Balanced Dynamical Theory

The primary and secondary circulations

Chapter 3

Inviscid equations of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \lambda} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \lambda} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = -\frac{1}{\rho r}\frac{\partial p}{\partial \lambda}, \tag{3.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \lambda} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{3.3}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho r u}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \lambda} + \frac{\partial \rho w}{\partial z} = 0, \tag{3.4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + \frac{v}{r}\frac{\partial \theta}{\partial \lambda} + w\frac{\partial \theta}{\partial z} = \dot{\theta} \tag{3.5}$$

$$\rho = p_* \pi^{\frac{1}{\kappa}-1}/(R_d\theta) \tag{3.6}$$

$\dot{\theta}$ is the diabatic heating rate $(1/c_p\pi)Dh/Dt$ (see Eq. 1.13), and $\pi = (p/p_*)^\kappa$ is the Exner function. The temperature is defined by $T = \pi\theta$.

Absolute angular momentum

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial r} + \frac{v}{r}\frac{\partial M}{\partial \lambda} + w\frac{\partial M}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{3.7}$$

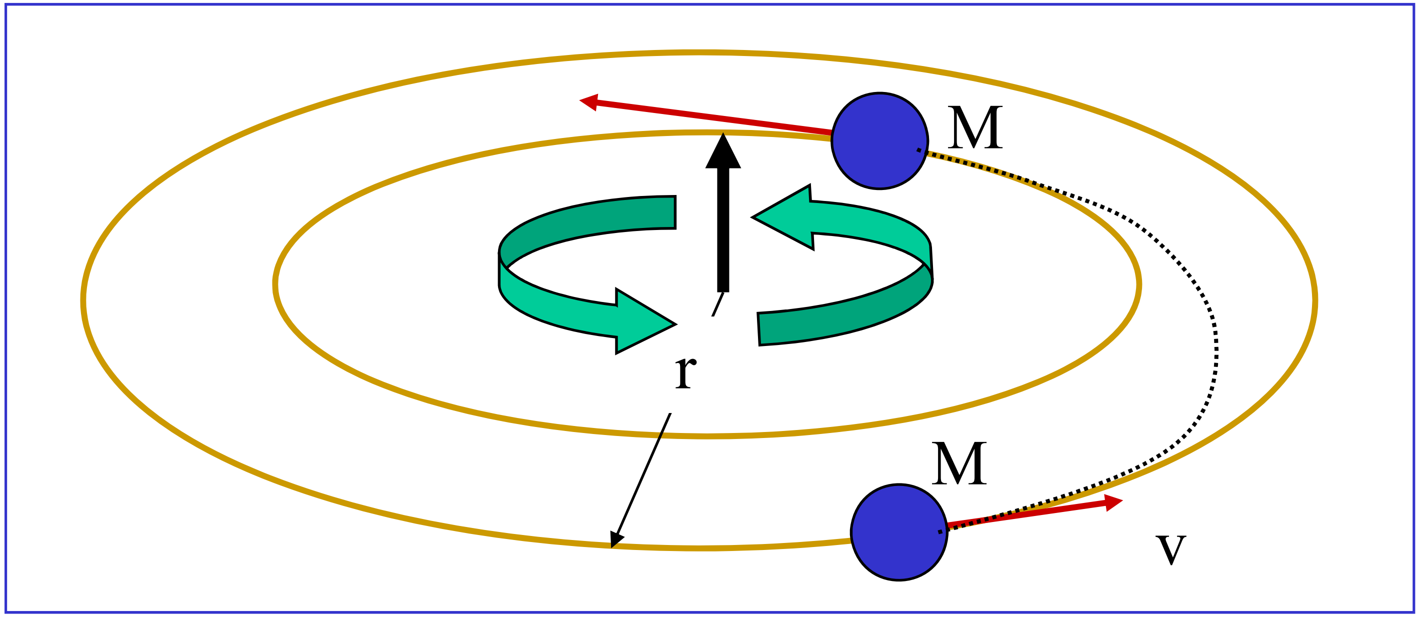
$$M = rv + \frac{1}{2}fr^2, \tag{3.8}$$

Eq. (3.7) follows from r **times Eq. (3.2)**

Tropical cyclone intensification

Basic principle: conservation of absolute angular momentum:

$$\boxed{M = rv + \tfrac{1}{2} f r^2}$$



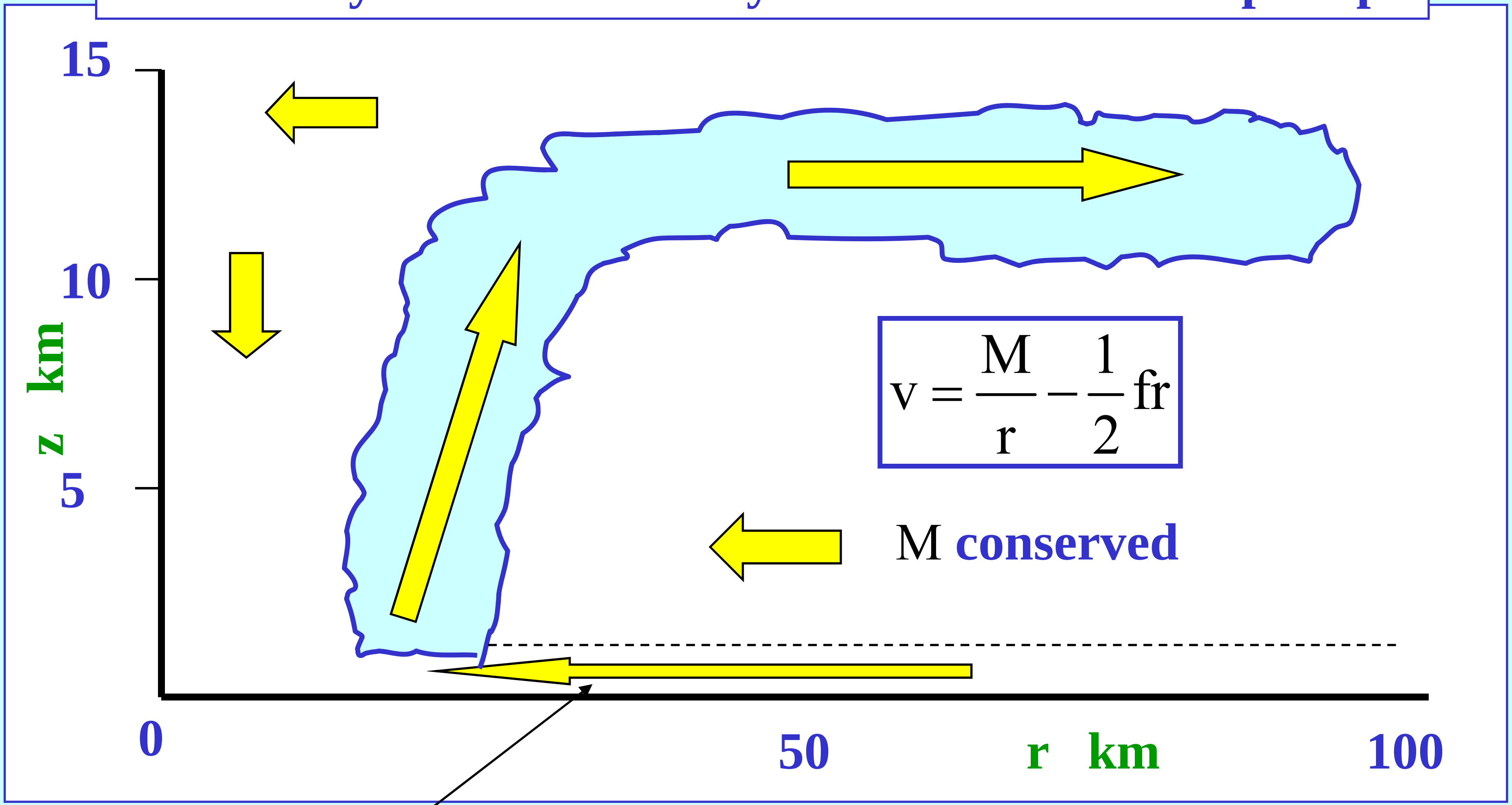
$$\boxed{v = \frac{M}{r} - \tfrac{1}{2} f r}$$

⇨ **If** r **decreases,** v **increases!**

⇨ **Spin up requires radial convergence**

Conventional view of intensification: axisymmetric

Thermally-forced secondary circulation leads to spin up



M not conserved, inflow feeds the clouds with moisture

Is that it? See later for a surprise

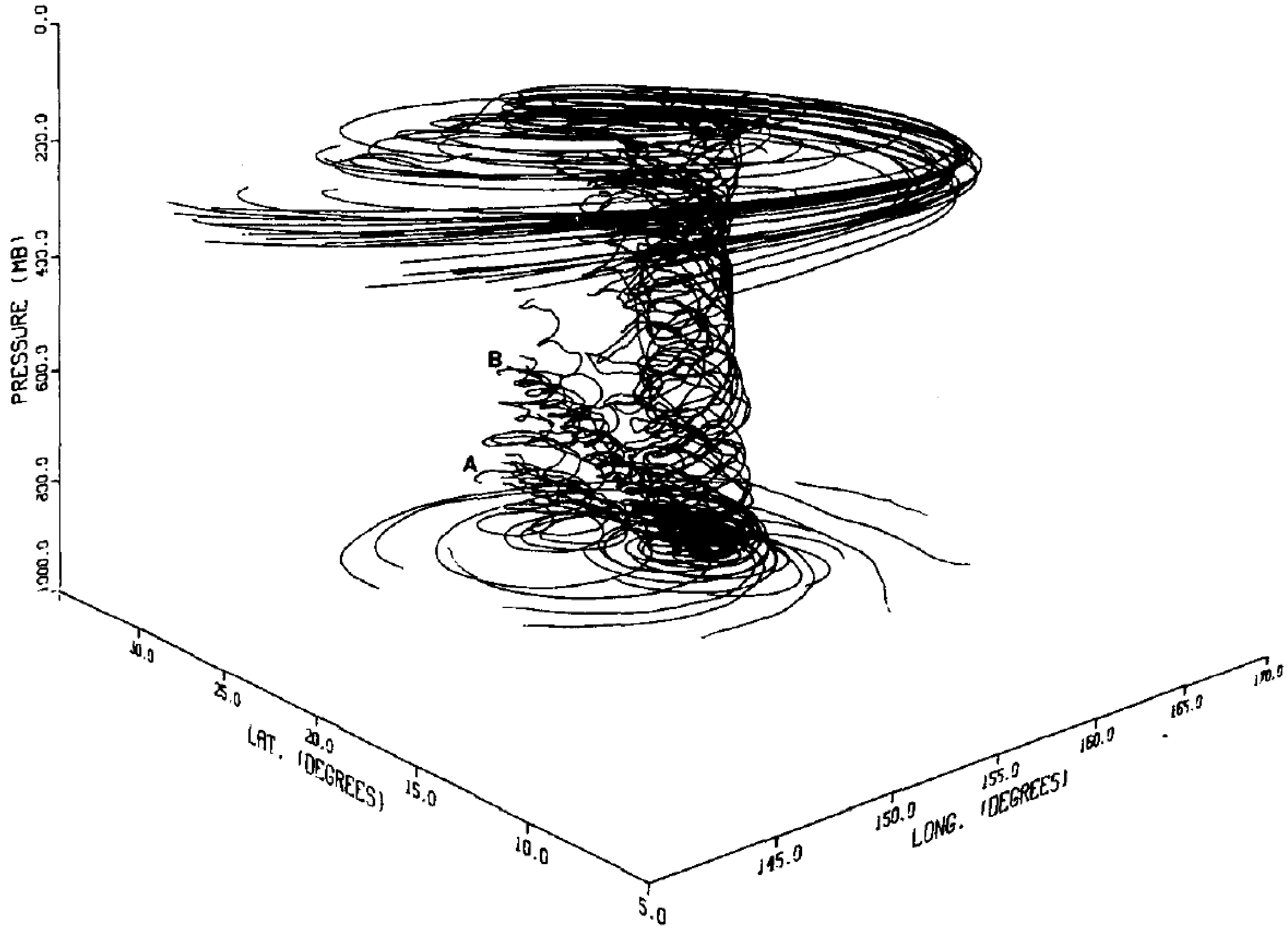


FIG. 9. Trajectories formed by particles released at various radii and pressure levels at $t = 0$. Most particles that reach the outflow level are transported outward by the outflow jet. Most particles released at radii of 20 km (A) and 100 km (B) are "trapped" inside the radius of the maximum wind and only rise slowly and drift toward the NW.

Primary circulation

Gradient wind

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} + fv, \tag{3.9}$$

Hydrostatic

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, \tag{3.10}$$

Thermal wind equation

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}. \tag{3.11}$$

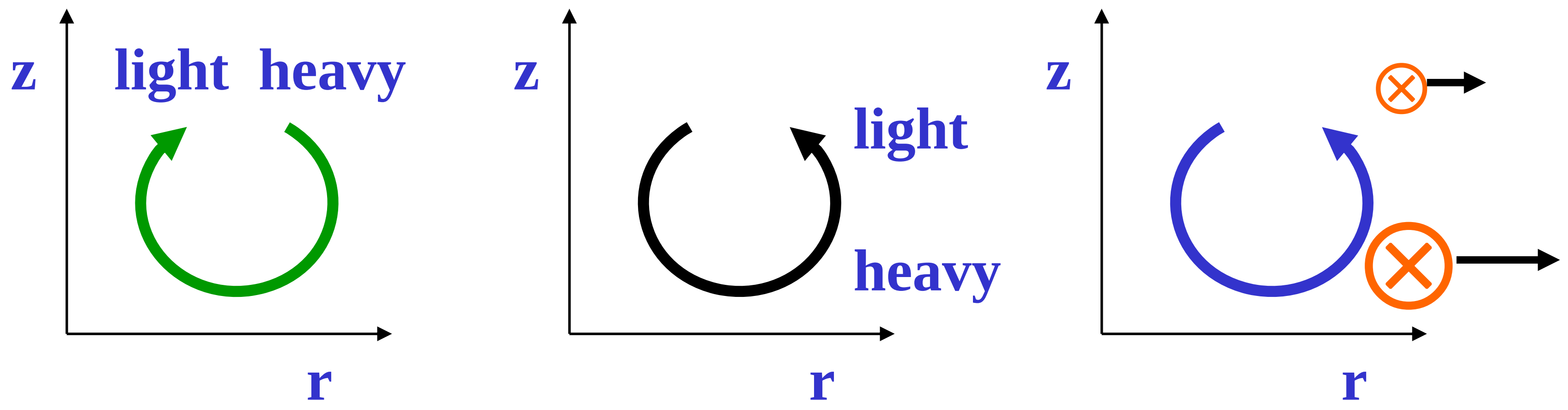
$$C = \frac{v^2}{r} + fv \tag{3.12}$$

Physical interpretation

Thermal wind equation

$$g\frac{\partial}{\partial r}\ln\rho+\left(\frac{v^2}{r}+fv\right)\frac{\partial}{\partial z}\ln\rho+\left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z}=0$$

Balance in toroidal circulation tendency



Thermal wind equation

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}. \tag{3.11}$$

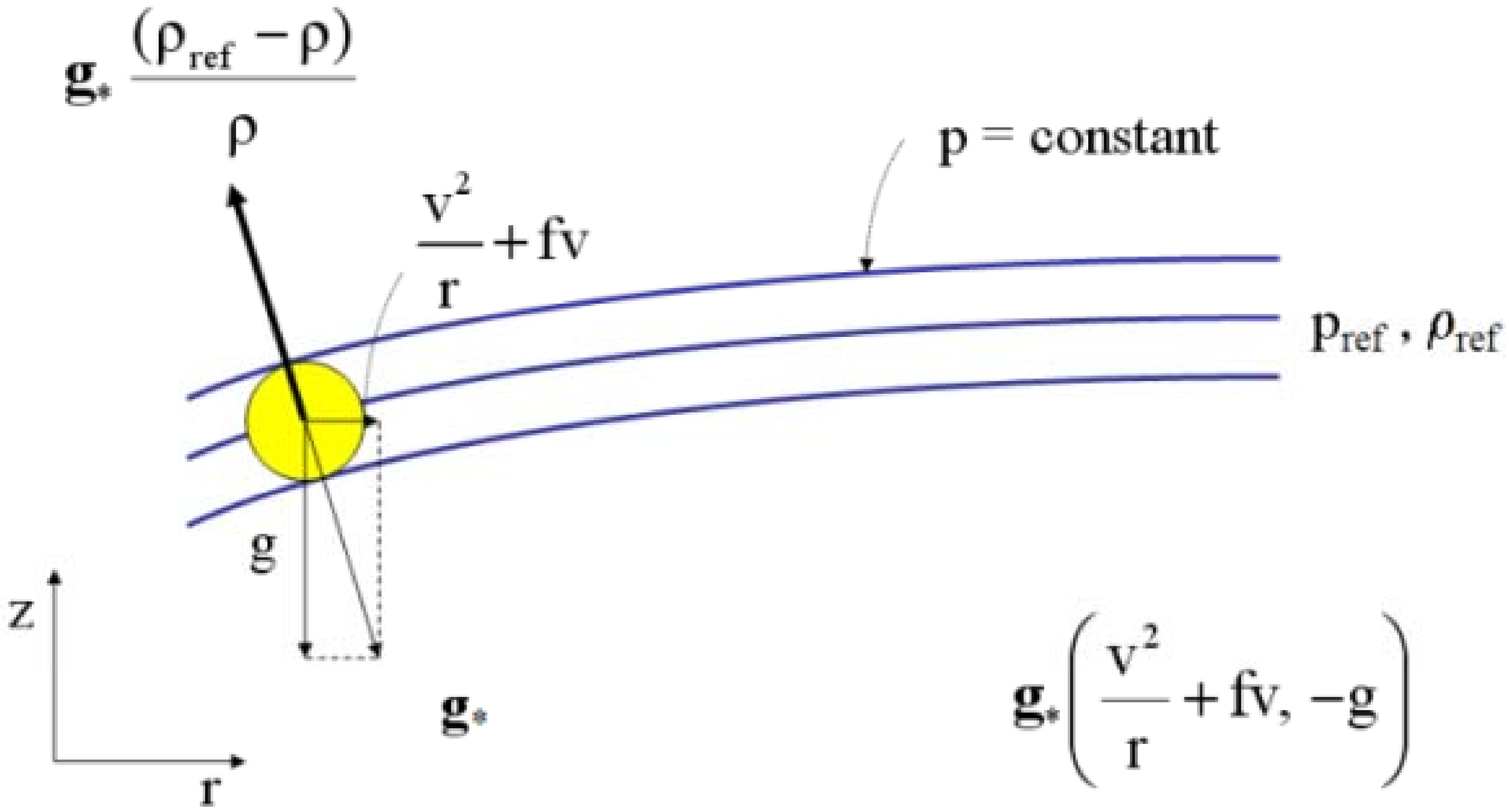
Characteristics

$$\frac{dz}{dr} = \frac{C}{g}.$$

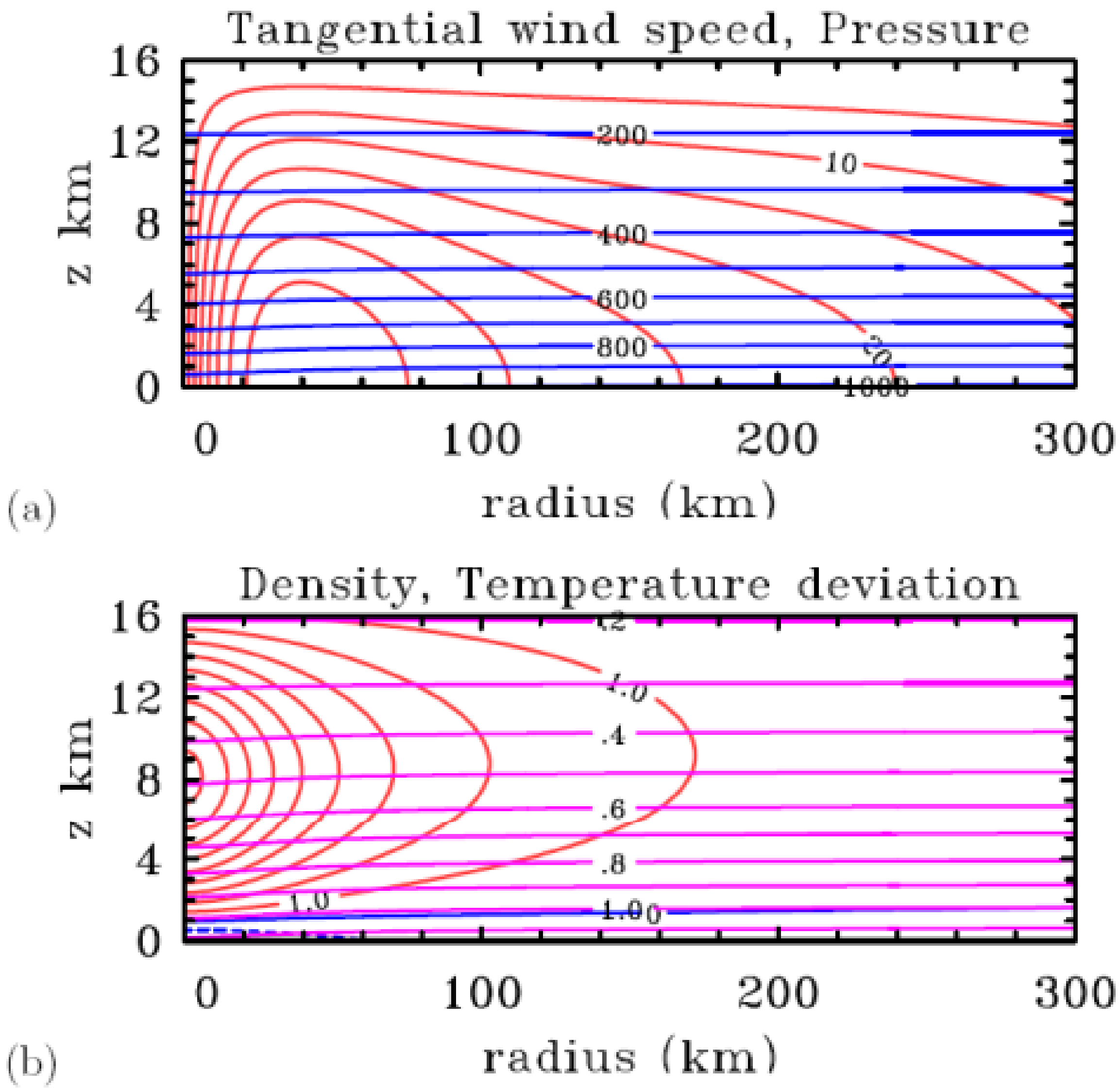
On a characteristic

$$\frac{d}{dr}\ln \rho = -\frac{1}{g}\frac{\partial C}{\partial z}. \tag{3.14}$$

Generalized buoyancy

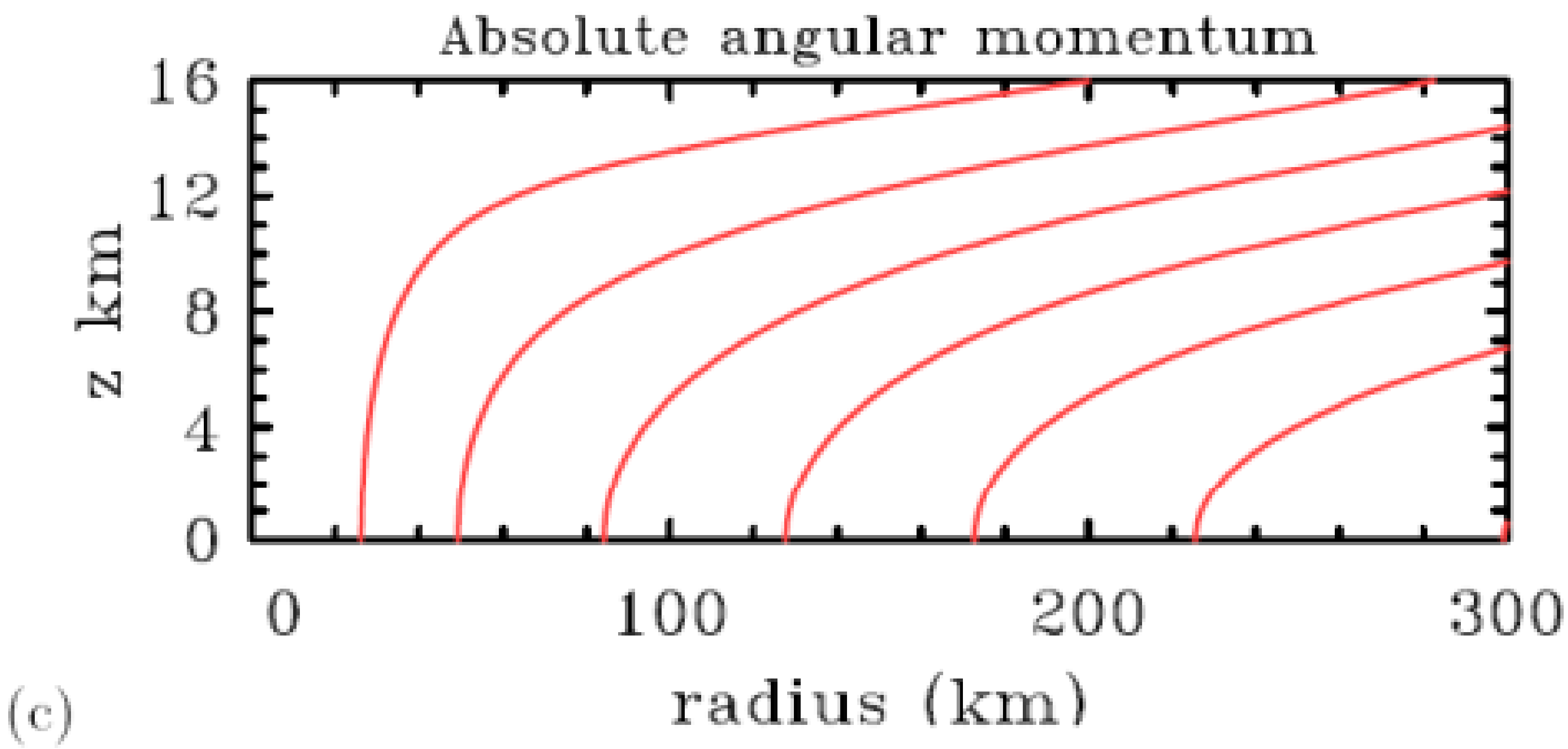


A typical vortex

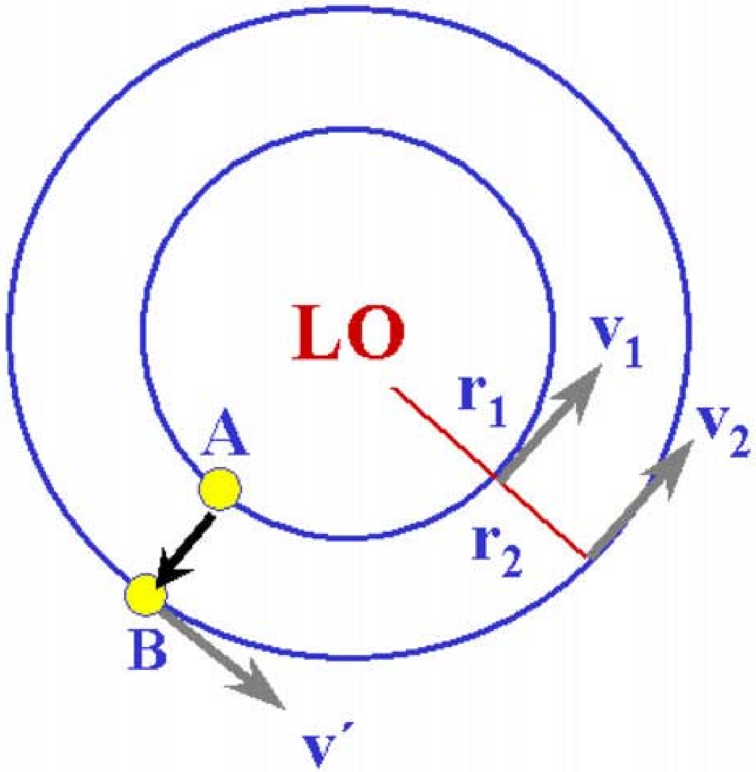


AAM in a typical vortex

M



Barotropic stability



The parcel at A conserves its angular momentum during its radial displacement to B

$$r_2 v' + \frac{1}{2} f r_2^2 = r_1 v_1 + \frac{1}{2} f r_1^2,$$

$$\boxed{v' = \frac{r_1}{r_2} v_1 + \frac{1}{2} \frac{f}{r_2} (r_1^2 - r_2^2) \qquad (3.17)}$$

Net radial force on a displaced air parcel

Radial pressure gradient at B

$$\frac{1}{\rho}\frac{dp}{dr}\bigg]_{r=r_2} = \frac{v_2^2}{r_2} + f v_2. \qquad (3.18)$$

Net force on parcel at B

$$F = \text{centrifugal} + \text{Coriolis force} - \text{radial pressure gradient}$$

$$= \frac{v'^2}{r_2} + f v' - \frac{1}{\rho}\frac{\partial p}{\partial r}\bigg]_{r=r_2}$$

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2} r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2} r_2^2 f)^2\right]. \qquad (3.19)$$

Net radial force on a displaced air parcel

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2}r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2}r_2^2 f)^2\right]. \tag{3.19}$$

In the special case of solid body rotation, $v = \Omega r$, and for a small displacement from radius $r_1 = r$ to $r_2 = r + r'$, (3.19) gives

$$F \approx -4(\Omega + \frac{1}{2}f)^2 r' \tag{3.20}$$

A scale analysis

$$continuity$$

$$\frac{1}{r}\frac{\partial \rho r u}{\partial r} \qquad \frac{\partial \rho w}{\partial z} \tag{3.4}$$

$$\rho\frac{U}{R} \qquad \rho\frac{W}{Z} \tag{1}$$

A scale analysis

$$u\text{-}momentum$$

$$\frac{\partial u}{\partial t} \quad + u\frac{\partial u}{\partial r} \quad + w\frac{\partial u}{\partial z} \quad - \frac{v^2}{r} \quad - fv \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3.1}$$

$$\frac{U}{T} \quad \frac{U^2}{R} \quad W\frac{U}{Z} \quad \frac{V^2}{R} \quad fV \quad \frac{\Delta p}{\rho R} \tag{2a}$$

$$S^2 \quad S^2 \quad S^2 \quad 1 \quad \frac{1}{Ro} \quad \frac{\Delta p}{\rho V^2} \tag{2b}$$

A scale analysis

$$v\text{-}momentum$$

$$\frac{\partial v}{\partial t} \quad + u\frac{\partial v}{\partial r} \quad + w\frac{\partial v}{\partial z} \quad + \frac{uv}{r} \quad + fu \quad = \tag{3.2}$$

$$\frac{V}{T} \quad U\frac{V}{R} \quad \frac{WV}{Z} \quad \frac{UV}{R} \quad fU \tag{3a}$$

$$S \quad S \quad S \quad S \quad \frac{S}{Ro} \tag{3b}$$

A scale analysis

w-momentum

$$\frac{\partial w}{\partial t} \quad + u\frac{\partial w}{\partial r} \quad + w\frac{\partial w}{\partial z} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad - g \qquad (3.3)$$

$$\frac{W}{T} \quad U\frac{W}{R} \quad W\frac{W}{Z} \quad \frac{\Delta p}{\rho Z} \quad g \qquad (4a)$$

$$\frac{WU}{Rg} \quad \frac{UW}{Rg} \quad \frac{WU}{Rg} \quad \frac{\Delta p}{\rho g Z} \quad 1 \qquad (4b)$$

The secondary circulation

Thermal wind equation

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}$$

Exercise 3.2 Show that Eq. (3.12) may be reformulated as

$$g\frac{\partial(\ln \chi)}{\partial r} + C\frac{\partial(\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}. \qquad (3.17)$$

where $\chi = 1/\theta$.

The secondary circulation

A balanced theory: The Sawyer-Eliassen equation

Continuity →

$$u = -\frac{1}{r\rho}\frac{\partial\psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial\psi}{\partial r}. \tag{3.57}$$

Thermal wind →

$$g\frac{\partial(\ln\chi)}{\partial r} + C\frac{\partial(\ln\chi)}{\partial z} = -\frac{\partial C}{\partial z}. \tag{3.17}$$

Prognostic equations

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V} \tag{3.58}$$

$$\frac{\partial\chi}{\partial t} + u\frac{\partial\chi}{\partial r} + w\frac{\partial\chi}{\partial z} = -\chi^2\dot{\theta} \tag{3.59}$$

A balanced theory: The Sawyer-Eliassen equation

∂/∂t (thermal wind equation)

$$g\frac{\partial}{\partial r}\frac{\partial \chi}{\partial t} + \frac{\partial}{\partial z}\left(C\frac{\partial \chi}{\partial t} + \chi\frac{\partial C}{\partial t}\right) = 0$$

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V}$$

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta}$$

A balanced theory: The Sawyer-Eliassen equation

∂/∂t **(thermal wind equation)**

$$g\frac{\partial}{\partial r}\left(u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} - Q\right) + \frac{\partial}{\partial z}\left[C\left(u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} - Q\right) + \chi\xi\left(u(\zeta + f) + wS - \dot{V}\right)\right] = 0$$

where $\chi = 1/\theta$ and $Q = -\chi^2\dot{\theta}$. Then

$$\frac{\partial}{\partial r}\left[g\frac{\partial \chi}{\partial z}w + g\frac{\partial \chi}{\partial r}u\right] +$$

$$\frac{\partial}{\partial z}\left[(\chi\xi(\zeta + f) + C\frac{\partial \chi}{\partial r})u + \frac{\partial}{\partial z}(\chi C)w\right] = g\frac{\partial Q}{\partial r} + \frac{\partial}{\partial z}(CQ) + \frac{\partial}{\partial z}(\chi\xi\dot{V})$$

or

$$\frac{\partial}{\partial r}\left[g\frac{\partial \chi}{\partial z}w - \frac{\partial}{\partial z}(\chi C)u\right] +$$

$$\frac{\partial}{\partial z}\left[(\chi\xi(\zeta + f) + C\frac{\partial \chi}{\partial r})u + \frac{\partial}{\partial z}(\chi C)w\right] = g\frac{\partial Q}{\partial r} + \frac{\partial}{\partial z}(CQ) + \frac{\partial}{\partial z}(\chi\xi\dot{V}) \qquad (3.60)$$

The Sawyer-Eliassen equation

$$\frac{\partial}{\partial r}\left[g\frac{\partial\chi}{\partial z}\frac{1}{\rho r}\frac{\partial\psi}{\partial r}+\frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial z}\right]-$$

$$\frac{\partial}{\partial z}\left[\left(\xi\chi(\zeta+f)+C\frac{\partial\chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial\psi}{\partial z}-\frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right]=g\frac{\partial Q}{\partial r}+\frac{\partial}{\partial z}(CQ)+\frac{\partial}{\partial z}(\chi\xi\dot{V}) \tag{3.61}$$

Discriminant

$$D=-g\frac{\partial\chi}{\partial z}\left(\xi\chi(\zeta+f)+C\frac{\partial\chi}{\partial r}\right)-\left[\frac{\partial}{\partial z}(\chi C)\right]^2 \tag{3.62}$$

Parameters

- the *static stability*

$$N^2=-g\frac{\partial ln\chi}{\partial z};$$

- the *inertial stability*

$$I^2=\frac{1}{r^3}\frac{\partial M^2}{\partial r}=\xi(\zeta+f);$$

- the *baroclinicity*

$$B_1=\frac{1}{r^3}\frac{\partial M^2}{\partial z}=\xi S.$$

The Sawyer-Eliassen equation

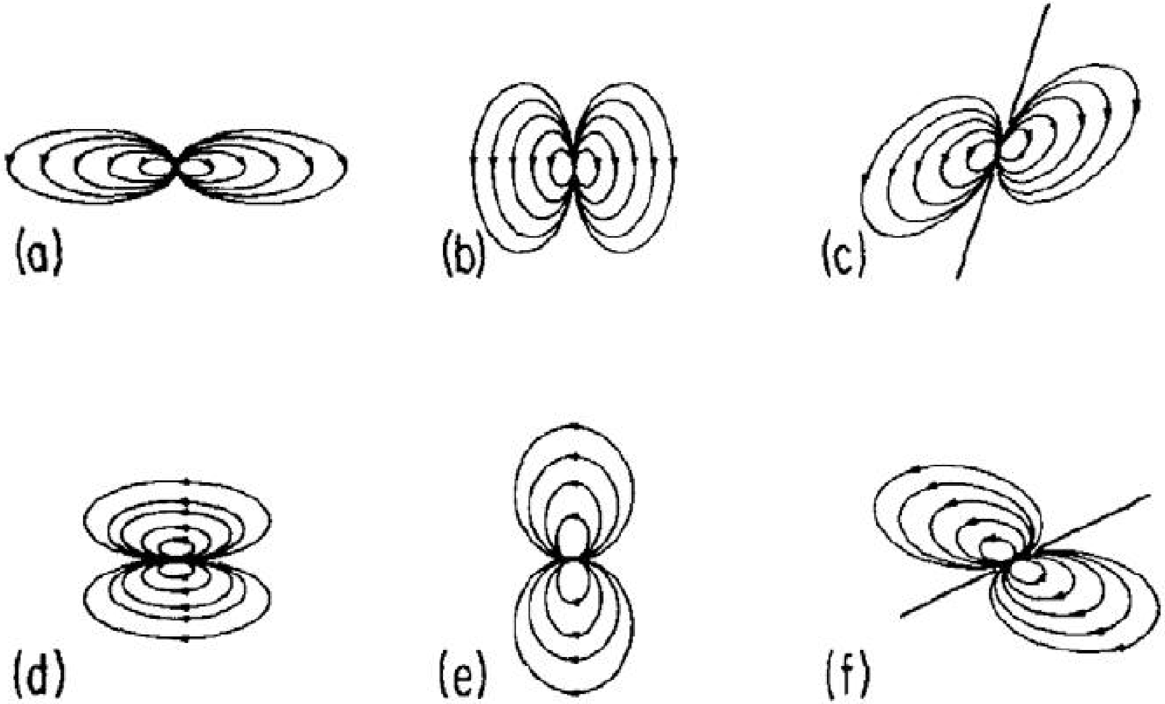


Figure 3.11: Streamfunction responses to point sources of: (a) Heat in a barotropic vortex with weak inertial stability, (b) heat in a barotropic vortex with strong inertial stability, (c) heat in a baroclinic vortex, (d) momentum in a barotropic vortex with weak inertial stability, (e) momentum in a barotropic vortex with strong inertial stability, and (f) momentum in a baroclinic vortex. (Based on Figs. 8, 9, 11, and 12

The Sawyer-Eliassen equation

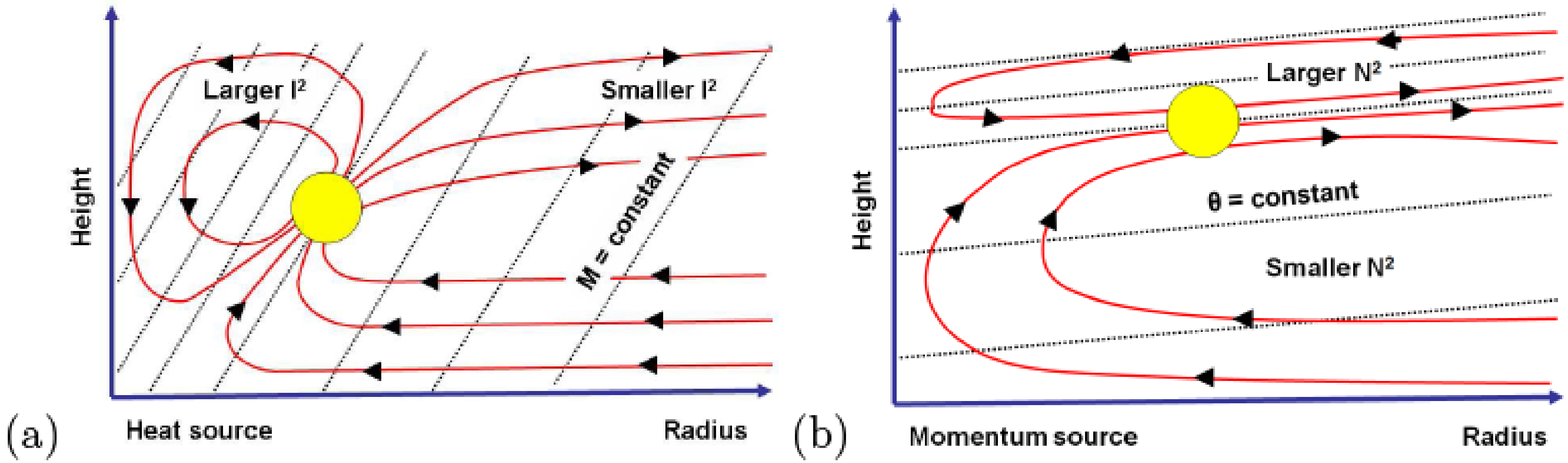


Figure 3.12: Secondary circulation induced in a balanced vortex by (a) a heat source and (b) a cyclonic momentum source showing the distortion induced by variation in inertial stability, I^2 and thermodynamic stability. N^2, and baroclinicity S^2. The strong motions through the source follow lines of constant angular momentum for a heat source and of constant potential temperature for a momentum source. From

Willoughby (1995)

The toroidal vorticity equation

The λ-component of vorticity, or toroidal vorticity is

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \tag{3.67}$$

The equation for η is derived as follows. Consider

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial t}\right)$$

This expression may be written

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial z}\left(-\mathbf{u}\cdot\nabla u + C - \frac{1}{\rho}\frac{\partial p}{\partial r} + F_u\right) - \frac{\partial}{\partial r}\left(-\mathbf{u}\cdot\nabla w - \frac{1}{\rho}\frac{\partial p}{\partial z} + F_w\right)$$

With some algebraic manipulation this becomes

$$\boxed{r\frac{D}{Dt}\left(\frac{\eta}{r\rho}\right) = \frac{1}{\rho}\frac{\partial C}{\partial z} + \frac{1}{\rho^2\chi}\left(\frac{\partial \chi}{\partial z}\frac{\partial p}{\partial r} - \frac{\partial \chi}{\partial r}\frac{\partial p}{\partial z}\right) + \frac{1}{\rho}\left(\frac{\partial F_u}{\partial z} - \frac{\partial F_w}{\partial r}\right),} \tag{3.68}$$

3.8.1 The Sawyer-Eliassen equation and toroidal vorticity equation

The Sawyer-Eliassen equation is an approximate form of the local time derivative of equation for the toroidal vorticity $\eta = \partial u/\partial z - \partial w/\partial r$. Assuming the most general form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u) + \frac{\partial}{\partial z}(\rho w) = 0$$

the toroidal vorticity equation may be written as

$$r\frac{D}{Dt}\left(\frac{\eta}{r\rho}\right) = \frac{1}{\rho}\frac{\partial C}{\partial z} + \frac{1}{\rho^2\chi}\left(\frac{\partial \chi}{\partial z}\frac{\partial p}{\partial r} - \frac{\partial \chi}{\partial r}\frac{\partial p}{\partial z}\right) \tag{3.63}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u}\cdot\nabla$ and $\eta/(r\rho)$ is a 'potential toroidal vorticity', where the analogous 'depth' is 'r', the radius of a toroidal vortex ring (see appendix). If thermal wind balance exists, the right-hand-side of (3.68) may be written as

$$\frac{1}{\rho\chi}\left(g\frac{\partial \chi}{\partial r} + \frac{\partial}{\partial z}(C\chi)\right).$$

The Sawyer-Eliassen equation

$$\frac{1}{\rho\chi}\left(g\frac{\partial\chi}{\partial r}+\frac{\partial}{\partial z}(C\chi)\right).$$

Then the time derivative of (3.63) is

$$\frac{\partial}{\partial t}\left[r\frac{D}{Dt}\left(\frac{\eta}{r\rho}\right)\right]=\frac{\partial}{\partial t}\left[\frac{1}{\rho\chi}\left(g\frac{\partial\chi}{\partial r}+\frac{\partial}{\partial z}(C\chi)\right)\right] \tag{3.64}$$

The right-hand-side of (3.64) gives the Sawyer-Eliassen equation when the thermal wind equation (3.12) is satisfied for all time. Then consistency requires that the left-hand-side is identically zero.

Exercise 3.12 Starting from the Boussinesq system of equations, show that the Sawyer-Eliassen equation takes the form

$$\frac{\partial}{\partial r}\left[\left(N^2+\frac{\partial b}{\partial z}\right)\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{S\xi}{r}\frac{\partial\psi}{\partial z}\right]+\frac{\partial}{\partial r}\left[\frac{I^2}{r}\frac{\partial\psi}{\partial z}-\frac{S\xi}{r}\frac{\partial\psi}{\partial r}\right]=-\frac{\partial\dot{B}}{\partial r}-\frac{\partial}{\partial z}(\xi\dot{V}), \tag{3.66}$$

where $dotB$ is the source of buoyancy in the Boussinesq form of the thermodynamic equation and I^2 is defined in Eq **(3.33).**

More on buoyancy

See 3.8.2 Buoyancy relative to a balanced vortex

3.8.3 Buoyancy in axisymmetric balanced vortices

Axisymmetric balanced models of tropical cyclone intensification appear to capture many important observed features of tropical cyclone behaviour. However, in an axisymmetric model that assumes exact thermal wind balance, $\mathrm{b_B}(r, z, t) \equiv 0$ and the corresponding $\partial p'/\partial z \equiv 0$, even though there may be heat sources or sinks present that generate buoyancy b. It is clear from the foregoing discussion that any diabatic heating or cooling in such models is incorporated directly into the balanced state, changing $\mathrm{b}(r, z, t)$, while $\mathrm{b_B}(r, z, t)$ remains identically zero by definition. Obviously, nonzero values of $\mathrm{b_B}$ relate to *unbalanced motions* provided that the appropriate reference state as defined above has been selected for the definition of buoyancy at any given time. It may be helpful to think of b as characterizing the *system buoyancy* and $\mathrm{b_B}$ as characterizing the *local buoyancy*.

See 3.9 Origins of buoyancy in tropical cyclones


End

Thermal wind equation

Gradient wind balance **Hydrostatic balance**

Write $$\frac{\partial p}{\partial r} = \rho\left(\frac{v^2}{r} + fv\right) \qquad \frac{\partial p}{\partial z} = -\rho g$$

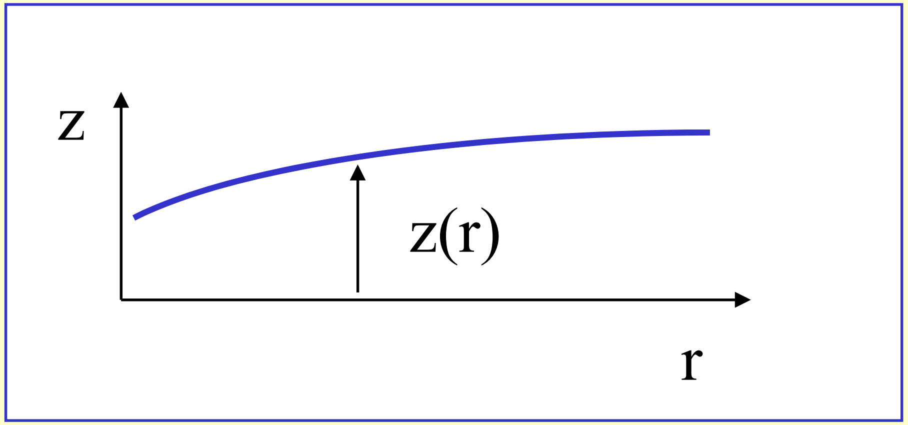
Eliminate p **using** $$\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial r}\right)$$

$$\frac{\partial}{\partial r}\ln\rho + \frac{1}{g}\left(\frac{v^2}{r} + fv\right)\frac{\partial}{\partial z}\ln\rho = -\frac{1}{g}\left(\frac{2v}{r} + f\right)\frac{\partial v}{\partial z}$$

Thermal wind equation

Mathematical solution

$$\frac{\partial}{\partial r}\ln\rho + \frac{1}{g}\left(\frac{v^2}{r} + fv\right)\frac{\partial}{\partial z}\ln\rho = -\frac{1}{g}\left(\frac{2v}{r} + f\right)\frac{\partial v}{\partial z}$$



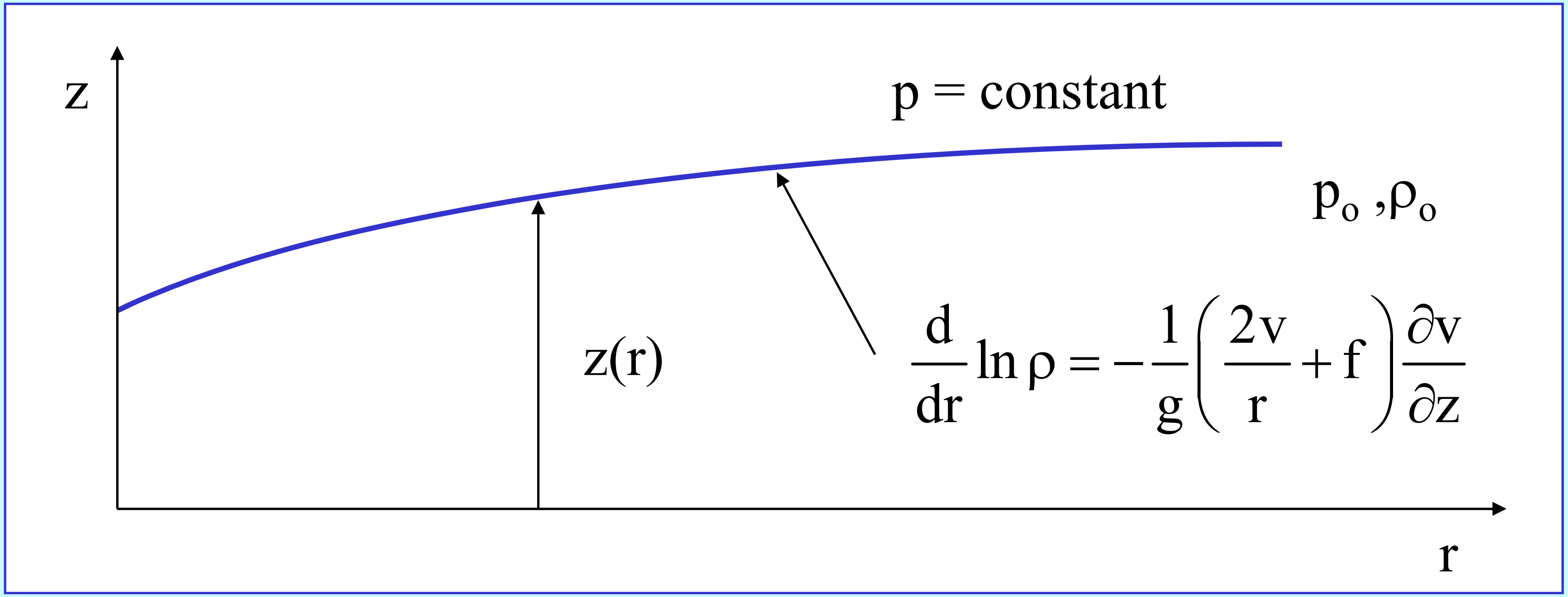
Characteristics

$$\frac{dz}{dr} = \frac{1}{g}\left(\frac{v^2}{r} + fv\right)$$

$$\frac{d}{dr}\ln\rho = -\frac{1}{g}\left(\frac{2v}{r} + f\right)\frac{\partial v}{\partial z}$$

Governs the variation of ρ along characteristics

Characteristics are isobaric surfaces

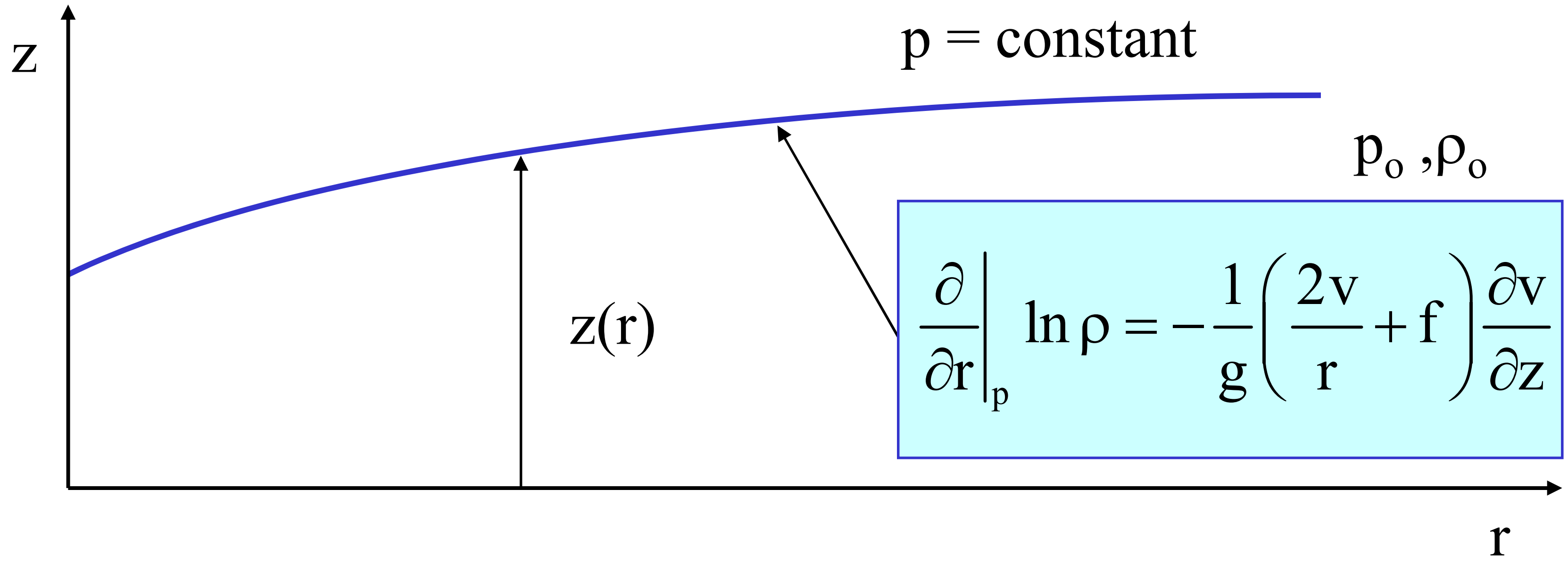


Along a characteristic

$$g\,dz = \left(\frac{v^2}{r} + fv\right)dr$$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz = \rho\left(\frac{v^2}{r} + fv\right)dr - \rho g\,dz = 0$$

Inferences



$$\left.\frac{\partial}{\partial r}\right|_p \ln\rho = -\frac{1}{g}\left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z}$$

Barotropic vortex $\frac{\partial v}{\partial z} = 0 \;\Rightarrow\; \left.\frac{\partial \rho}{\partial r}\right|_p = 0 \;\Rightarrow\; \left.\frac{\partial T}{\partial r}\right|_p = 0$

Baroclinic vortex $\frac{\partial v}{\partial z} < 0 \;\Rightarrow\; \left.\frac{\partial \rho}{\partial r}\right|_p > 0 \;\Rightarrow\; \left.\frac{\partial T}{\partial r}\right|_p < 0$

Equation of state $T = p / R\rho$

Summary

- A barotropic vortex is cold cored if temperature contrasts are measured at constant height.
- A baroclinic vortex is warm cored if temperature contrasts are measured at constant height and if $-\partial v/\partial z$ is large enough.

A sample calculation =>