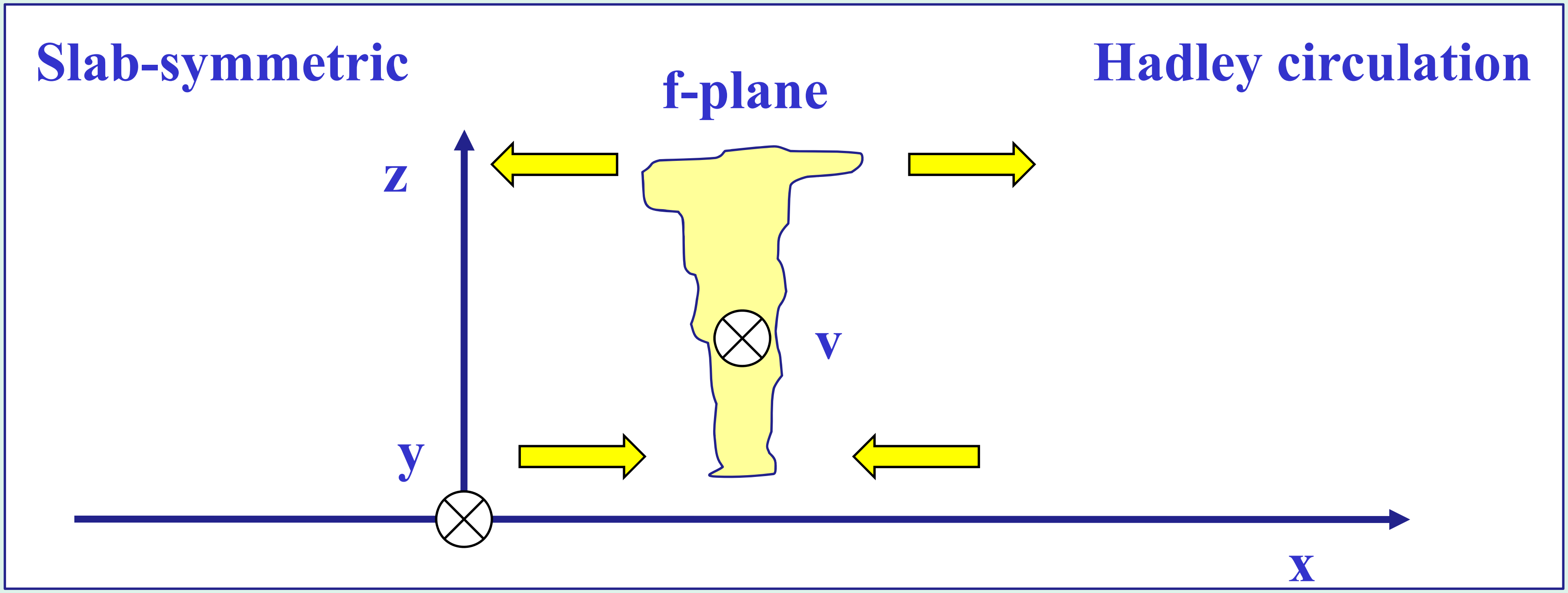
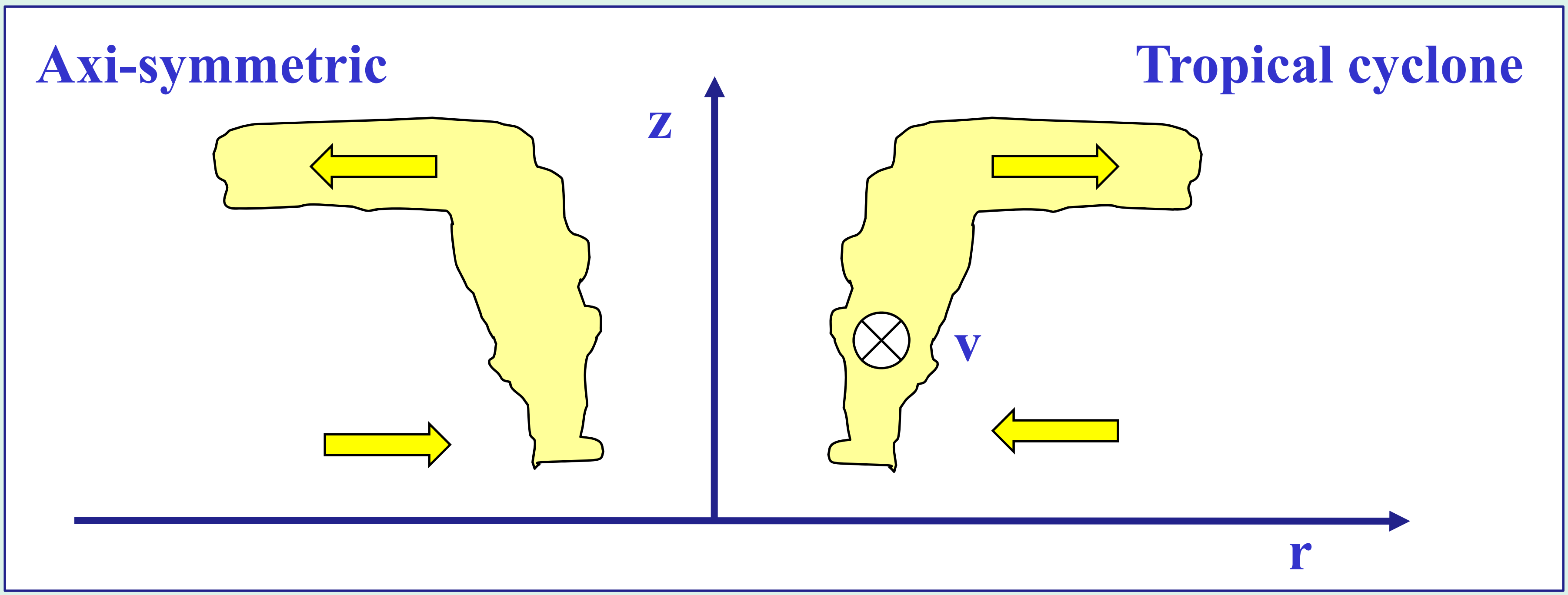
Slab-symmetric and axi-symmetric models

Balanced evolution of tropical cyclones


Slab-symmetric
f-plane
Hadley circulation
z
v
y
x


Axi-symmetric
Tropical cyclone
z
v
r

Slab-symmetric

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + fu = \dot{V}$$

$$\zeta = \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + w\left(N^2 + \frac{\partial b}{\partial z}\right) = \dot{B}$$

$$fS = f\frac{\partial v}{\partial z} = \frac{\partial b}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = \frac{\partial \psi}{\partial z} \qquad w = -\frac{\partial \psi}{\partial x}$$

Thermal wind

Axi-symmetric

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{v^2}{r} + fu = 0$$

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r}$$

$$\chi = 1/\theta$$

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta}$$

$$g\frac{\partial(\ln \chi)}{\partial r} + C\frac{\partial(\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}$$

$$\frac{\partial}{\partial r}\rho r u + \frac{\partial}{\partial r}\rho r z = 0$$

$$u = -\frac{1}{r\rho}\frac{\partial \psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial \psi}{\partial r}$$

Potential vorticity

Ertel PV

$$P = \frac{(\omega + \mathbf{f}) \cdot \nabla \theta}{\rho}$$

Slab-symmetric form

$$q = \omega_{\mathbf{a}} \cdot \nabla b = \left(N^2 + \frac{\partial b}{\partial z} \right) \zeta_a - \frac{\partial v}{\partial z} \frac{\partial b}{\partial x} = \left(N^2 + \frac{\partial b}{\partial z} \right) \zeta_a - f S^2$$

Sawyer-Eliassen Equation

Slab-symmetric

$$\left(N^2+\frac{\partial b}{\partial z}\right)\frac{\partial^2\psi}{\partial x^2}-2fS\frac{\partial^2\psi}{\partial x\partial z}+f\zeta_a\frac{\partial^2\psi}{\partial z^2}=\frac{\partial \dot{V}}{\partial z}-\frac{\partial \dot{B}}{\partial x}$$

Transform $X=x+v/f, \qquad Z=z$

$$\frac{\partial}{\partial X}\left(q\frac{\partial\psi}{\partial X}\right)+f^3\frac{\partial^2\psi}{\partial Z^2}=\frac{f^2}{\zeta_a}\left(\frac{\partial \dot{V}}{\partial Z}+\frac{S}{f}\frac{\partial \dot{V}}{\partial X}\right)-\frac{\partial \dot{B}}{\partial X}$$

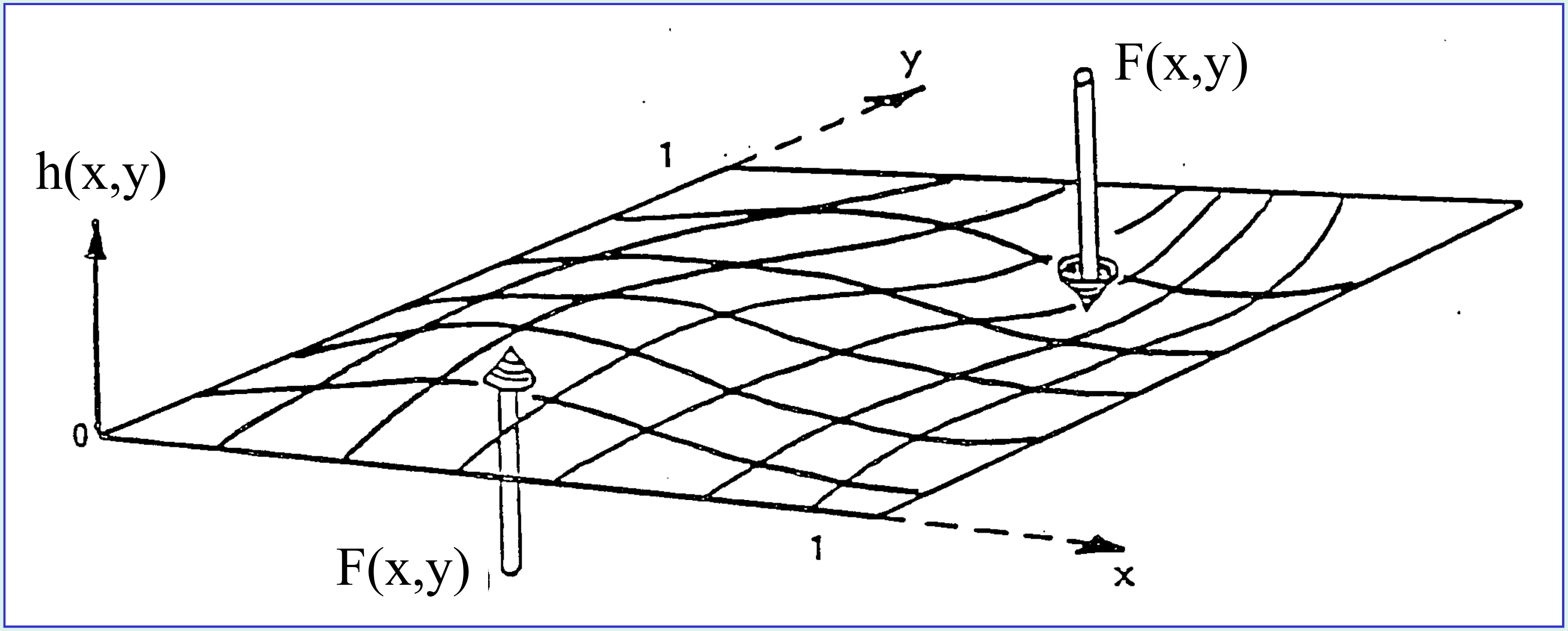
Special case q = constant

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial \bar{Z}^2}=\frac{f^2}{q\zeta_a}\left(\frac{\partial \dot{V}}{\partial Z}+\frac{S}{f}\frac{\partial \dot{V}}{\partial X}\right)-\frac{1}{q}\frac{\partial \dot{B}}{\partial X}$$

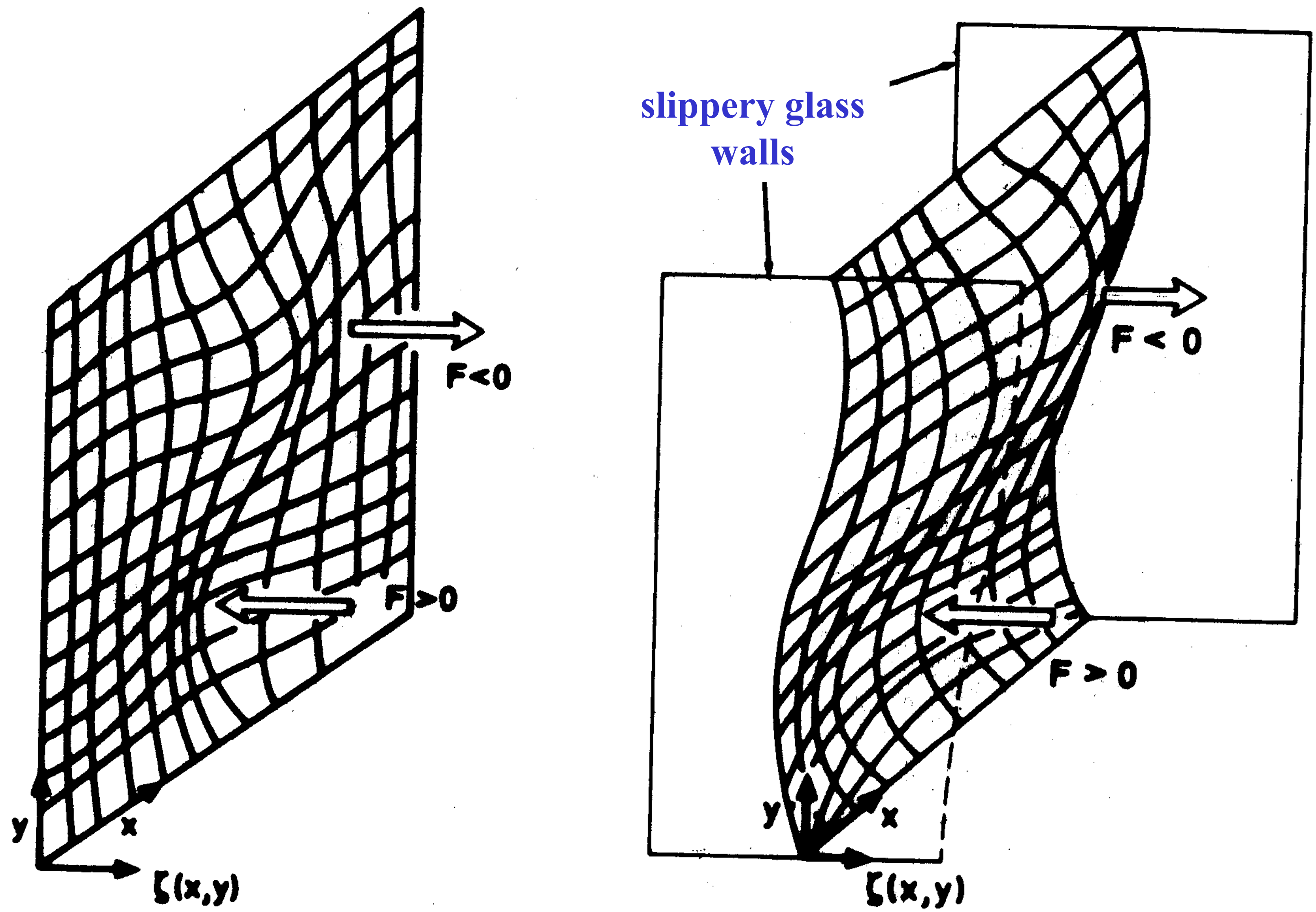
where $\bar{Z}=\sqrt{q/f}Z/f$ is a vertical length scale

The membrane analogy

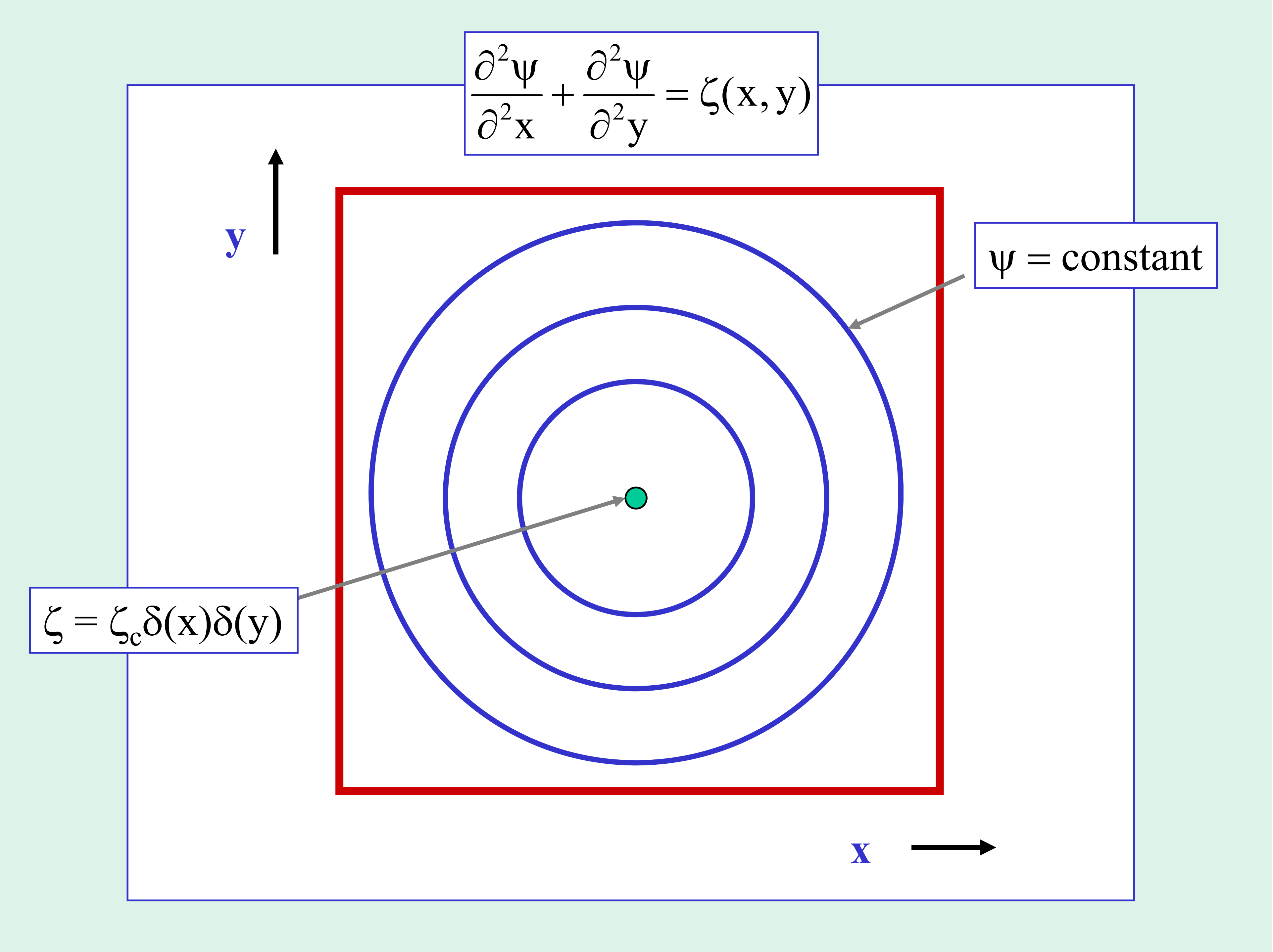
$$\frac{\partial^2 h}{\partial^2 x} + \frac{\partial^2 h}{\partial^2 y} = -F(x, y)$$



Equilibrium displacement of a stretched membrane over a square under the force distribution F(x,y).



F<0
F>0
y
x
ζ(x,y)
slippery glass
walls
F < 0
F > 0
y
x
ζ(x,y)

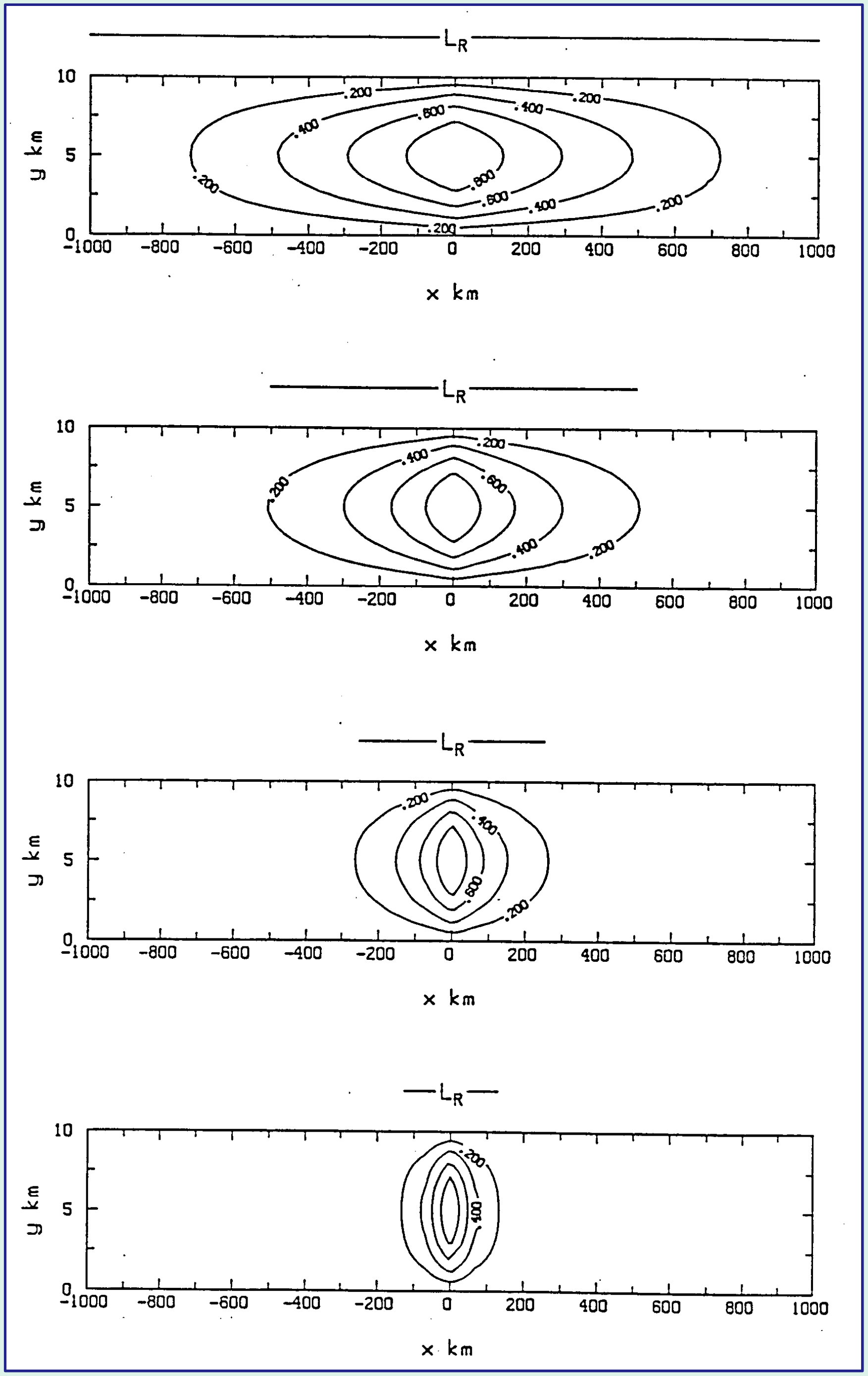

$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = \zeta(x, y)$
y
ψ = constant
$\zeta = \zeta_c \delta(x)\delta(y)$
x

$$N^2 \frac{\partial^2 \psi}{\partial^2 x} + f^2 \frac{\partial^2 \psi}{\partial^2 z} = F(x,z)$$

Put $\overline{z} = \frac{N}{f} z \quad \Rightarrow$

$$\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial^2 \overline{z}} = \frac{1}{N^2} \overline{F}(x,\overline{z})$$

$$L_R = \frac{NH}{f}$$



The Sawyer-Eliassen equation

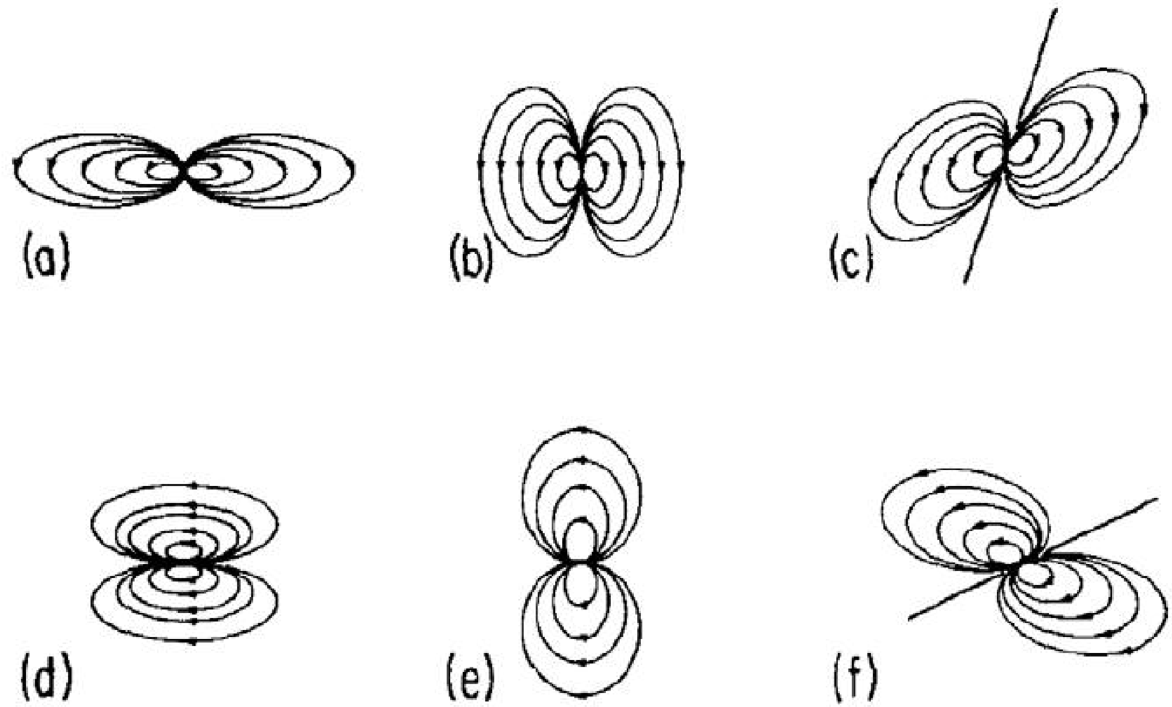


Figure 3.11: Streamfunction responses to point sources of: (a) Heat in a barotropic vortex with weak inertial stability, (b) heat in a barotropic vortex with strong inertial stability, (c) heat in a baroclinic vortex, (d) momentum in a barotropic vortex with weak inertial stability, (e) momentum in a barotropic vortex with strong inertial stability, and (f) momentum in a baroclinic vortex. (Based on Figs. 8, 9, 11, and 12

Sawyer-Eliassen Equation

Axi-symmetric

$$\frac{\partial}{\partial r}\left[g\frac{\partial \chi}{\partial z}\frac{1}{\rho r}\frac{\partial \psi}{\partial r} + \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial z}\right] -$$

$$\frac{\partial}{\partial z}\left[\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial \psi}{\partial z} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right] = g\frac{\partial Q}{\partial r} + \frac{\partial}{\partial z}(CQ) + \frac{\partial}{\partial z}(\chi\xi\dot{V})$$

Discriminant $$D = -g\frac{\partial \chi}{\partial z}\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right) - \left[\frac{\partial}{\partial z}(\chi C)\right]^2$$

SE equation is elliptic if D > 0

$$u = -\frac{1}{r\rho}\frac{\partial \psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial \psi}{\partial r}$$

Parameters in SE Equation

- the *static stability*

$$N^2 = -g\frac{\partial ln\chi}{\partial z};$$

- the *inertial stability*

$$I^2 = \frac{1}{r^3}\frac{\partial M^2}{\partial r} = \xi(\zeta + f)$$

- the *baroclinicity*

$$B^2 = \frac{1}{r^3}\frac{\partial M^2}{\partial z} = \xi S.$$

$$\xi = \frac{2v}{r} + f$$

Potential vorticity

Ertel PV

$$P = \frac{(\omega + \mathbf{f}) \cdot \nabla\theta}{\rho}$$

Slab-symmetric

$$q = \omega_{\mathbf{a}} \cdot \nabla b = \left(N^2 + \frac{\partial b}{\partial z}\right)\zeta_a - \frac{\partial v}{\partial z}\frac{\partial b}{\partial x} = \left(N^2 + \frac{\partial b}{\partial z}\right)\zeta_a - fS^2$$

Axi-symmetric

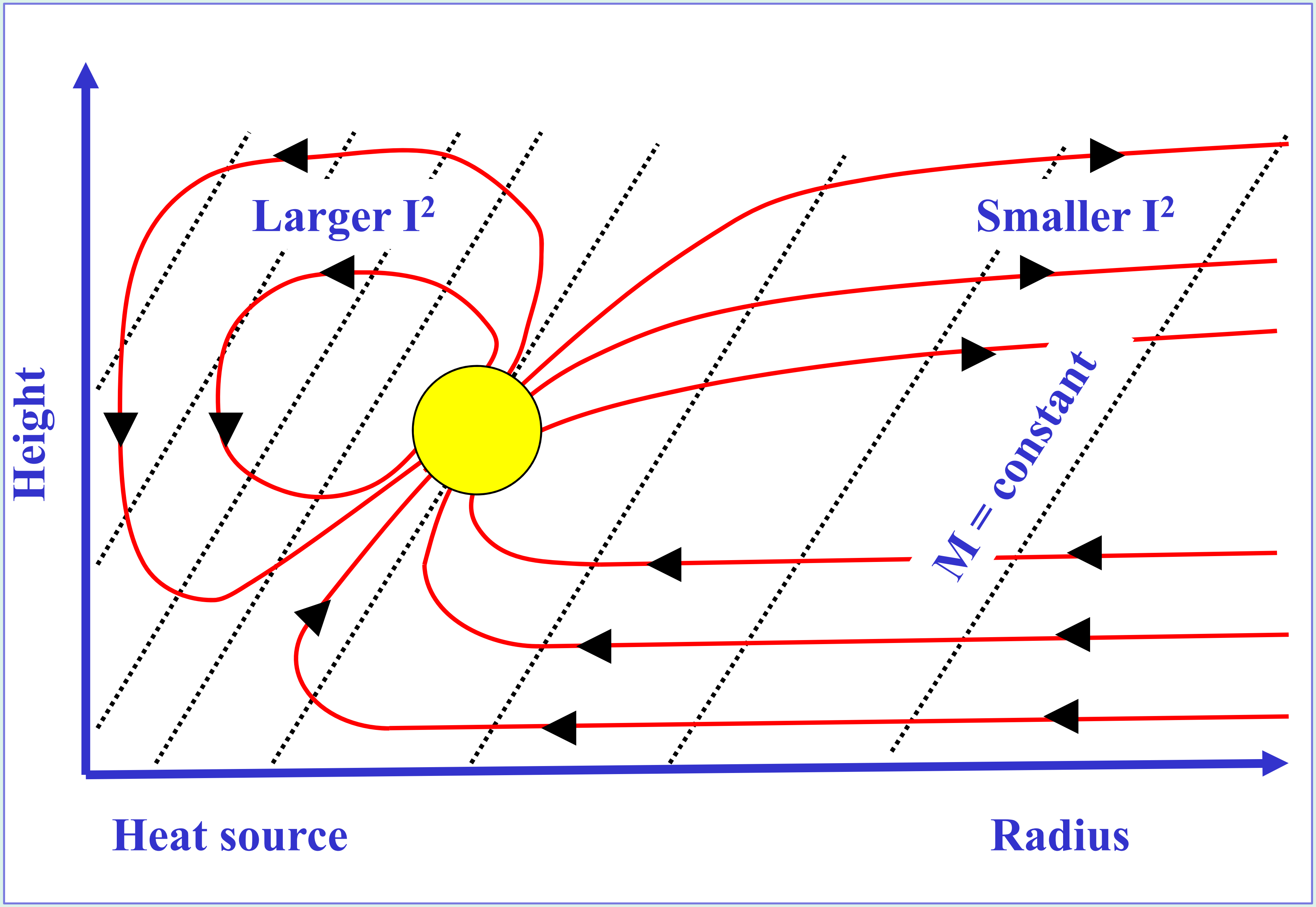
$$P = \frac{1}{\rho\chi^2}\left[\frac{\partial v}{\partial z}\frac{\partial \chi}{\partial r} - (\zeta + f)\frac{\partial \chi}{\partial z}\right]$$

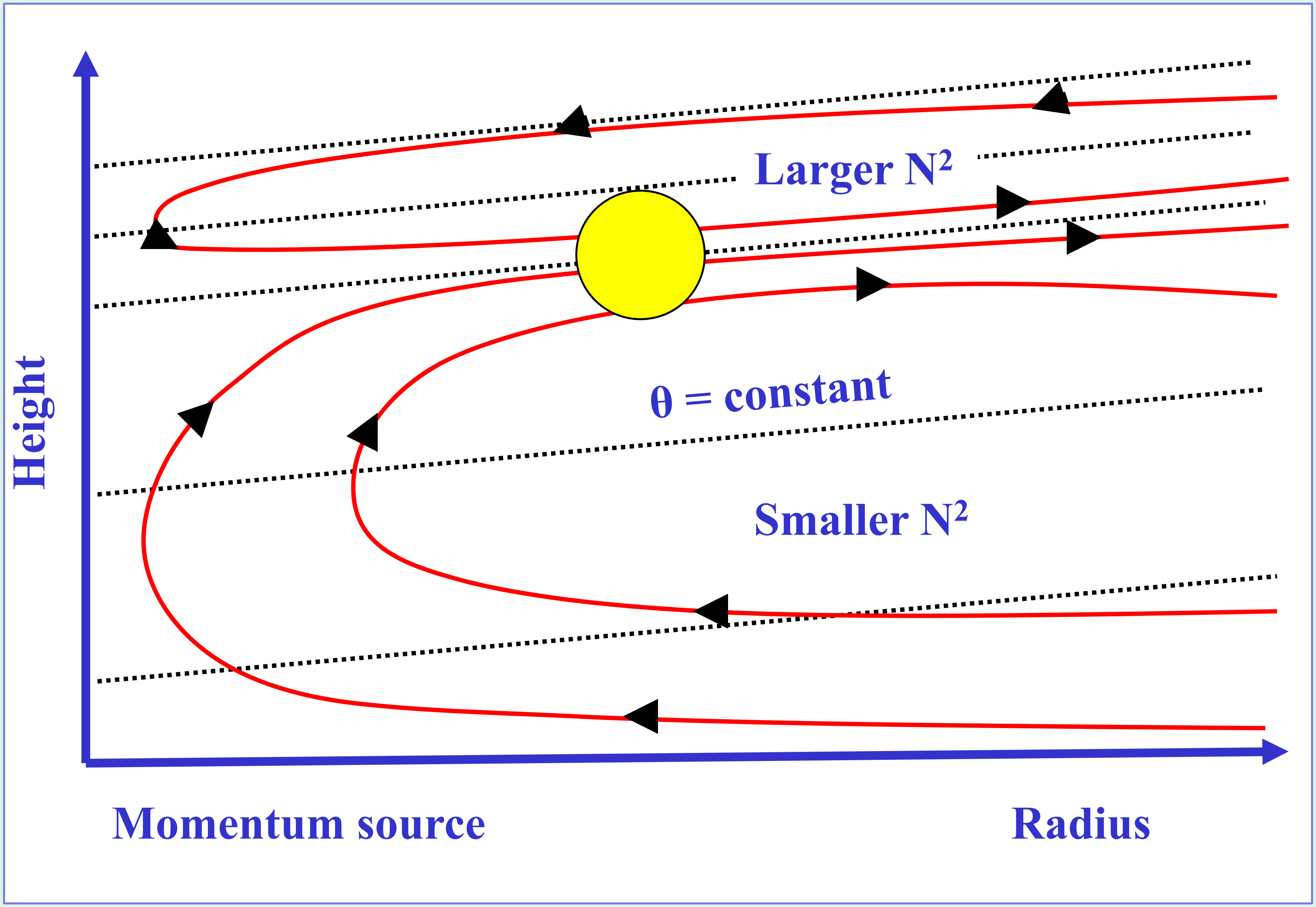
Discriminant

$$D = -g\frac{\partial \chi}{\partial z}\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right) - \left[\frac{\partial}{\partial z}(\chi C)\right]^2$$

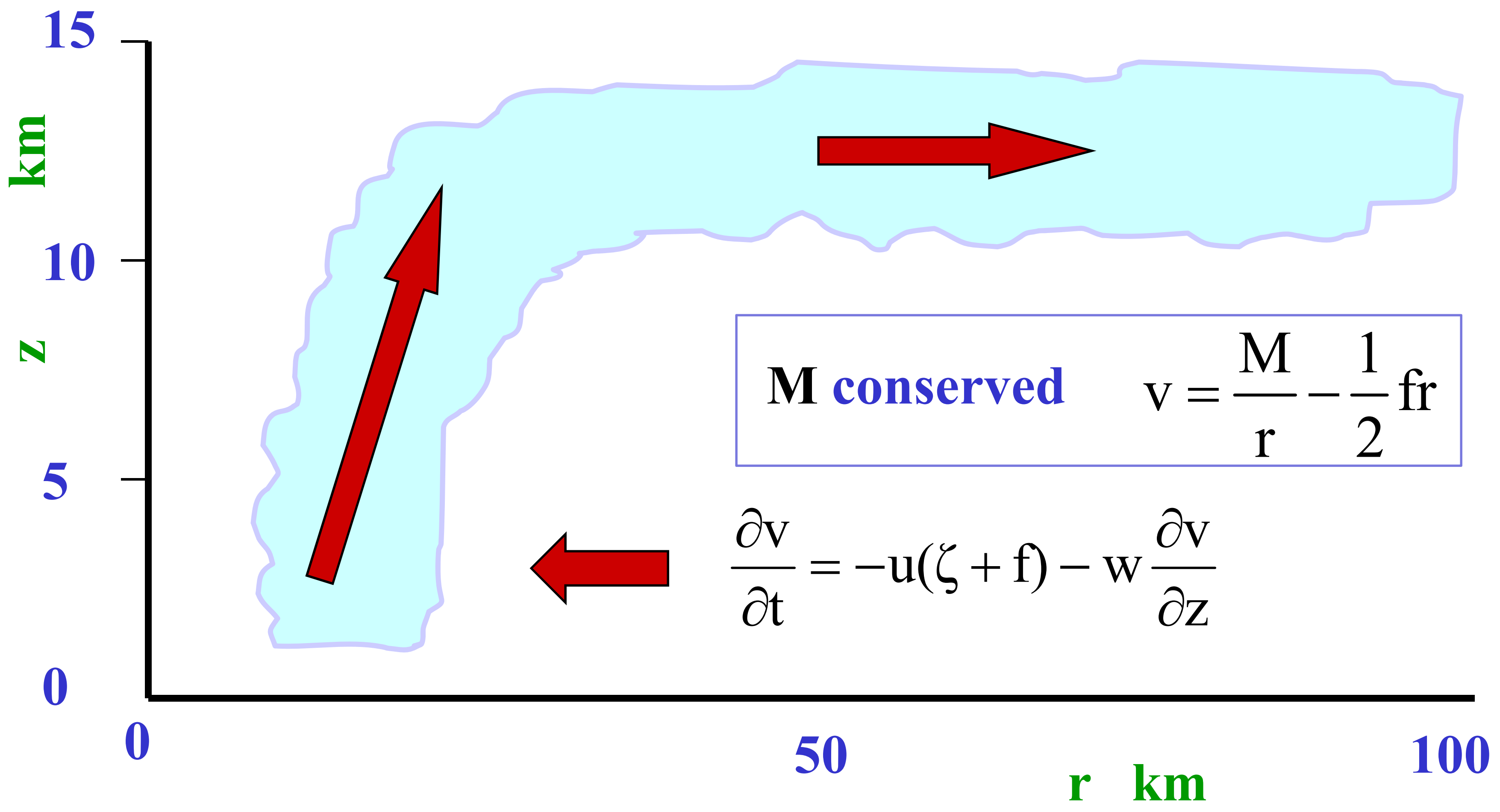
Can show that

$$g\rho\chi^3\xi P = D$$


Height
Larger I^2
Smaller I^2
M = constant
Heat source
Radius


Larger N²
θ = constant
Smaller N²
Height
Momentum source
Radius

Thermally-forced secondary circulation leads to spin up



$$v = \frac{M}{r} - \frac{1}{2} fr$$

$$\frac{\partial v}{\partial t} = -u(\zeta + f) - w\frac{\partial v}{\partial z}$$

Prediction method

Initial condition: v(r,z,0) **given**

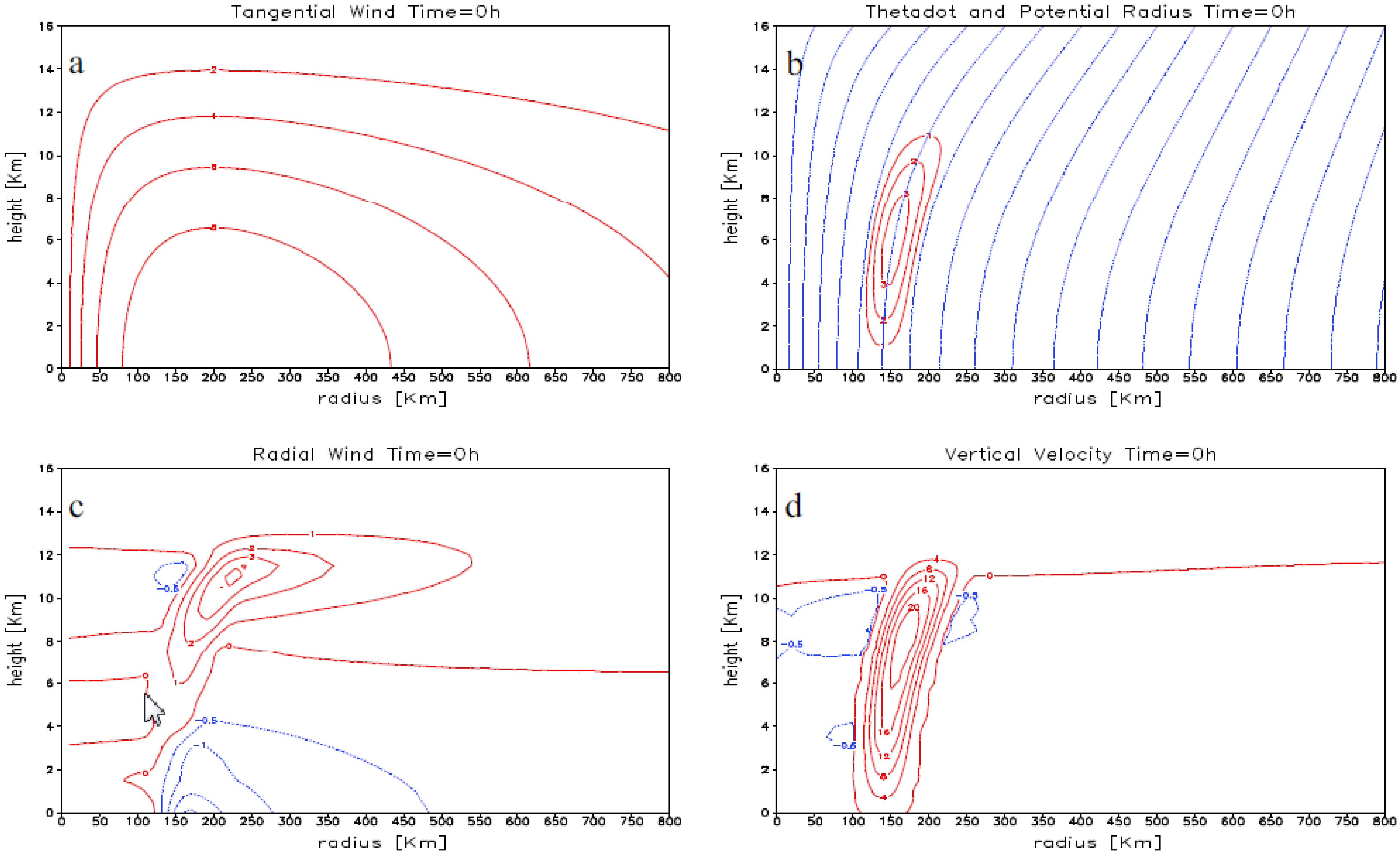
Heating and friction distributions given

Solve $$g\frac{\partial(\ln\chi)}{\partial r}+C\frac{\partial(\ln\chi)}{\partial z}=-\frac{\partial C}{\partial z}$$ **for** χ

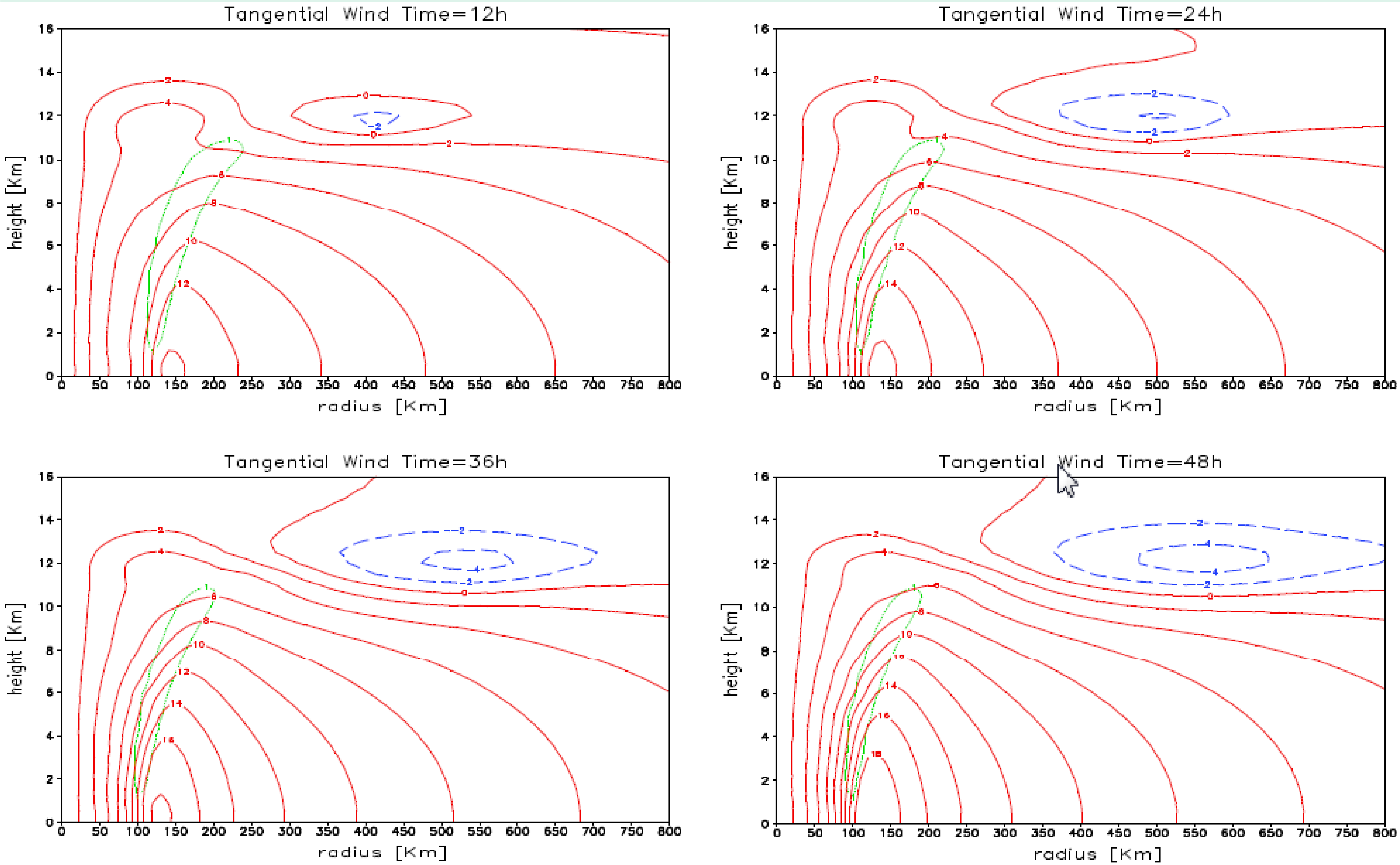
Solve SE-equation for ψ

Solve $$u=-\frac{1}{r\rho}\frac{\partial\psi}{\partial z}\qquad w=\frac{1}{r\rho}\frac{\partial\psi}{\partial r}$$ **for** u **and** w

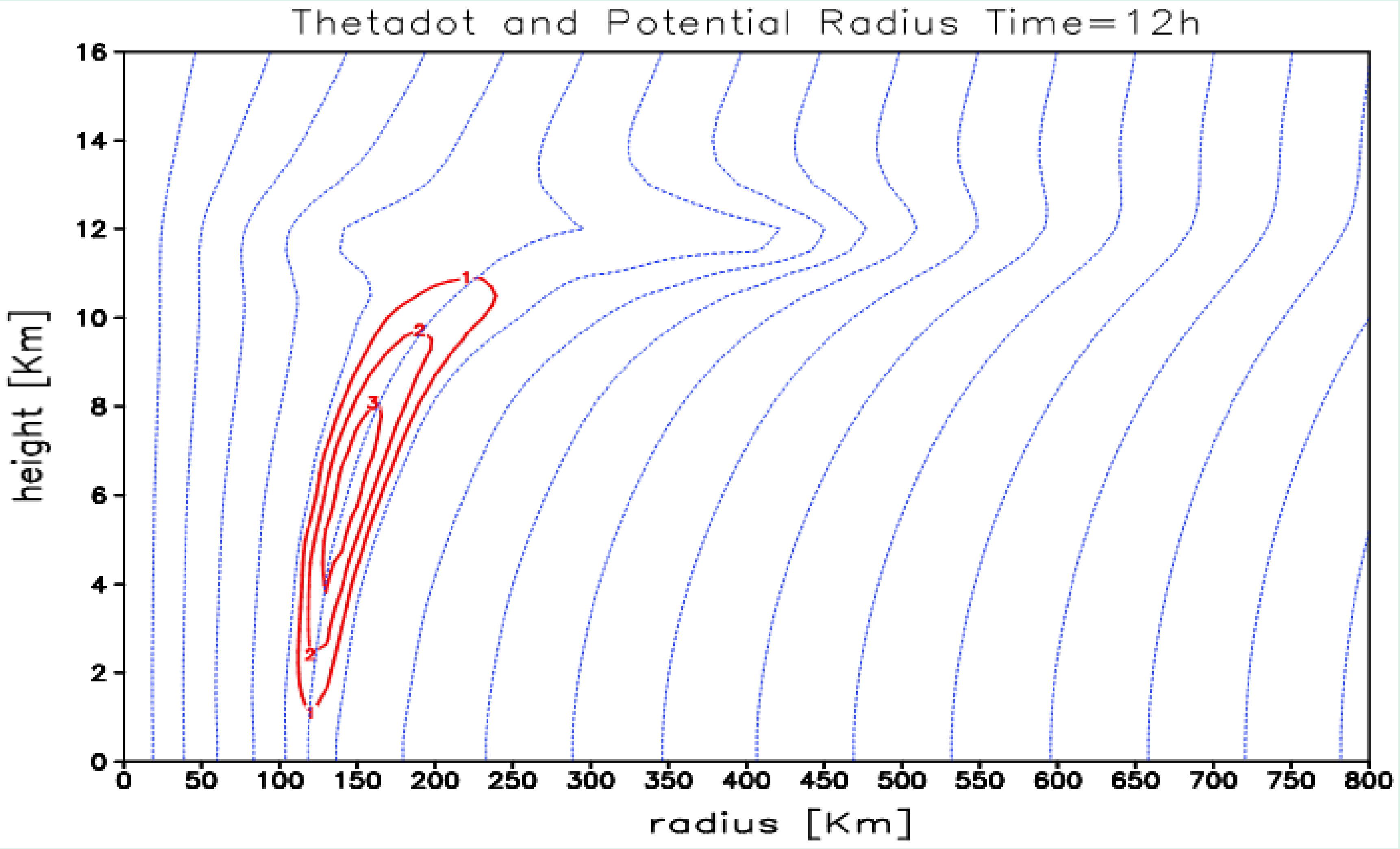
Integrate $$\boxed{\frac{\partial v}{\partial t}+u(\zeta+f)+wS=\dot{V}}$$ **for** v(r,z,Δt)



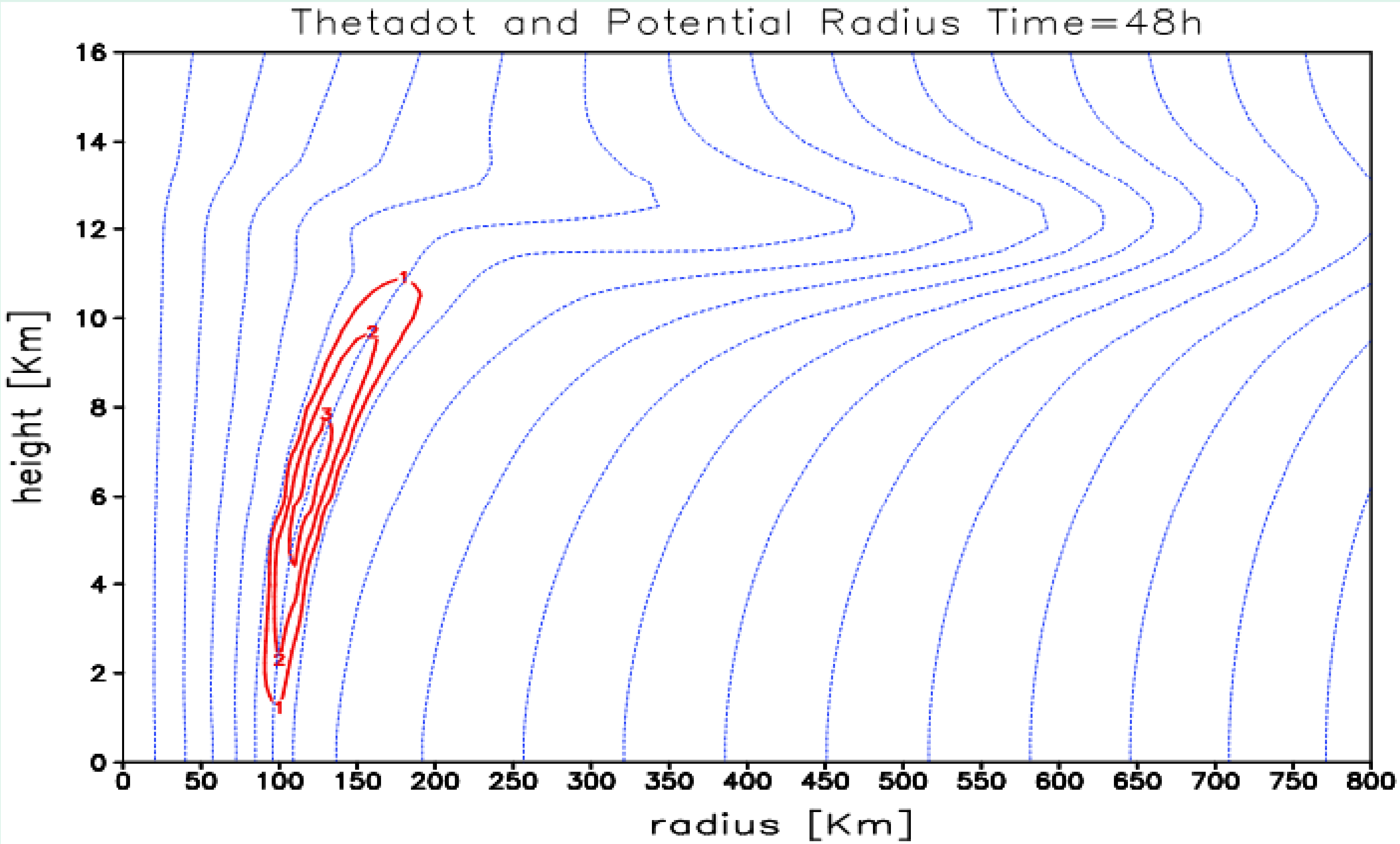
Calculations by Dr. Hai Bui (Hanoi Uni)



Calculations by Dr. Hai Bui (Hanoi Uni)

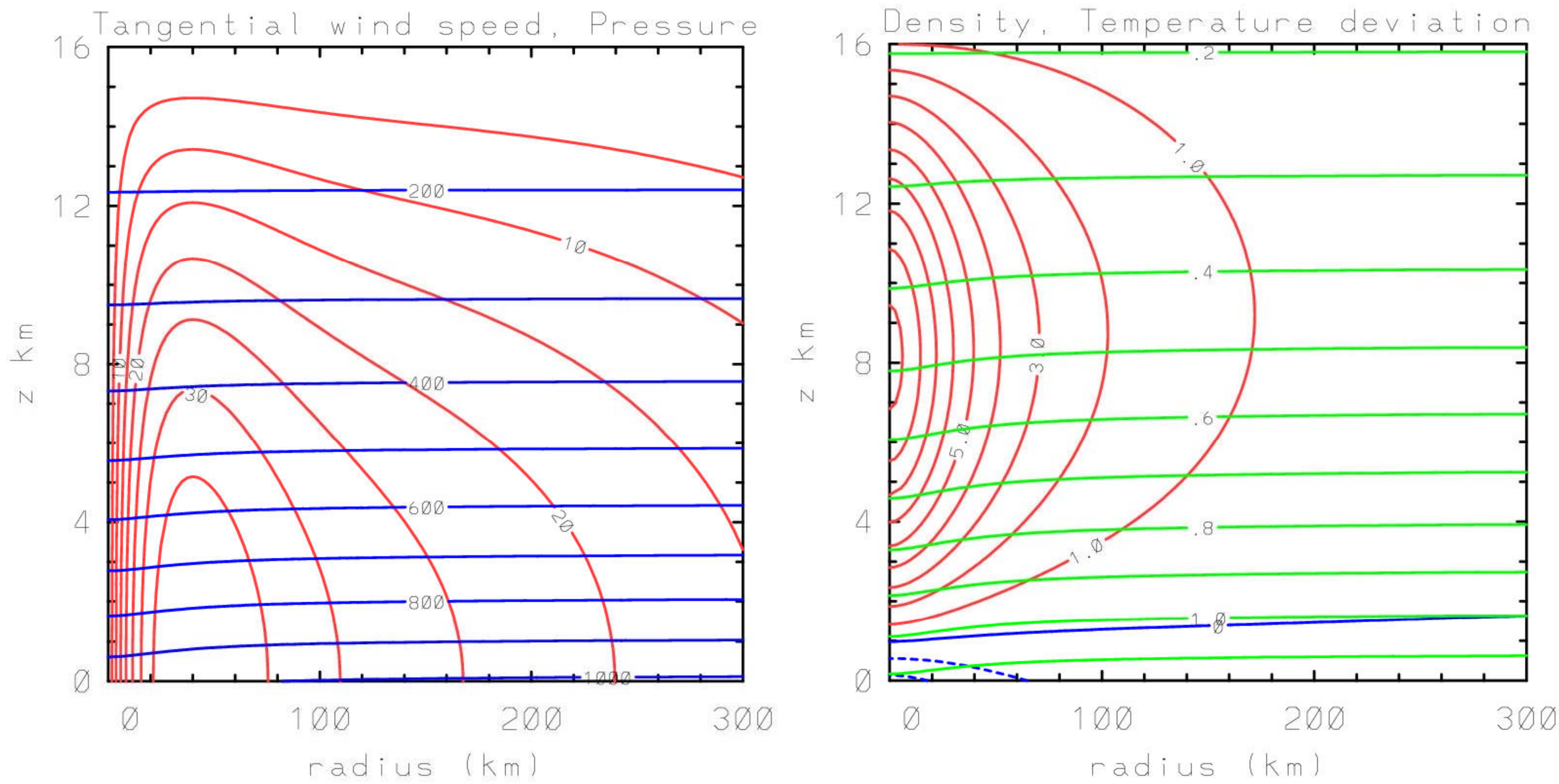


Calculations by Dr. Hai Bui (Hanoi Uni)

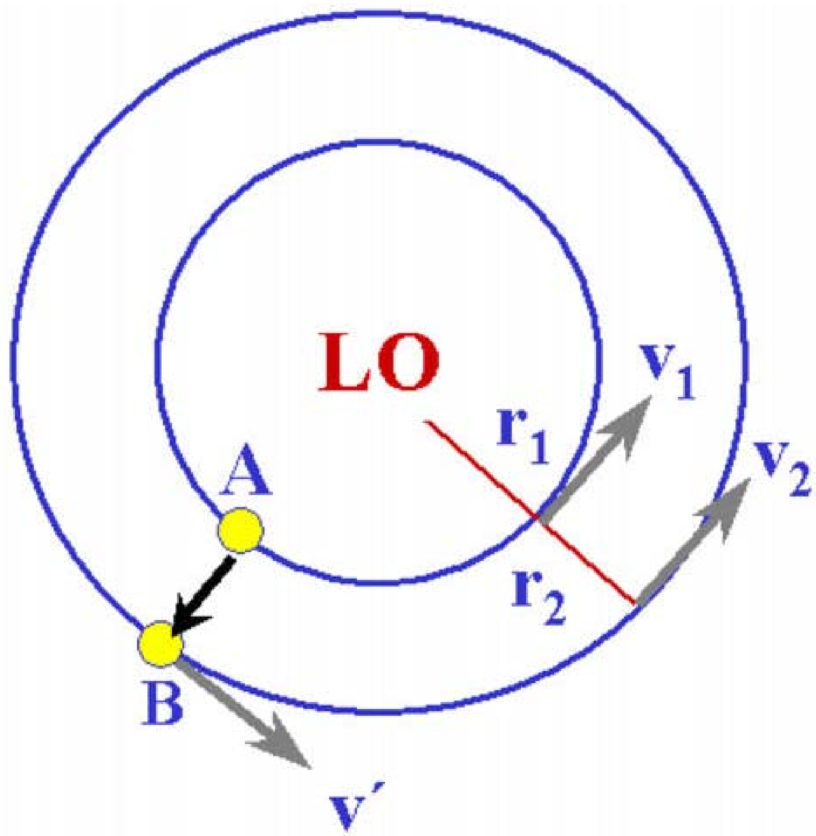


Calculations by Dr. Hai Bui (Hanoi Uni)

A warm-cored vortex



Barotropic stability



The parcel at **A** conserves its angular momentum during its radial displacement to **B**

$$r_2 v' + \frac{1}{2} f r_2^2 = r_1 v_1 + \frac{1}{2} f r_1^2,$$

$$v' = \frac{r_1}{r_2} v_1 + \frac{1}{2} \frac{f}{r_2} (r_1^2 - r_2^2) \qquad (3.17)$$

Net radial force on a displaced air parcel

Radial pressure gradient at B

$$\frac{1}{\rho}\frac{dp}{dr}\bigg]_{r=r_2} = \frac{v_2^2}{r_2} + f v_2. \tag{3.18}$$

Net force on parcel at B

$$\begin{aligned} F &= \text{centrifugal} + \text{Coriolis force} - \text{radial pressure gradient} \\ &= \frac{v'^2}{r_2} + f v' - \frac{1}{\rho}\frac{\partial p}{\partial r}\bigg]_{r=r_2} \end{aligned}$$

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2} r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2} r_2^2 f)^2\right]. \tag{3.19}$$

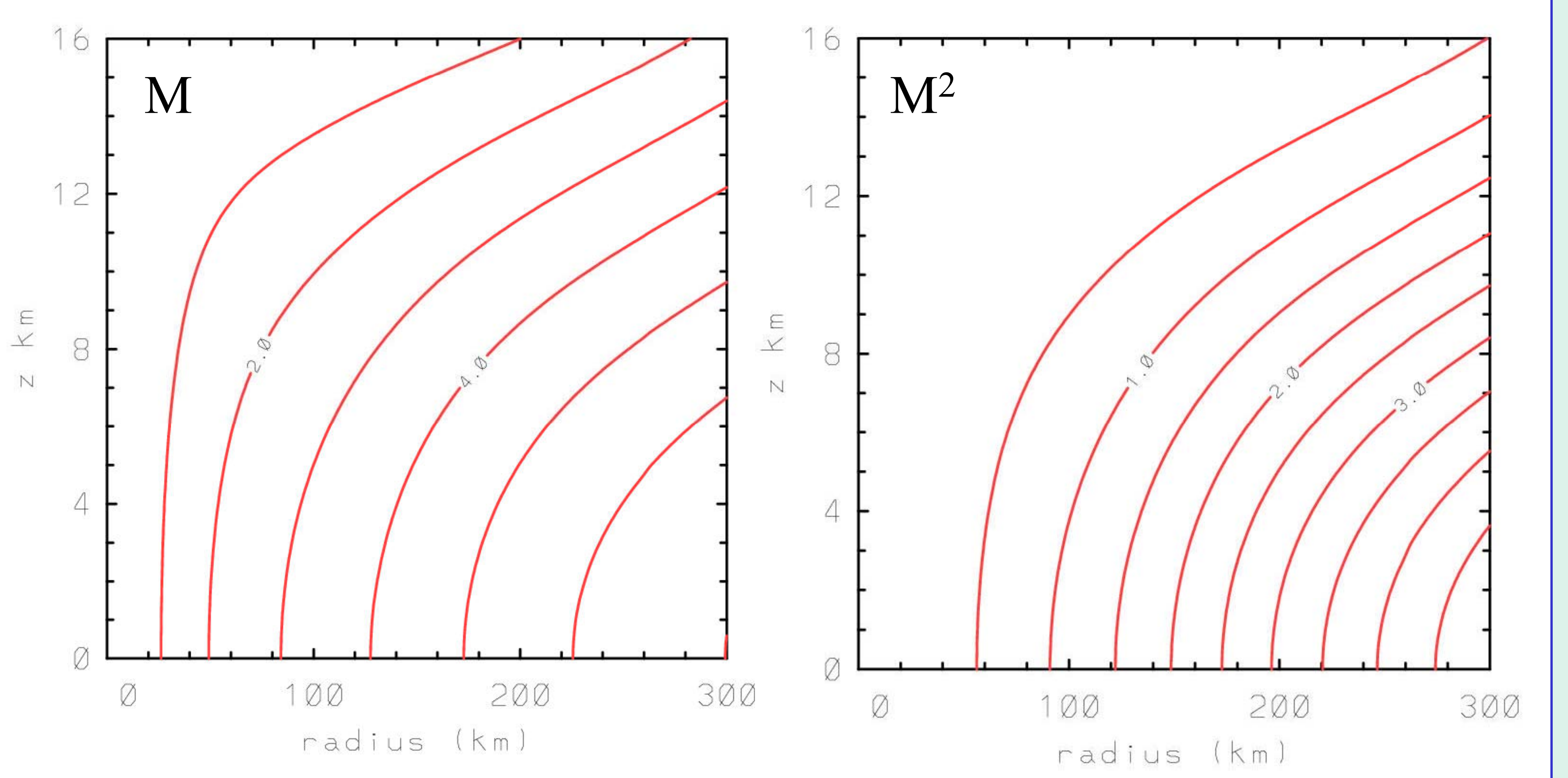
Net radial force on a displaced air parcel

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2}r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2}r_2^2 f)^2\right]. \tag{3.19}$$

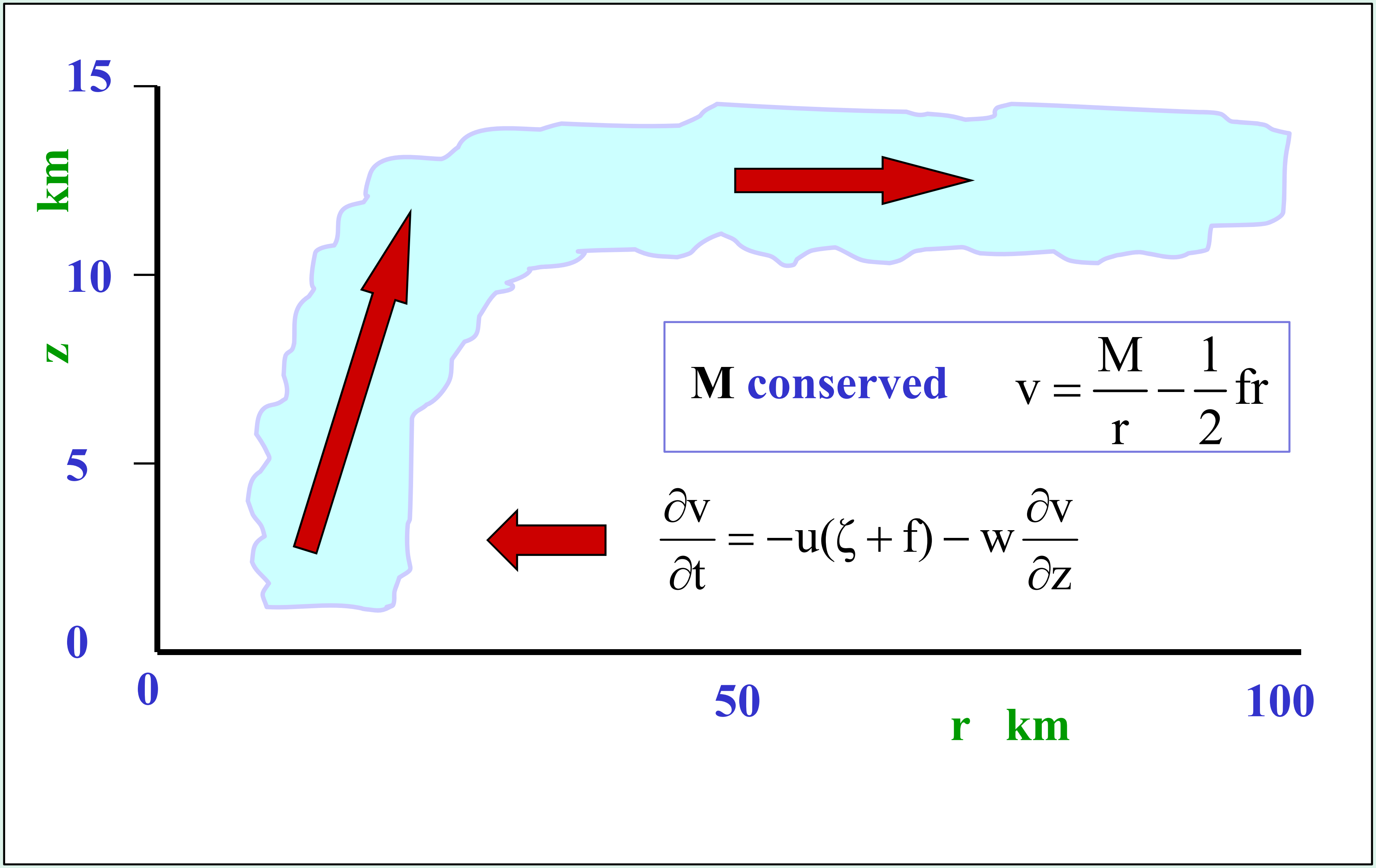
In the special case of solid body rotation, $v = \Omega r$, and for a small displacement from radius $r_1 = r$ to $r_2 = r + r'$, (3.19) gives

$$F \approx -4(\Omega + \frac{1}{2}f)^2 r' \tag{3.20}$$

AAM in a typical vortex

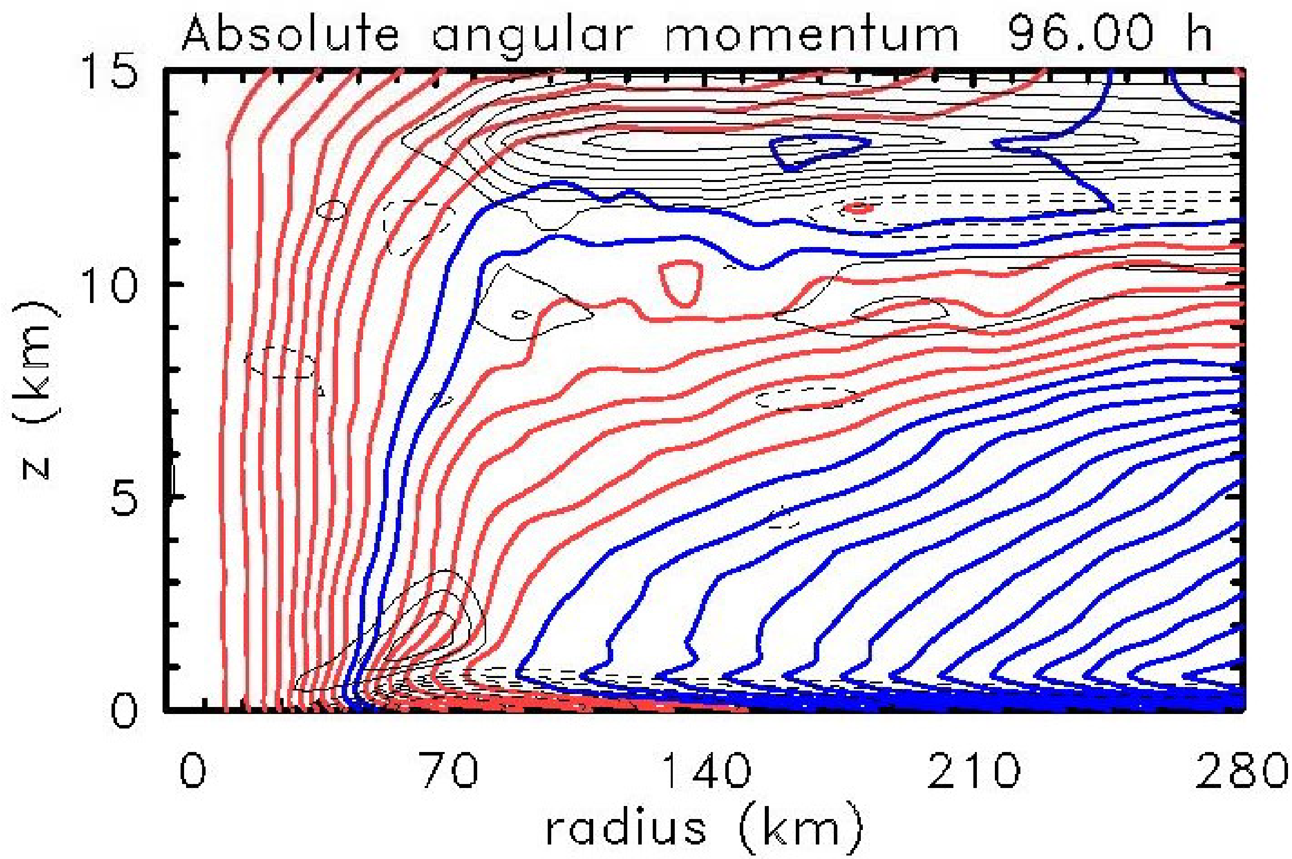


Thermally-forced secondary circulation leads to spin up

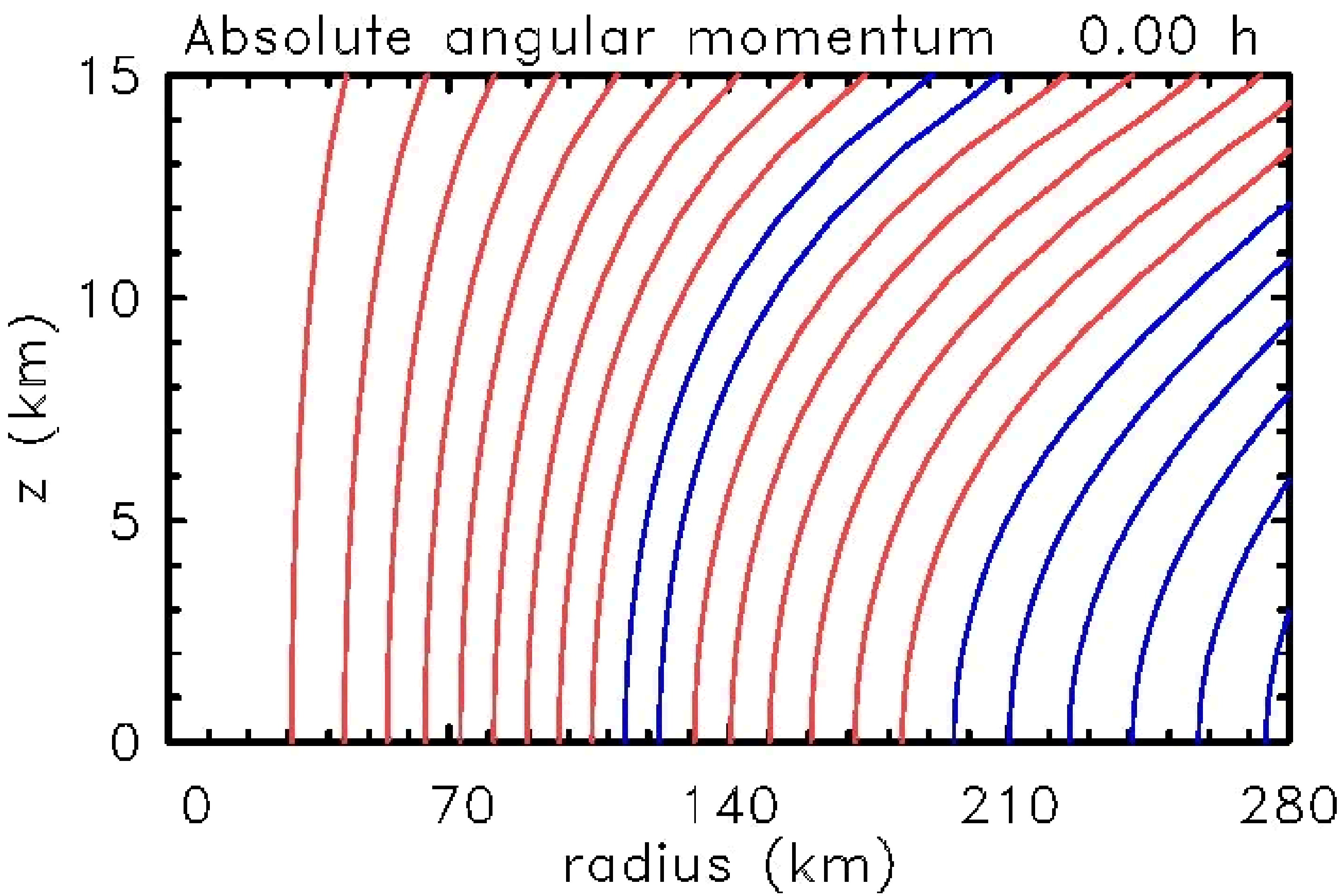


Movie

Time-height sequence of Absolute Angular Momentum



$$M = rv + \frac{1}{2} f r^2 \quad \Rightarrow \quad v = \frac{M}{r} - \frac{1}{2} f r$$


Absolute angular momentum 0.00 h
15
10
5
0
z (km)
0
70
140
210
280
radius (km)