Exercise 1

Show that in a vortex in gradient wind balance and thermal wind balance, the pressure field satisfies the partial differential equation

$$g\frac{\partial p}{\partial r} + C\frac{\partial p}{\partial z} = 0,$$

where

$$C = \frac{v^2}{r} + fv.$$

Show also that an arbitrary function of pressure satisfies the same equation.

Given a tangential wind field $v(r, z)$ and the vertical pressure profile at large radius, $p_o(z)$,, explain how you could solve this equation to find the surface pressure distribution, $p(r, 0)$.

Exercise 2

The thermal wind equation for a vortex with a tangential wind field $v(r, z)$ is:

$$g\frac{\partial(\ln \rho)}{\partial r} + C\frac{\partial(\ln \rho)}{\partial z} = -\frac{\partial C}{\partial z}.$$

where ρ is the density and

$$C = \frac{v^2}{r} + fv.$$

Show that this equation may be rewritten as

$$g\frac{\partial(\ln \chi)}{\partial r} + C\frac{\partial(\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}.$$

where χ is the inverse of the potential temperature, θ.

Exercise 3

Show that the tangential momentum equation for an inviscid, axisymmetric flow in cylindrical coordinates (r, λ, z) may be written in the two forms:

$$\frac{\partial v}{\partial t} + u\zeta_a + w\frac{\partial v}{\partial z} = 0,$$

or

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial r} + w\frac{\partial M}{\partial z} = 0,$$

where ζ_a is the *absolute vorticity* and $M = rv + \frac{1}{2}fr^2$ is the *absolute angular momentum* per unit mass. Explain why vortex spin up requires that air parcels move inwards.

Exercise 4

Show that the vertical momentum equation can be written in terms of *the perturbation pressure* $p' = p_T + p_{ref}(z)$ and *buoyancy* $b = -g(\rho - \rho_{ref}(z))/\rho$ as:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b,$$

where p_T is the total pressure, $p_{ref}(z)$ and $\rho_{ref}(z)$ are a *reference pressure* and *reference density*, respectively, that satisfy hydrostatic balance, and g is the acceleration due to gravity.

Show that the same equation holds if p_{ref} and ρ_{ref} are functions of both radius r and height z, such as the balanced pressure and density fields in a baroclinic vortex. How would you decide whether a cloud within a balanced warm-cored vortex had any buoyancy?

Exercise 5

The Boussinesq forms of the thermal wind equation, tangential momentum equation and thermodynamic equation are:

$$\frac{\partial b}{\partial r} = \frac{\partial C}{\partial z}, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = \dot{V}, \tag{2}$$

and

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial r} + wN^2 = \dot{B}, \tag{3}$$

respectively, where

$$\xi = \frac{2v}{r} + f, \qquad S = \frac{\partial v}{\partial z}, \qquad C = \frac{v^2}{r} + fv.$$

Show that the Boussinesq form of the Sawyer-Eliassen equation for the streamfunction of the secondary circulation is

$$\frac{\partial}{\partial r}\left[N^2\frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{S\xi}{r}\frac{\partial \psi}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{I^2}{r}\frac{\partial \psi}{\partial z} - \frac{S\xi}{r}\frac{\partial \psi}{\partial r}\right] = -\frac{\partial \dot{B}}{\partial r} - \frac{\partial}{\partial z}(\xi\dot{V}), \tag{4}$$

where $I^2 = \xi\zeta_a$ and ζ_a is the absolute vorticity.