rightarrow$ **in middle latitudes**

$$\frac{\delta\rho}{\rho_0} \approx \frac{\delta p}{p_0} \approx \frac{\delta\theta}{\theta_0} \approx 10^{-2}$$

$\Rightarrow$ **For geostrophic motions, fluctuations in** p, ρ, **and** θ **are small.**

Low-latitude scaling (continued)

At low latitudes, $f \approx 10^{-5}\ s^{-1}$**, so that for the same scales of motion as above,** $Ro = 1$**.** $\Rightarrow$ **the advection terms in the momentum equation are comparable with the horizontal pressure gradient.**

But the foregoing scaling remains valid $\Rightarrow$

$$\frac{\delta\rho}{\rho_0} \approx \frac{\delta p}{p_0} \approx \frac{\delta\theta}{\theta_0} \approx 10^{-3}$$

$\Rightarrow$ **In the tropics, fluctuations in** p, ρ, **and** θ **are an order of magnitude smaller than in middle latitudes.**

$\Rightarrow$ **The adjustment to a pressure gradient imbalance is less constrained by rotation in the tropics .**

Adiabatic scaling

Adiabatic form of $\left(\partial_t + \mathbf{V}\cdot\nabla_h\right)\ln\theta + w\,\partial_z \ln\theta = \cancel{Q}/\left(c_p T\right)$

put $Q = 0$

Scaling $\Rightarrow$ $$\frac{U}{L}\frac{\delta\theta}{\theta_0} \approx W\frac{1}{\theta_0}\frac{d\theta_0}{dz}$$

Define the Brunt-Väisälä frequency $$N^2 = \frac{g}{\theta_0}\frac{d\theta_0}{dz}$$

Define the Richardson number $$Ri = \frac{N^2 H_s{}^2}{U^2}$$

$$\frac{U}{L}\frac{F^2}{Ro} \approx W\frac{N^2}{g}, \quad \text{or} \quad \frac{W}{D} \approx \frac{U}{L}\frac{1}{Ro\,Ri}$$ **valid for** $Ro \approx 1$

Vertical velocities tiny

$$\frac{W}{D} \approx \frac{U}{L}\frac{1}{Ro\,Ri}$$

For the same scales of motion and in the absence of convective processes of substantial magnitude, we may expect the vertical velocity in the equatorial regions to be considerably smaller than in the middle latitudes.

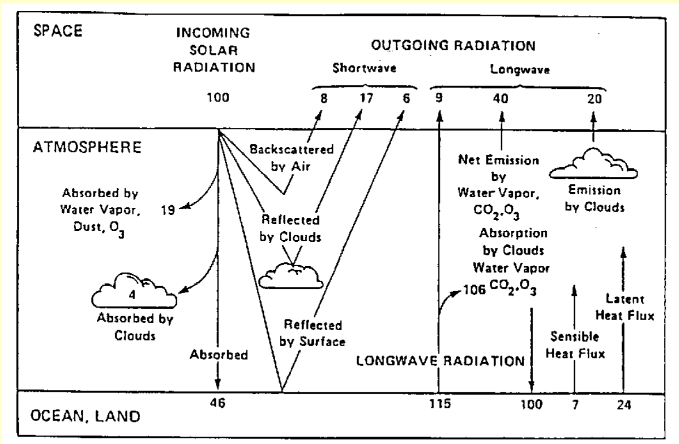
Typical values for synoptic scale systems:

$U = 10$ **ms**$^{-1}$, $D = 10$ **km,** $L = 1000$ **km,** $H_s = 10$ **km,** $N = 10^{-2}$ **s**$^{-1}$**,**
$R = 10^2$ $\Rightarrow$ $W = 10^{-3} / Ro$ **ms**$^{-1}$**. In the tropics,** $Ro \approx 1$
$\Rightarrow$ **vertical velocities on the order of** 1 **mm/s (exceedingly tiny!)**

Diabatic processes

- In the tropics it is important to consider diabatic processes.
- Consider first the diabatic contribution in regions away from active convection.
- Then the *net* diabatic heating is associated primarily with radiative cooling to space alone.
- The next figure shows the annual heat balance of the earth´s atmosphere.

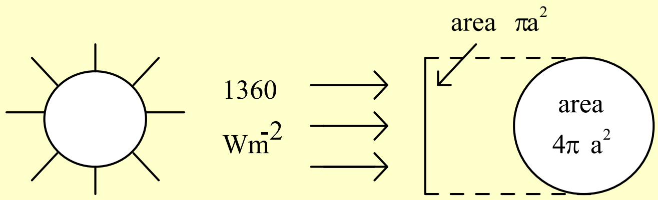
The atmospheric heat balance



Units are percent of incoming solar radiation. The solar fluxes are shown on the left-hand side, and the long wave (thermal IR) fluxes are on the right-hand side.

(from Lindzen, 1990)

Distribution of incoming solar radiation



The incoming solar radiation of 1360 **Wm^{-2} (the solar constant) intercepted by the earth** ($\pi a^2 \times 1360$ **W) is distributed, when averaged over a day or longer, over an area** $4\pi a^2$ **.**

Outgoing terrestrial radiation

- **The atmosphere loses heat by radiation over** 1 **day or longer at the rate** $\Delta Q = 0.31 \times 0.25 \times 1360$ **Wm^{-2}.**
- **In unit time, this corresponds to a temperature change** ΔT **given by** $\Delta Q = c_p M \Delta T$**, where** M **is the mass of a column of atmosphere** 1 **m^2 in cross-section.**
- **Since** M = (mean surface pressure) /g, **we find that**

$$\Delta T = -\frac{0.31 \times 0.25 \times 1360 \times 24 \times 3600}{1005 \times 1.013 \times 10^4} \approx -0.9 \mathrm{K / day}$$

- **Actually, the rate of cooling varies with latitude.**

Latitudinal variation in radiative cooling

- **From the surface to 150 mb (i.e. for ≈ 85% of the atmosphere´s mass):**
  - $\Delta T = -1.2$ **K/day from** 0 - 30° **lat.,**
  - $\Delta T = -0.88$ **K/day from** 30 - 60° **lat.,**
  - $\Delta T = -0.57$ **K/day from** 60 - 90° **lat.**
- **The stratosphere and mesosphere warm a little on average, but even together they have relatively little mass.**

The estimate

$$\frac{\delta\rho}{\rho_0} \approx \frac{\delta p}{p_0} \approx \frac{\delta\theta}{\theta_0} \approx 10^{-3}$$

suggests that for synoptic scale systems in the tropics, we can expect potential temperature changes associated with adiabatic changes of no more than a fraction of a degree.

The estimate

$$\frac{W}{D} \approx \frac{U}{L}\frac{1}{Ro\,Ri}$$

shows that associated vertical motions are on the order DU/(LRi) **which is typically**

$$10^4 \times 10 \div (10^6 \times (10^{-4} \times 10^8 \div 10^2)) \approx 10^{-3}\ \mathrm{ms}^{-1}$$

Radiative cooling at the rate $Q/c_p = -1.2$ **K/day would lead to a subsidence rate which we estimate from**

$$\left(\partial_t + \mathbf{V}\cdot\nabla_h\right)\ln\theta + w\,\partial_z \ln\theta = Q/\left(c_p T\right)$$

as $$WN^2/g \approx \left(Q/c_p\right)/T$$

$$W \approx -\frac{g}{N^2}\times\frac{1\cdot 2}{300}\times\frac{1}{24\times 3600} = -0.5\ \mathrm{cm/sec}$$

We may expect slow subsidence over much of the tropics and the vertical velocities associated with radiative cooling are somewhat larger than those arising from synoptic scale adiabatic motions.

Implications of the scaling

Vertical component of the vorticity equation

$$\underset{A}{\left(\partial_t + \mathbf{V}\cdot\nabla\right)\zeta} + \underset{B}{\left[\zeta\nabla\cdot\mathbf{V} + w\,\partial_z\zeta + \mathbf{k}\cdot\nabla w\wedge\partial_z\mathbf{V}\right]} + \underset{C}{\mathbf{V}\cdot\nabla f} + \underset{D}{f\,\nabla\cdot\mathbf{V}}$$

$$= \underset{E}{\mathbf{k}\wedge\left[(1/\rho)\nabla\rho\wedge(1/\rho)\nabla p\right]}$$

Compare the scales of each term with the scale for term A for $Ro \ll 1$ **and** $Ro \approx 1$**.**

$$(\partial_t + \mathbf{V}\cdot\nabla)\zeta + [\zeta\nabla\cdot\mathbf{V} + w\partial_z\zeta + \mathbf{k}\cdot\nabla w \wedge \partial_z\mathbf{V}] + \mathbf{V}\cdot\nabla f + f\nabla\cdot\mathbf{V}$$

A B C D

$$= \mathbf{k}\wedge\left[(1/\rho)\nabla\rho \wedge (1/\rho)\nabla p\right]$$

E

Term	A	B	C	D	E
General	1	$\frac{L}{U}\frac{W}{D}$	$\frac{2\Omega}{U}\frac{L^2}{a}\cos\phi$	$\frac{L}{U}\frac{W}{D}\frac{1}{Ro}$	$\frac{F^2}{Ro^2}$
Ro << 1	1	$\frac{1}{Ri\,Ro}$	(..)	$\frac{1}{Ri\,Ro^2}$	$\frac{F^2}{Ro^2}$
Ro ≈ 1	1	$\frac{1}{Ri}$	(..)	$\frac{1}{Ri}$	F^2

$$(\partial_t + \mathbf{V}\cdot\nabla)\zeta + [\zeta\nabla\cdot\mathbf{V} + w\partial_z\zeta + \mathbf{k}\cdot\nabla w \wedge \partial_z\mathbf{V}] + \mathbf{V}\cdot\nabla f + f\nabla\cdot\mathbf{V}$$

A B C D

$$= \mathbf{k}\wedge\left[(1/\rho)\nabla\rho \wedge (1/\rho)\nabla p\right]$$

E

Typical values: $Ri = 10^2$, $F^2 = 10^{-3}$

Term	A	B	C	D	E
Ro << 1	1	10^{-1}	1	1	10^{-1}
Ro ≈ 1	1	10^{-2}	1	10^{-2}	10^{-3}

Middle latitudes: Ro << 1 ⇒ $(\partial_t + \mathbf{V}\cdot\nabla)(\zeta + f) + f\nabla\cdot\mathbf{V} = 0$

Low latitudes: Ro ≈ 1 ⇒ $(\partial_t + \mathbf{V}\cdot\nabla)(\zeta + f) = 0$

Barotropic versus baroclinic

Middle latitudes: Ro << 1 ⇒ $\left(\partial_t + \mathbf{V}\cdot\nabla\right)\left(\zeta + f\right) + f\nabla\cdot\mathbf{V} = 0$

Or, using continuity, $\left(\partial_t + \mathbf{V}\cdot\nabla\right)\left(\zeta + f\right) = \frac{f}{\rho}\frac{\partial}{\partial z}(\rho w)$

⇒ Vertical gradients of vertical mass flux can generate absolute vorticity.

Low latitudes: Ro ≈ 1 ⇒ $\left(\partial_t + \mathbf{V}\cdot\nabla\right)\left(\zeta + f\right) = 0$

⇒ There is no generation of absolute vorticity in the absence of diabatic processes. Air parcels are confined to a particular level, where they move around conserving their absolute vorticity.

Latent heat release

$$\left(\partial_t + \mathbf{V}\cdot\nabla_h\right)\ln\theta + w\,\partial_z\ln\theta = Q/\left(c_p T\right)$$

$$WN^2/g \approx \left(Q/c_p\right)/T$$

- **Budget studies ⇒ three quarters of the radiative cooling of the tropical troposphere is balanced by latent heat release.**
- **From figures given earlier, this means for 0-30° latitude, the warming rate is about 0.9 K/day.**
- **Gray (1973) estimated that tropical weather systems cover about 20% of the tropical belt.**
- **This would imply a warming rate $Q/c_p \approx 5 \times 0.9 = 4.5$ K/day in weather systems.**

Implied rainfall

- A rainfall rate of 1 cm/day (i.e. 10^{-2} m/day) implies 10^{-2} m/day per unit area (i.e. m^2) of vertical column.
- This would imply a latent heat release $\Delta Q \approx L\Delta m$ per unit area per day, where $L = 2.5 \times 10^6$ J/kg is the latent heat of condensation and Δm is the mass of condensed water.
- Since the density of water is 10 kg/m^{-3}, we have

$$\Delta Q \approx 2.5 \times 10^6 \frac{J}{kg} \times 10^{-2} m^3 \times 10^3 \frac{kg}{m^3}$$
$$= 2.5 \times 10^7 \text{ J / unit area / day}$$

Implied temperature rise

- A heating rate of 2.5×10^7 J/unit area/day is equivalent to a mean temperature rise ΔT in a column extending from the surface to 150 mb given by $c_p m_a \Delta T \approx 2.5 \times 10^7$ J/unit area/day.

$m_a = (1000 - 150)$ mb/g = mass of air unit area in the column

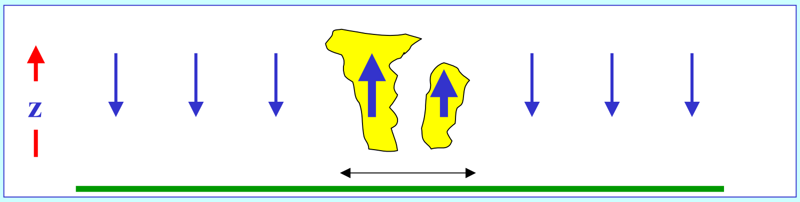
- With $c_p = 1005$ J/K/kg we obtain $\Delta T \approx 2.9°$/day.
- Therefore, a heating rate of 0.9° K/day requires a rainfall of about 1/3 cm/day averaged over the tropics, or 1.5 cm/day averaged over weather systems.

Implied vertical velocities

Again using $WN^2/g \approx (Q/c_pT)$ with the same parameters as before ⇒ a heating rate Q = 4.5 K/day ⇔ vertical velocity of about 1.5 cm/sec.

But note that the effective N is smaller in regions of moist convection ⇒ the estimate for w is a conservative one.

Area occupied by precipitation



- We can use these simple concepts to obtain an estimate for the horizontal area occupied by precipitating disturbances
- From mass conservation, the ratio of the area of ascent to descent must be inversely proportional to the ratio of the corresponding vertical velocities.
- Using the figures given above, this ratio is 1/3, but allowing for a smaller N in convective regions will decrease this somewhat, closer to Gray´s estimate of 1/5.

Additional notes on the scaling at low latitudes (1)

- **In mid-latitudes** Ro << 1 **and it is a convenient small parameter for asymptotic expansion.**
- **Generally at low latitudes as** $f \to 0$, $Ro \approx 1$ **and we must seek other parameters.**
- **One such parameter,** $(RiRo)^{-1}$ **is always** << 1, **even if** $L \approx 10^7$ **m.**
- **The vorticity equation gives useful information: it states that synoptic-scale phenomena** ($L \approx 10^6$ **m) are nearly uncoupled in the vertical except under circumstances that limit the derivation. These are:**
  - **When** Q/c_p **is large. Then** w **is scaled using** $wN^2/g \approx Q/(c_pT)$.

Additional notes on the scaling at low latitudes (2)

- **For planetary-scale motions** ($L \approx 10^7$) **of the type discussed in Chapter 1. Then, again** Ro << 1 **and if** $D \approx H_s$ **as before, the quasi-geostrophic scaling applies once more. Moreover, the appropriate vorticity equation includes the divergence term and coupling in the vertical is re-established; i.e.**

$$(\partial_t + \mathbf{V}\cdot\nabla)(\zeta + f) + f\nabla\cdot\mathbf{V} = 0$$

- **If the motions involve vertically-propagating gravity waves with** $D << H_s$, **but still with** $L \approx 10^7$ **and if** $U \to 0$, **then again** Ro <<1 **and vertical coupling occurs.**

Barotropic features

As a consequence of $\left(\partial_t + \mathbf{V}\cdot\nabla\right)\left(\zeta + f\right) = 0$

the atmosphere is governed by barotropic processes.

- ⇒ the usual baroclinic way of producing kinetic energy from potential energy, i.e., the lifting of warm air and the lowering of cold air, does not occur.
- ⇒ energy transfers are strictly limited.
- How then can the kinetic energy be generated in the tropics?
- The answer lies in convective processes.
- But if this is so, why are the thermal gradients so small?

Rate of potential energy production

Íf w is approximated by $wN^2/g \approx Q/(c_pT)$, then

$$< w'T' > \approx g < Q'T' > / \left(N^2 c_p T\right) \qquad [*]$$

the rate of production of kinetic energy

As $<Q'T'>$ is proportional to the rate of production of potential energy (i.e. there is heating where it is hot and cooling where it is cold) the statement [*] is a reflection of the fact that in the tropics, potential energy is converted to kinetic energy as soon as it is generated:

⇒there is no storage of potential energy.

The weak temperature gradient approximation

A balanced theory for motions in the deep tropics:

Assume that $\partial\theta/\partial t$ **and** $\mathbf{V}\cdot\nabla\theta$ **are much less than** $w(\partial\theta/\partial z)$

$$w\frac{\partial\theta}{\partial z} = \frac{D\theta}{Dt} \equiv S_\theta$$

$$\pi = (p/p^*)^\kappa$$

$$S_\theta = Q/(c_p\pi)$$

Vorticity equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(\zeta + f) = (f+\zeta)D$$

$$D = \nabla\cdot\mathbf{V} = -\frac{1}{\rho}\frac{\partial}{\partial z}(\rho w)$$

The weak temperature gradient approximation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(\zeta + f) = \frac{(f+\zeta)}{\rho}\frac{\partial}{\partial z}\left(\frac{\rho S_\theta}{\partial\theta/\partial z}\right)$$

If there were no diabatic heating ($S_\theta = 0$**), the RHS would be zero** $\Rightarrow$ **the absolute vorticity is simply advected around by the horizontal wind.**

Heating produces horizontal divergence $\Rightarrow$

$D > 0 \Rightarrow \zeta + f$ **decreases,** $D < 0 \Rightarrow \zeta + f$ **increases**

Solution method

The weak temperature gradient approximation

Given the diabatic heating $S_\theta = 0$**, update** $\zeta + f$ **in time using:**

$$\frac{\partial}{\partial t}(\zeta + f) = -\mathbf{V}\cdot\nabla(\zeta + f) + \frac{(f+\zeta)}{\rho}\frac{\partial}{\partial z}\left(\frac{\rho S_\theta}{\partial\theta/\partial z}\right)$$

$$\mathbf{V} = k \wedge \nabla\psi + \nabla\chi$$

Diagnose ψ **and** χ **by solving the two Elliptic PDEs:**

$$\nabla^2\psi = \zeta$$

$$\nabla^2\chi = D$$

$$D = \nabla\cdot\mathbf{V} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{\rho S_\theta}{\partial\theta/\partial z}\right)$$

The End