Chapter 9

Appendices

9.1 Transformation of Euler's equation to an accelerating frame of reference

The Euler equation may be written

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f} \wedge \mathbf{u} = -\frac{1}{\rho} \nabla p$$

and

$$\mathbf{u} \cdot \nabla \mathbf{u} = \omega \wedge \mathbf{u} + \nabla (\frac{1}{2} \mathbf{u}^2)$$

Therefore

$$\frac{\partial \mathbf{u}}{\partial t} + (\omega + \mathbf{f}) \wedge \mathbf{u} = -\frac{1}{\rho} \nabla \left(p + \frac{1}{2} \mathbf{u}^2 \right)$$

Now transform the coordinate system ($\mathbf{x}$, t) to ($\mathbf{X}$, T), where

$$\mathbf{x} = \mathbf{X} + \mathbf{x_c}(\mathbf{t}), \quad t = T$$

and

$$\frac{d\mathbf{x_c}}{dt} = \mathbf{c}(t) = (c_1(t), c_2(t)).$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} \underbrace{\frac{\partial X}{\partial x}}_{=1} + \frac{\partial}{\partial Y} \underbrace{\frac{\partial Y}{\partial x}}_{=0} + \frac{\partial}{\partial T} \underbrace{\frac{\partial T}{\partial x}}_{=0}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial X} \underbrace{\frac{\partial X}{\partial y}}_{=0} + \frac{\partial}{\partial Y} \underbrace{\frac{\partial Y}{\partial y}}_{=1} + \frac{\partial}{\partial T} \underbrace{\frac{\partial T}{\partial y}}_{=0}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial X} \underbrace{\frac{\partial X}{\partial t}}_{=-c_1} + \frac{\partial}{\partial Y} \underbrace{\frac{\partial Y}{\partial t}}_{=-c_2} + \frac{\partial}{\partial T} \underbrace{\frac{\partial T}{\partial t}}_{=1}$$

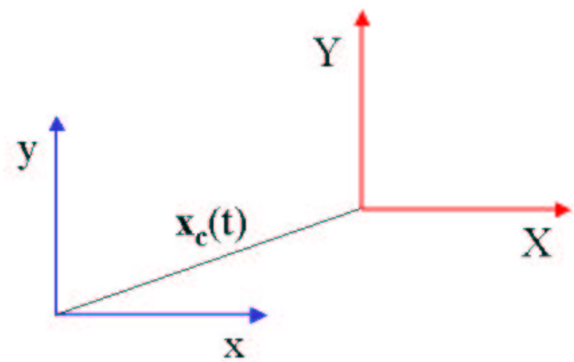

Figure 9.1:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c_1 & -c_2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \\ \frac{\partial}{\partial T} \end{pmatrix},$$

or

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T} - \mathbf{c} \cdot \nabla_X, \quad \nabla_{\mathbf{x}} = \nabla_{\mathbf{X}}$$

Note that although

$$T = t, \qquad \partial/\partial T \neq \partial/\partial t.$$

Let $\mathbf{U}$ be the velocity in the moving frame, i.e., $\mathbf{U} = \mathbf{u} - \mathbf{c}$. Then the Euler equation transforms to

$$\left(\frac{\partial}{\partial T} - \mathbf{c} \cdot \nabla\right)(\mathbf{U} + \mathbf{c}) + (\omega + \mathbf{f}) \wedge (\mathbf{U} + \mathbf{c}) = \frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}(\mathbf{U} + \mathbf{c})^2\right),$$

where

$$\omega = \nabla_{\mathbf{x}} \wedge \mathbf{u} = \nabla_{\mathbf{X}} \wedge \mathbf{U} \qquad \text{and} \qquad \mathbf{U} \wedge \mathbf{c} = 0.$$

Further reduction gives

$$\frac{\partial \mathbf{U}}{\partial T} + (\omega + \mathbf{f}) \wedge \mathbf{U} = -\frac{1}{\rho}\nabla p - \nabla(\tfrac{1}{2}\mathbf{U}^2) - \frac{\partial \mathbf{c}}{\partial t} + \mathbf{c} \cdot \nabla \mathbf{U} - (\omega + \mathbf{f}) \wedge \mathbf{c} - \nabla(\mathbf{U} \cdot \mathbf{c}),$$

using the fact that $\nabla \mathbf{c} = \mathbf{0}$ because $\mathbf{c} = \mathbf{c}(t)$. Now

$$\nabla(\mathbf{U} \cdot \mathbf{c}) = \mathbf{U} \cdot \underbrace{\nabla \mathbf{c}}_{=0} + \mathbf{c} \cdot \nabla \mathbf{U} + \mathbf{U} \wedge \underbrace{(\nabla \wedge \mathbf{c})}_{=0} + \mathbf{c} \underbrace{(\nabla \wedge \mathbf{U})}_{=\omega}$$

$$\therefore \qquad \frac{\partial \mathbf{U}}{\partial T} + (\omega + f) \wedge \mathbf{U} = -\frac{1}{\rho}\nabla(p + \tfrac{1}{2}\rho \mathbf{U}^2) - \mathbf{f} \wedge \mathbf{c} - \frac{d\mathbf{c}}{dt}. \tag{9.1}$$

The vorticity equation takes the form

$$\frac{\partial \omega}{\partial T} + \mathbf{U} \cdot \nabla(\omega + \mathbf{f}) = (\omega + \mathbf{f}) \cdot \nabla \mathbf{U} - \mathbf{c} \cdot \nabla f \tag{9.2}$$

using

$$\nabla \wedge (\mathbf{f} \wedge \mathbf{c}) = \mathbf{f} \underbrace{(\nabla \cdot \mathbf{c})}_{=0} \; - \; \underbrace{(\nabla \cdot \mathbf{f})\mathbf{c}}_{\substack{=0 \\ \mathbf{f}=(0,0,f) \\ \text{and } \partial f/\partial z = 0.}} \; + \mathbf{c} \cdot \nabla \mathbf{f} - \underbrace{\mathbf{f} \cdot \nabla}_{=0} \mathbf{c}$$