Chapter 5

BAROCLINIC VORTEX FLOWS

Vorticity tendency for a baroclinic vortex $v(r, z)$ in a zonal shear flow $U(z)$.

Consider the velocity vector:

$$\mathbf{u} = U(z)\hat{\mathbf{i}} + v(r, z)\hat{\boldsymbol{\theta}} = U\cos\theta\hat{\mathbf{r}} + (v - U\sin\theta)\hat{\boldsymbol{\theta}} \tag{5.1}$$

The vorticity in cylindrical coordinates is

$$\boldsymbol{\omega} = \frac{1}{r}\left(\frac{\partial u_z}{\partial\theta} - \frac{\partial r u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial r u_\theta}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\hat{\mathbf{k}},$$

so that for the velocity vector (5.1),

$$\boldsymbol{\omega} = \left(-\frac{\partial}{\partial z}(v - U\sin\theta)\right)\hat{\mathbf{r}} + \left(\frac{\partial}{\partial z}U\cos\theta\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}r(v - U\sin\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}U\cos\theta\right)\hat{\mathbf{k}}$$

or

$$\boldsymbol{\omega} = \left(\frac{dU}{dz}\sin\theta - \frac{\partial v}{\partial z}\right)\hat{\mathbf{r}} + \frac{dU}{dz}\cos\theta\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial r v}{\partial r}\hat{\mathbf{k}}$$

Let us write

$$\boldsymbol{\omega} = \left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\mathbf{r}} + \frac{dU}{dz}\cos\theta\hat{\boldsymbol{\theta}} + \zeta\hat{\mathbf{k}} \tag{5.2}$$

Now, in cylindrical coordinates (see Batchelor, 1970, p602)

$$\mathbf{u}.\nabla\boldsymbol{\omega} = \left(\mathbf{u}.\nabla\omega_r - \frac{u_\theta\omega_\theta}{r}\right)\hat{\mathbf{r}} + \left(\mathbf{u}.\nabla\omega_\theta + \frac{u_\theta\omega_r}{r}\right)\hat{\boldsymbol{\theta}} + (\mathbf{u}.\nabla\omega_z)\hat{\mathbf{k}}$$

Then for the velocity vector (5.1),

$$\mathbf{u}.\nabla\boldsymbol{\omega} = \left(\mathbf{u}.\nabla\left(\xi + \frac{dU}{dz}\sin\theta\right) - \frac{u_\theta}{r}\frac{dU}{dz}\sin\theta\right)\hat{\mathbf{r}} +$$

$$\left(\mathbf{u}.\nabla\omega_\theta + \frac{u_\theta}{r}\left(\xi + \frac{dU}{dz}\sin\theta\right)\right)\hat{\boldsymbol{\theta}} + (\mathbf{u}.\nabla\zeta)\,\hat{\mathbf{k}}$$

The three components of this equation are:

$$\begin{aligned}
\left(\mathbf{u}.\nabla\xi - \frac{u_\theta\omega_\theta}{r}\right) &= U\cos\theta\frac{\partial}{\partial r}\left(\xi + \frac{dU}{dz}\sin\theta\right) \\
&+ (v - U\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\xi + \frac{dU}{dz}\sin\theta\right) - \frac{(v - U\sin\theta)}{r}\frac{dU}{dz}\cos\theta \\
&= U\cos\theta\frac{\partial\xi}{\partial r}
\end{aligned}$$

$$\begin{aligned}
\left(\mathbf{u}.\nabla\omega_\theta + \frac{u_\theta\omega_r}{r}\right) &= U\cos\theta\frac{\partial}{\partial r}\left(\frac{dU}{dz}\cos\theta\right) \\
&+ (v - U\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{dU}{dz}\cos\theta\right) + \frac{(v - U\sin\theta)}{r}\left(\xi + \frac{dU}{dz}\sin\theta\right) \\
&= \frac{(v - U\sin\theta)}{r}\xi
\end{aligned}$$

$$\mathbf{u}.\nabla\omega_z = U\cos\theta\frac{\partial\zeta}{\partial r} + \frac{(v - U\sin\theta)}{r}\frac{\partial\zeta}{\partial\theta} = U\cos\theta\frac{\partial\zeta}{\partial r}$$

Therefore

$$\mathbf{u}.\nabla\boldsymbol{\omega} = U\cos\theta\frac{\partial\xi}{\partial r}\hat{\mathbf{r}} + \frac{(v - U\sin\theta)}{r}\xi\hat{\boldsymbol{\theta}} + U\cos\theta\frac{\partial\zeta}{\partial r}\hat{\mathbf{k}} \tag{5.3}$$

Now

$$\boldsymbol{\omega}.\nabla\mathbf{u} = \left(\boldsymbol{\omega}.\nabla u_r - \frac{\omega_\theta u_\theta}{r}\right)\hat{\mathbf{r}} + \left(\boldsymbol{\omega}.\nabla u_\theta + \frac{\omega_\theta u_r}{r}\right)\hat{\boldsymbol{\theta}} + (\boldsymbol{\omega}.\nabla u_z)\,\hat{\mathbf{k}}. \tag{5.4}$$

The first component of this equation is

$$\boldsymbol{\omega}.\nabla u_r - \frac{\omega_\theta u_\theta}{r} = \left[\left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\mathbf{r}} + \frac{dU}{dz}\cos\theta\hat{\boldsymbol{\theta}} + \zeta\mathbf{k}\right] \quad \times$$

$$\left(\frac{\partial}{\partial r}(U\cos\theta)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}(U\cos\theta)\hat{\boldsymbol{\theta}} + \frac{\partial}{\partial z}(U\cos\theta)\hat{\mathbf{k}}\right) - \frac{v - U\sin\theta}{r}\frac{dU}{dz}\cos\theta$$

$$= -\frac{U}{r}\frac{dU}{dz}\cos\theta\sin\theta + \zeta\frac{dU}{dz}\cos\theta - \frac{v - U\sin\theta}{r}\frac{dU}{dz}\cos\theta$$

or, finally

$$\boldsymbol{\omega}.\nabla u_r - \frac{\omega_\theta u_\theta}{r} = \zeta\frac{dU}{dz}\cos\theta - \frac{v}{r}\frac{dU}{dz}\cos\theta = \frac{dv}{dr}\frac{dU}{dz}\cos\theta \tag{5.5}$$

The second component of (5.5) is

$$\boldsymbol{\omega}.\nabla u_\theta + \frac{\omega_\theta u_r}{r} = \left[\left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\mathbf{r}} + \frac{dU}{dz}\cos\theta\hat{\boldsymbol{\theta}} + \zeta\mathbf{k}\right] \quad \times$$

$$\left[\frac{\partial}{\partial r}(v - U\sin\theta)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}(v - U\sin\theta)\hat{\boldsymbol{\theta}} + \frac{\partial}{\partial z}(v - U\sin\theta)\hat{\mathbf{k}}\right] + \frac{U\cos\theta}{r}\frac{dU}{dz}\cos\theta$$

$$= \left[\left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\mathbf{r}} + \frac{dU}{dz}\cos\theta\hat{\boldsymbol{\theta}} + \zeta\mathbf{k}\right] \quad \times$$

$$\left[\frac{\partial v}{\partial r}\hat{\mathbf{r}} - \frac{U}{r}\cos\theta\hat{\boldsymbol{\theta}} + \left(\frac{\partial v}{\partial z} - \frac{dU}{dz}\sin\theta\right)\hat{\mathbf{k}}\right] + \frac{U}{r}\frac{dU}{dz}\cos^2\theta$$

$$= \left(\xi + \frac{dU}{dz}\sin\theta\right)\frac{\partial v}{\partial r} - \zeta\left(\xi + \frac{dU}{dz}\sin\theta\right) = \left(\frac{\partial v}{\partial r} - \zeta\right)\left(\xi + \frac{dU}{dz}\sin\theta\right),$$

or, finally,

$$\boldsymbol{\omega}.\nabla u_\theta + \frac{\omega_\theta u_r}{r} = -\frac{v}{r}\left(\xi + \frac{dU}{dz}\sin\theta\right) \tag{5.6}$$

The third component of (5.5) is simply

$$\boldsymbol{\omega}.\nabla u_z = 0 \tag{5.7}$$

The (5.4) may be written

$$\boldsymbol{\omega}.\nabla\mathbf{u} = \frac{dv}{dr}\frac{dU}{dz}\cos\theta\hat{\mathbf{r}} \;\; - \frac{v}{r}\left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\boldsymbol{\theta}} \tag{5.8}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = -\mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{u}$$

$$\mathbf{u}.\nabla\boldsymbol{\omega} = U\cos\theta\frac{\partial\xi}{\partial r}\hat{\mathbf{r}} + \frac{(v - U\sin\theta)}{r}\left(\xi - \frac{dU}{dz}\sin\theta\right)\hat{\boldsymbol{\theta}} + U\cos\theta\frac{\partial\zeta}{\partial r}\hat{\mathbf{k}}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = -\left(U\cos\theta\frac{\partial\xi}{\partial r}\hat{\mathbf{r}} + \frac{(v - U\sin\theta)}{r}\xi\hat{\boldsymbol{\theta}} + U\cos\theta\frac{\partial\zeta}{\partial r}\hat{\mathbf{k}}\right) +$$

$$\frac{dv}{dr}\frac{dU}{dz}\cos\theta\hat{\mathbf{r}} \;\; - \frac{v}{r}\left(\xi + \frac{dU}{dz}\sin\theta\right)\hat{\boldsymbol{\theta}}$$

$$= \left(-U\frac{\partial\xi}{\partial r} + \frac{dv}{dr}\frac{dU}{dz}\right)\cos\theta\hat{\mathbf{r}} - \left[\left(\frac{v}{r} + \frac{(v - U\sin\theta)}{r}\right)\xi - \frac{v}{r}\frac{dU}{dz}\sin\theta\right]\hat{\boldsymbol{\theta}} - U\cos\theta\frac{\partial\zeta}{\partial r}\hat{\mathbf{k}},$$

or finally,

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \left(-U\frac{\partial\xi}{\partial r} + \frac{dv}{dr}\frac{dU}{dz}\right)\cos\theta\hat{\mathbf{r}} - \left[\left(\frac{2v}{r} - \frac{U\sin\theta}{r}\right)\xi + \frac{v}{r}\frac{dU}{dz}\sin\theta\right]\hat{\boldsymbol{\theta}} - U\cos\theta\frac{\partial\zeta}{\partial r}\hat{\mathbf{k}} \tag{5.9}$$

Special cases:

1. Uniform flow ($U =$ constant), barotropic vortex, $v = v(r) \Rightarrow \xi = 0$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -U \cos\theta \frac{\partial \zeta}{\partial r}\hat{\mathbf{k}} \quad \Rightarrow \quad \frac{\partial \zeta}{\partial t} = -U\frac{\partial \zeta}{\partial x}$$

In this case there is only vertical vorticity and this is simply advected by the basic flow as discussed in Chapter 3.

2. No basic flow ($U = 0$), baroclinic vortex, $v = v(r, z)$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\frac{2v}{r}\xi\hat{\boldsymbol{\theta}}$$

$$\frac{\partial \xi}{\partial t} = 0, \quad \frac{\partial \eta}{\partial t} = -\frac{2v}{r}\xi, \quad \frac{\partial \zeta}{\partial t} = 0$$

In this case there are initially two components of vorticity, a radial component and vertical vertical component, but in general, the vortex does not remain stationary as there is generation of toroidal vorticity. The exception is, of course, when the vortex is in thermal-wind balance in which case there is generation of toroidal vorticity of the opposite sign by the horizontal density gradient so that the net rate-of-generation of toroidal vorticity is everywhere zero.

3. Uniform flow ($U =$ constant), baroclinic vortex, $v = v(r, z)$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -U \cos\theta \frac{\partial \xi}{\partial r}\hat{\mathbf{r}} - \left(\frac{2v}{r} - \frac{U\sin\theta}{r}\right)\xi\hat{\boldsymbol{\theta}} - U\cos\theta\frac{\partial \zeta}{\partial r}\hat{\mathbf{k}}$$

$$\frac{\partial \xi}{\partial t} = -U\frac{\partial \xi}{\partial x} \quad \frac{\partial \eta}{\partial t} = -\left(\frac{2v}{r} - \frac{U\sin\theta}{r}\right)\xi \quad \frac{\partial \zeta}{\partial t} = -U\frac{\partial \zeta}{\partial x}$$

Again there are initially two components of vorticity, a radial component and vertical vertical component, and again there is generation of toroidal vorticity unless the vortex is in thermal-wind balance. However, even in the latter case there would appear to be a generation of toroidal vorticity at the rate $(U\sin\theta/r)\xi$. It can be shown that this rate-of-generation is associated with the coordinate system represented by the unit vectors $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}}$, is fixed (see Exercise 5.1). Thus as the vortex moves away from the origin of coordinates, the radial component of vorticity in the moving frame projects onto the $\hat{\theta}$-component in the fixed frame.

4. Uniform shear flow (dU/dz = constant = U'), barotropic vortex, $v = v(r) \Rightarrow \xi = 0$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \frac{dv}{dr}\frac{dU}{dz}\cos\theta\hat{\mathbf{r}} + \frac{v}{r}\frac{dU}{dz}\sin\theta\hat{\boldsymbol{\theta}} + U\cos\theta\frac{\partial \zeta}{\partial r}\hat{\mathbf{k}}$$

$$\frac{\partial \xi}{\partial t} = -U\frac{\partial \xi}{\partial x} + \frac{dv}{dr}\frac{dU}{dz}\cos\theta \quad \frac{\partial \eta}{\partial t} = \frac{v}{r}\frac{dU}{dz}\sin\theta \quad \frac{\partial \zeta}{\partial t} = -U\frac{\partial \zeta}{\partial x}$$

Translation of the balanced density field

Let $\rho = p_0(r, z)$ at time $t = 0$. Then

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{u}) = -\mathbf{u}\cdot\nabla\rho - \rho(\nabla\cdot\mathbf{u}).$$

Now the velocity field $\mathbf{u} = (U(z)\cos\theta,\ v - U(z)\sin\theta,\ 0)$ is nondivergent ($\nabla\cdot\mathbf{u} = 0$) and therefore

$$\frac{\partial \rho}{\partial t} = -U(z)\cos\theta\frac{\partial \rho}{\partial r} - \frac{(v - U(z)\sin\theta)}{r}\frac{\partial \rho}{\partial \theta}.$$

The second term on the right-hand-side is zero because ρ is dependent of θ whereupon

$$\frac{\partial \rho}{\partial t} = -U(z)\frac{\partial \rho}{\partial x}$$

and the density field is simply advected at speed $U(z)$ at height z.

Exercise 5.1 Show that the term $(U\sin\theta/r)\xi$ in Eqs. (**??**) is the rate-of-

generation of toroidal vorticity in the fixed coordinate system represented by the unit vectors $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}}$ due to the subsequent displacement of the vortex centre from the coordinate origin.

Exercise 5.2 Show that

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},$$

and

$$\frac{1}{r}\frac{\partial}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y},$$

Deduce that

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta},$$

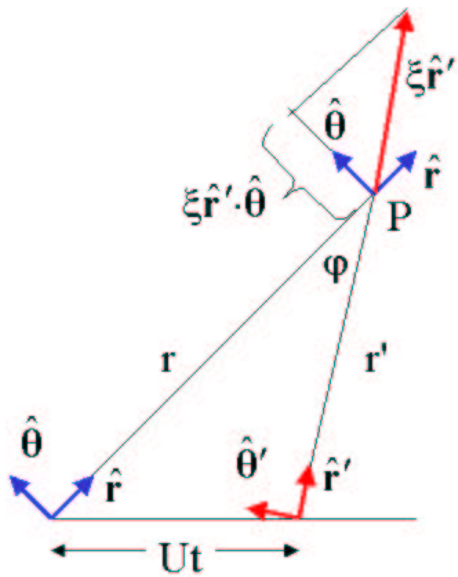

Figure 5.1: See text for discussion.

and

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Solution to Exercise 5.1

Let the vortex be centred at the origin at time $t = 0$ and at a position Ut from the origin at time t (Fig. 5.1). At time t, the radial component of vorticity is $\omega_r'\hat{\mathbf{r}}' = \xi\hat{\mathbf{r}}'$ and we are interested in the projection of this vector in the $\hat{\boldsymbol{\lambda}}'$ direction. In particular we want to calculate its rate of change

$$\Lambda = \frac{d}{dt}(\xi\hat{\mathbf{r}}' \cdot \hat{\boldsymbol{\lambda}}) = \xi\frac{d}{dt}\sin\phi$$

Consider $\mathbf{r} \wedge \mathbf{r}' = |\mathbf{r}||\mathbf{r}'|\sin\phi\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector normal to the plane of $\mathbf{r}$ and $\hat{\mathbf{r}}'$ and note that $\mathbf{r}' = \mathbf{r} - X\hat{\mathbf{i}}$, where $\hat{\mathbf{i}}$ is a unit vector in the x-direction

$$\hat{\mathbf{k}}\sin\phi = \frac{X\hat{\mathbf{i}} \wedge \mathbf{r}'}{|\mathbf{r}|\,|\mathbf{r}'|} = \hat{\mathbf{k}}\frac{X}{|r|}\sin\lambda$$

so that

$$\frac{d}{dt}\sin\phi = \frac{1}{r}\sin\lambda\frac{dX}{dt} = \frac{U}{r}\sin\lambda$$

and therefore

$$\Lambda = \frac{U}{r}\xi\sin\lambda$$

as required.