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Is asymptotic non-divergence of the large-scale tropical atmosphere consistent with equatorial wave theories?

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ABSTRACT

Observations suggest that the large-scale tropical atmospheric circulations, associated with intraseasonal variabilties, are dominated more by the vorticity than the divergence. The present paper examines the consistency of the above observations with linear equatorial wave theories. Both free and forced linear waves are considered. The free equatorial waves are classified into two major categories: (1) the Rossby waves, strongly dominated by vorticity and (2) the inertial-gravity waves, relatively dominated by the divergence. Both the Kelvin and the mixed-Rossby gravity waves are intermediate of these two major categories.

In the forced case, the wave response is predominantly inertial-gravity wave-like for periods less than 5 d, thus predominantly divergent. On the other hand, for forcing with the longer periods, the wave response closely following free Rossby-wave structures, asymptotically approaches to a non-divergent state. The asymptotic tendency for non-divergence is found to be much stronger than observed. The difference is so stark that, notably, the tropical intraseasonal variability cannot be consistent with linear equatorial waves theories.

1. Introduction

In a linear dry limit, the primitive equation system over the tropics is characterized by a set of solutions called equatorial waves (Matsuno, 1966). For this reason, popular approaches take the large-scale tropical atmospheric dynamics as dictated by these equatorial waves under interactions with moist convection (cf. Wheeler and Kiladis, 1999).

Arguably, the wave-based description of the large-scale tropical atmosphere deals with an inherently divergent flow in contrast to the mid-latitude large-scale flows. The latter is typically treated as a non-divergent flow to a leading-order approximation by introducing a geostrophy on a *f*-plane. A weak divergence is introduced only to a higher-order of an asymptotic expansion where a weak unsteadiness and a deviation from a *f*-plane approximation (i.e. β-effect) are taken into account. Thus, the mid-latitude large-scale flows are asymptotically non-divergent. In contrast, in equatorial waves theories no such asymptotic expansion is introduced, thus in this respect, the flows are inherently divergent even to a leading order approximation.

Against this conventional point of view, it was Charney (1963) who pointed out by a heuristic scale analysis that the large-scale tropical atmosphere could be considered to be non-divergent to a leading-order approximation. (see also Yano and Bonazzola, 2009). More recently, Yano et al. (2009) showed by an analysis of the Tropical Ocean Global Atmosphere–Coupled Ocean Atmosphere Response Experiment (TOGA-COARE) Large-Scale Array (LAS) data that the large-scale tropical tropospheric flows may be considered to be 'asymptotically' non-divergent, though at the tropopause level, the divergence is equally important as vorticity.

They quantified the asymptotic non-divergence by the ratio of the root-mean square (RMS) between the divergence and the vorticity

$$r \equiv \langle \delta^2 \rangle^{1/2} / \langle \zeta^2 \rangle^{1/2}. \tag{1}$$

Here, the angle brackets $\langle \ \rangle$ designate the space-time average. According to them, the RMS ratio ranges 0.3–0.4 at the lower troposphere 550–850 hPa, substantially a small value that may justify a development of an asymptotic-expansion theory (cf. Bender and Orszag, 1978) based on a non-divergent approximation to the leading order. They further show that the RMS ratio further reduces to 0.2 for the intraseasonal timescales.

In turn, this paper poses rather a basic question, but not answered in the literature in authors' best knowledge: to how much extent Yano et al.'s (2009) observational analysis is consistent with the standard equatorial wave theories? In order to answer this question, the RMS ratio between the divergence and the vorticity is simply calculated from the linear equatorial-waves

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solutions. Both free and forced waves are considered. The analysis result is presented in the next two sections for these two cases. The results are discussed in Section 4.

2. Analysis: free-wave case

The analysis is performed for a linear shallow-water system on an equatorial β-plane (cf. section 11.4, Holton, 2004) with the parameter β designating a linear tendency of the Coriolis parameter around the equator. The mean depth h_E of the shallow-water system is re-interpreted as a parameter called the 'equivalent depth' in order to translate the solution for a fully three-dimensional primitive equation system. In the latter context, the equivalent depth h_E characterizes a vertical structure of a wave mode (e.g. Kasahara and Puri, 1981; Fulton and Schubert, 1985).

The equivalent depth h_E is further related to the gravity-wave speed c_g by $c_g = (gh_E)^{1/2}$ with g the accelerations of the gravity. In the present analysis, by taking c_g as a free parameter of the problem, we namely consider the two cases: (1) $c_g = 50$ m s^{-1} corresponding to the free waves (e.g. Milliff and Madden, 1996) and (2) $c_g = 12$ m s^{-1} corresponding to the so-called convectively coupled equatorial waves (cf. Wheeler and Kiladis, 1999).

The analytical solutions for equatorial waves are provided in various textbooks (e.g. section 11.4, Holton, 2004), thus they are not repeated here. From these solutions, we have calculated the RMS ratio between the divergence δ and the vorticity ζ averaged over the whole β-plane, as defined by eq. (1). The system is assumed to be periodic in longitudinal direction with the period equal to the length of Earth's equatorial circle, and extending to the infinity in the latitudinal direction. The analytical reduction is fairly standard, and only a brief summary is provided in the Appendix.

As usual, the Kelvin wave requires a special treatment, which also provides the most notable result. In that case, the RMS ratio is given by a linear function of the longitudinal wavenumber k

$$r = k\frac{\lambda_\beta}{\sqrt{2}}, \tag{2}$$

where the equatorial radius of deformation $\lambda_\beta = (c_g/\beta)^{1/2}$ is 1480 and 730 km for $c_g = 50$ and 12 m s^{-1}, respectively.

The result (2) shows that the Kelvin wave is only weakly divergent (i.e. $r \ll 1$) for the scales $\lambda \gg \lambda_\beta \sim 10^3$ km with $\lambda = 2\pi/k$ the wavelength. Notably, the Kelvin waves are only weakly divergent for the planetary scale $\lambda \sim 10^4$ km. The result may appear rather a surprise, because the Kelvin wave is often considered as if like gravity waves, satisfying exactly the same momentum equation in the longitudinal direction. At the same time, the Kelvin wave satisfies an exact geostrophy in the latitudinal direction. The geostrophy leads to a vorticity (zonal-wind shear) of the scale λ_β, whereas the gravity-wave induces a divergence proportional to the zonal wavenumber k. Thus, it leads to a result $r \sim k\,\lambda_\beta$ being consistent with eq. (2).

It may also be worthwhile to recall that the Kelvin wave has no potential vorticity $q = \zeta - \beta y\eta/h_E$, though it has a relatively strong vorticity ζ associated with a perturbation η of surface height. The condition of the vanishing potential vorticity, that is, $q = 0$, leads to $\zeta/\eta = \beta y/h_E$. Furthermore, by noting that the surface height perturbation η is related to the divergence δ by $ikc_g\,\eta = h_E\delta$ from mass conservation, we obtain a local ratio $\delta/\zeta = ik\lambda_\beta^2/y$ with y the distance from equator. At a distance $y \sim \lambda_\beta$ away from the equator, where a major contribution in averaging is expected, we obtain $\delta/\zeta \sim k\lambda_\beta$, again, consistent with eq. (2).

Note that the RMS ratio is proportional to $\lambda_\beta \sim c_g^{1/2} \sim h_E^{1/4}$, thus the non-divergent tendency of the Kelvin wave is more pronounced for slower gravity-wave speeds, and for shallower systems. It follows that the convectively coupled Kelvin waves, representing smaller effective gravity-wave speed $c_g \sim h_E^{1/2}$ than free waves, are less divergent, probably against the intuition. Interestingly, the tendency for non-divergence towards the larger scales is consistent with the observational analysis of Yano et al. (2009, see their fig. 7).

Full results of the analysis are depicted in Fig. 1 by plotting the RMS ratio, r (eq. 1), as a function of the zonal-wavenumber index m, for the cases with $c_g = 50$ and 12 m s^{-1}, respectively, in (a) and (b). The RMS ratio is calculated by eqs (A1a) and (A1b) in the Appendix by substituting the frequency ν for a corresponding equatorial wave and for a given wavenumber k, or the zonal-wavenumber index m. Recall that the zonal-wavenumber index m is related to the zonal wavenumber k by $m = ka$ with a the radius of the Earth. By convention, we assume that a positive and a negative m correspond to eastward and westward propagations, respectively.

The result can be separated into the two major categories of waves: the inertial-gravity waves, those tend to be more divergent, and the Rossby waves, those tend to be non-divergent. The RMS ratio for the inertial-gravity waves always exceeds unity, reflecting highly divergent nature of these waves. However, it is important to note that the ratio remains the order of unity: a contribution of vorticity is always comparable to the divergence,

On the other hand, the equatorial Rossby waves are predominantly rotational: the RMS ratio remains less than 0.1 almost for all the wavenumbers, and it decreases for higher wavenumbers. The result is rather surprising considering a smallness of the Coriolis parameter over the equator.

Two particular waves cross over these two major categories. The mixed-Rossby gravity wave behaves like an inertial-gravity wave to a limit of high positive wavenumbers (eastward propagation), and like a Rossby wave to a limit of high negative wavenumbers (westward propagation). It gradually transits from one limit to another over a small wavenumber zone. Another particular case already discussed is the Kelvin wave, whose RMS ratio simply linearly increases with the wavenumber. Thus, the value is comparable to those of Rossby waves in a small

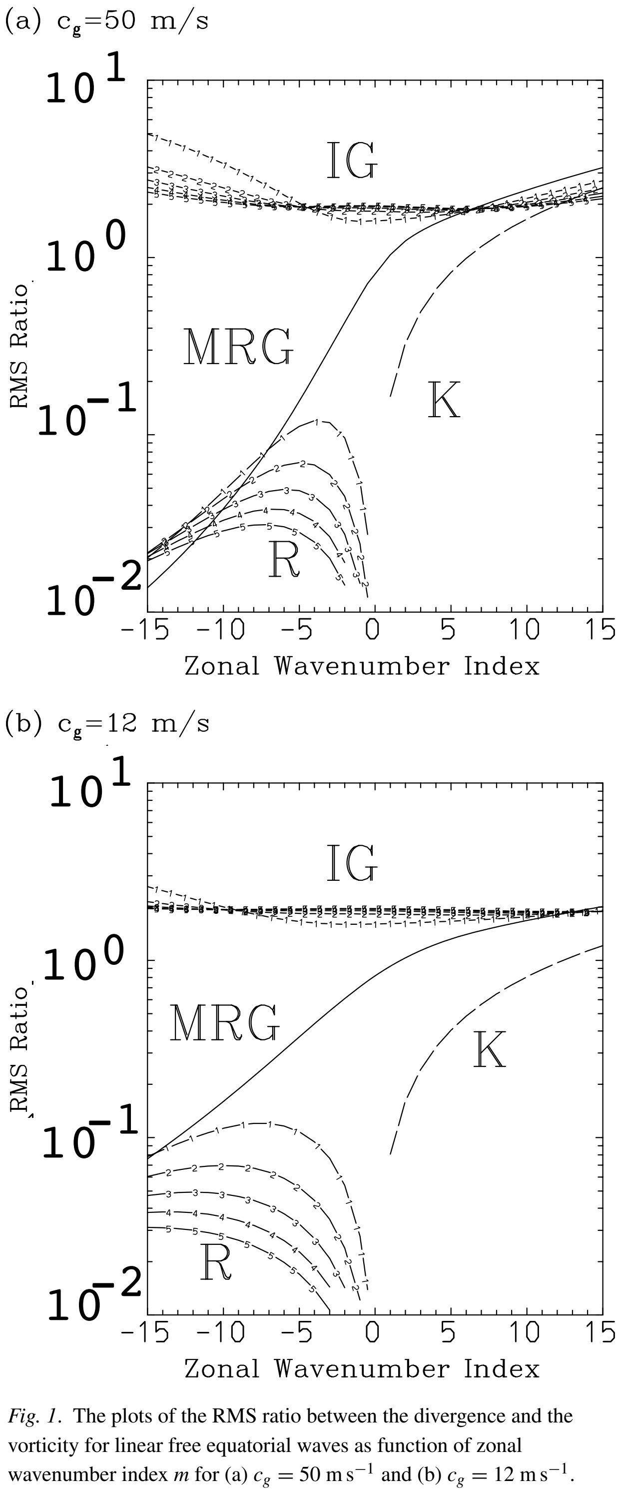

Fig. 1. The plots of the RMS ratio between the divergence and the vorticity for linear free equatorial waves as function of zonal wavenumber index m for (a) $c_g = 50\ \mathrm{m\,s^{-1}}$ and (b) $c_g = 12\ \mathrm{m\,s^{-1}}$. Types of the waves are marked by K (Kelvin wave), R (Rossby waves), MRG (mixed Rossby-gravity waves), and IG (inertial-gravity waves) with the corresponding latitudinal-mode index n marked on curves. The positive and the negative wavenumbers correspond to the eastward and the westward propagating waves, respectively.

wavenumber limit, and it is comparable to those of inertial-gravity waves in high wavenumber limit.

Last to remark may be a relatively weak dependence of the results on the gravity-wave speed c_g. Contrast between the inertial-gravity and Rossby waves for the high wavenumber limit is somehow more pronounced for faster gravity waves, but no qualitative change is seen for the RMS-ratio curves.

3. Analysis: forced-wave case

It is often interpreted that various large-scale tropical features are generated by interactions between the large-scale circulations and moist convection. Extensive efforts have been made to interpret these features in terms of linear waves 'forced' by convection. There are two major approaches to consider this problem. The first approach solely focuses on a dynamical response of waves against a prescribed convective forcing (e.g. Webster, 1972; Gill, 1980; Lau and Lim, 1982), whereas the second attempts a closed formulation in which the convective forcing is determined in a self consistent manner within a model (e.g. Hayashi, 1970; Lindzen, 1974; Emanuel, 1987; Neelin et al., 1987; Yano and Emanuel, 1991). However, for the present purpose of defining the ratio between the divergence and the vorticity, the second approach reduces to the first for the reason explained immediately below.

The forced problem can be considered by adding a forcing term S_n to an equation for the meridional wind

$$\left[\frac{\partial^2}{\partial y^2} - \left(\frac{\beta y}{c_g}\right)^2 + \left(\frac{\nu}{c_g}\right)^2 - k^2 - \frac{k\beta}{\nu}\right] v_n = S_n \tag{3}$$

in the linear free-wave problem considered in the last section, where we assume $S_n = \tilde{S}_n H_n(y/\lambda_\beta)\mathrm{e}^{-(y/\lambda_\beta)^2/2}\mathrm{e}^{i(kx-\nu t)}$ for the forcing with $\tilde{S}_n$ a constant. A general problem may be considered by taking a sum of n for forcing. Note that the forcing S_n includes all types of physical forcing on the system, not only the convective heating. Furthermore, this 'integrated' forcing S_n for the nth mode does not necessarily correspond to a forcing of the nth mode in the original shallow-water equation, and *vice verse*. For example, diabatic heating (a source term is in mass conservation equation) with the nth mode is defined by a linear combination of S_{n-1}, S_n and S_{n+1}.

The meridional wind solution is given by $v_n = H_n(y/\lambda_\beta)\mathrm{e}^{-(y/\lambda_\beta)^2/2}\mathrm{e}^{i(kx-\nu t)}$ under the forced problem as for the free-wave problem. However, a major difference now is that both the wavenumber k and the frequency ν are prescribed externally by a given forcing, rather than being prescribed by a wave dispersion relation. Thus, the same formula (eqs A1a and A1b) is applicable as before for computing the RMS ratio, but with the frequency ν and the wavenumber k externally varied.

The principle for calculating the RMS ratio does not change, either, when a theory is considered in which the convective forcing is self-consistently defined. For the present purpose, such a self-consistent solution is reproduced simply by substituting a wavenumber and a frequency obtained by a given theory into eq. (3), then into eqs (A1a) and (A1b). In other words, regardless of either approach is taken, the ratio between the divergence and the vorticity of these solutions can be evaluated simply by solving a forced wave problem with a fixed forcing period, as long as the imposed forcing period corresponds to the one obtained by a given theory.

Many of a latter type of theories consider a linear growth rate of the perturbations. Here, for re-interpreting their results in terms of a linear forced waved problem, we simply suppose that these linear growing modes would arrive at an equilibrated state under a weak non-linearity, thus only the obtained preferred periods of the modes do matter in the end. It may also be emphasized that regardless of the actual horizontal structure of the solutions obtained by a linear stability analysis, a solution can be decomposed by Hermite polynomials due to their completeness, thus a generality of the analysis here is guaranteed also in this respect.

The RMS ratio, r (eq. 1), as functions of the externally-imposed period and scale (horizontal wavelength) evaluated in this manner is presented in Fig. 2 for the first three latitudinal modes: (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$. Here, we take $c_g = 50\ \mathrm{m\,s^{-1}}$. Though the case with $c_g = 12\ \mathrm{m\,s^{-1}}$ is not shown, what is going to be said for the case with $c_g = 50\ \mathrm{m\,s^{-1}}$ also equally applies for the case with $c_g = 12\ \mathrm{m\,s^{-1}}$. The period and the wavelength are used as coordinates in place of the frequency ν and the wavenumber k in order to enable a direct comparison with fig. 7 of Yano et al. (2009), which presents the observational RMS ratio as functions of time and space scales. Here, the positive and negative periods correspond to the eastward and the westward propagations, respectively. For a graphical purpose, both the period and the horizontal scale are changed in a reasonably continuous manner.

For all the latitudinal modes considered, a strongly divergence response is obtained only for the shortest periods (order to magnitude of period less than a day) for all the horizontal scales. The RMS ratio rapidly decreases with increasing periods, and

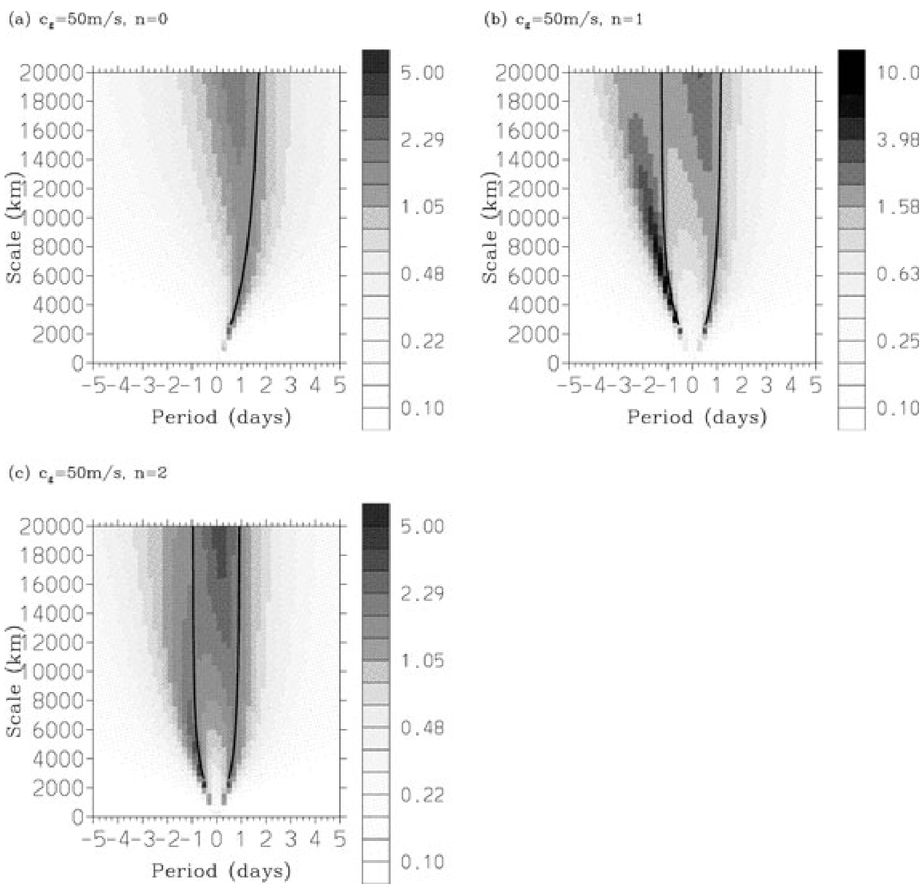

Fig. 2. The RMS ratio between the divergence and the vorticity for forced linear equatorial waves are shown by grey tones as a function of periods (horizontal axis) and the scale (wavelength: vertical axis) for the latitudinal-mode index $n = 0$ (a), $n = 1$ (b) and $n = 2$ (c) for the cases with $c_g = 50\ \mathrm{m\,s^{-1}}$. Superposed are the dispersion curves for the eastward-propagating mixed-Rossby gravity waves in (a), and those for the inertial-gravity waves in (b) and (c). The positive and the negative periods correspond to the eastward and the westward propagations, respectively.

beyond the 5-d period (not shown), the wave response becomes predominantly rotational with the divergent contribution much less than the former (well below 0.1).

The obtained result appears to be a rather simple extrapolation of the results with the free waves: the RMS ratio is larger when the forced response is close to that of the inertial-gravity waves and eastward-propagating mixed Rossby-gravity waves, and substantially smaller when the forced response is closer to that of the Rossby waves. Thus, the largest RMS ratio is expected to a vicinity of the dispersion curve for the former type of waves.

In support of this interpretation, dispersion curves for the eastward-propagating mixed-Rossby gravity waves (for $n = 0$) and the inertial-gravity waves (for $n = 1, 2$) are superposed on the plots of the RMS ratio. Zone of strong divergence response closely follow the dispersion curves for the eastward-propagating mixed-Rossby and the inertial-gravity waves. Consequently, high RMS values can loosely be understood as a consequence of 'resonance' with free inertial-gravity waves and eastward-propagating mixed Rossby-gravity waves. However, note that this 'resonance' interpretation is valid only qualitatively, and the actual maximum of the RMS ratio does not exactly coincide with the dispersion curves for the divergent waves.

The strongly non-divergent response for longer periods reflects the nature of the equatorial Rossby waves as already well depicted by Fig. 1. The degree of non-divergence is much stronger than the one obtained by an observational analysis of Yano et al. (2009). They found that the RMS ratio decreases to the order of 0.2 for the intraseasonal timescale, compared to a value 0.3 obtained for shorter and smaller scales. On the other hand, the RMS ratio decreases to a value much smaller in a limit of long periods under the linear forced-wave problem considered here.

The discrepancy is so substantial that it seems reasonable to conclude the observed so-called convectively coupled equatorial waves cannot be interpreted as linearly 'forced' waves for the periods longer than 5 d. A major missing element in the present analysis is the advective non-linearity. Advective non-linearity is likely critically contributing in determining the observed horizontal structures. In other words, these so-called convectively coupled equatorial waves are strongly non-linear.

4. Discussions

The present analysis for the free linear waves shows that the observed asymptotic non-divergence of the large-scale tropical atmosphere is 'qualitatively' consistent with the calculated RMS ratio both for the Kelvin and the Rossby waves. On the other hand, the inertial-gravity and the mixed-Rossby gravity waves are too divergent to be consistent with the observed large-scale flows. The conclusions are found to be insensitive regardless of whether a typical free gravity-wave speed (i.e. $c_g = 50\,\mathrm{m\,s^{-1}}$, e.g. Milliff and Madden, 1996) or an effective gravity-wave speed estimated for the so-called convectively coupled equatorial waves (i.e. $c_g = 12\,\mathrm{m\,s^{-1}}$, Wheeler and Kiladis, 1999) is chosen.

Hence, the result implies that the large-scale tropical atmospheric dynamics must be dominated by Kelvin and Rossby waves, if they are to be 'effectively' interpreted in terms of the free equatorial waves. It may be worthwhile to note that a longwave approximation introduced by Lighthill (1969) and adopted, for example, by Gill (1980), Lau and Lim (1982) and Stevens et al. (1990) provides an optimal method for filtering out less important mixed Rossby-gravity and the inertial-gravity waves. The present analysis, in turn, shows that both Kelvin and the Rossby waves can be considered to be asymptotically non-divergent. An asymptotically non-divergent formulation for the Kelvin and the Rossby waves is still to be developed. It may turn out that a traditional longwave approximation could simply be re-interpreted qualitatively as an asymptotically non-divergent approximation.

Alternatively, an asymptotically non-divergent approximation may be introduced into a global system defined on a sphere. A generalization of the quasi-geostrophic system equally applicable to the low latitudes is obtained as a result (Verkley, 2001, 2009). However, a problem with this approach is that only the Rossby waves are retained and the Kelvin waves are filtered out.

The results for the forced waves can be understood in a relatively straightforward manner from those for the free waves. In a short timescale limit (for periods less than a day), the wave response is dominated by that of inertial-gravity waves, thus highly divergent. On the other hand, for long timescales (say, periods longer than 5 d), the wave response is predominantly of Rossby waves, thus only weakly divergent. The degree of non-divergence obtained from the linear forced waves of intraseasonal timescales is much stronger than the one found observationally (Yano et al., 2009). Note that the same conclusion also follows from the linear free-wave analysis. The obtained RMS for the linear Rossby waves is much smaller than those observed. Hence, we should conclude that non-linearity is critical in order to explain the observed degree of ratio between the vorticity and the divergence.

Note that the results of any linear convectively coupled equatorial wave theory is reproduced under the present framework for the forced-wave response analysis, once the preferred period and scale obtained by the theory is substituted. Consequently, our analysis suggests that the large-scale tropical circulation is strongly non-linear in the sense that linear wave theories do not provide even a leading-order approximation as assumed under weakly non-linear theories. Rather the non-linearity makes both the wavenumber-frequency relation (dispersion) and the horizontal structure of the waves qualitatively different from those predicted from linear theories. A major non-linear effect playing a role here is likely to be self-advection of waves, because that is the major effect neglected under the linear analysis.

Though the importance of non-linearity in the tropical large-scale atmospheric dynamics has been investigated by for example, Gill and Phlips (1986), Van Tuyl (1986, 1987), only the weakly non-linear regimes have been considered so far by limiting their attention to a circulation generated as a *direct* response to a given diabatic heating. On the other hand, the present study suggests that unforced background flows, which are not necessarily zonal, substantially contribute to the whole circulations of the large-scale tropical atmosphere in defining the ratio between the vorticity and the divergence. It may turn out that the modon theory that describes strongly non-linear free Rossby waves driven by self advection (Flierl et al., 1980; Flierl, 1987; Butchart et al., 1989) also provides a good analogue for those so-called convectively coupled equatorial waves.

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6. Appendix

By following (section 11.4 Holton, 2004), we set a solution for the meridional wind for the *n*th latitudinal mode as

$$v_n = H_n(\xi) \mathrm{e}^{-\xi^2/2} \mathrm{e}^{i(kx-\nu t)},$$

where $H_n(\xi)$ is the *n*th Hermite polynomial, k the wavenumber, ν the frequency. The latitudinal dependence is presented in terms of a non-dimensionalized coordinate $\xi = y/\lambda_\beta$ with y the *dimensional* distance from the equator, $\lambda_\beta = (c_g/\beta)^{1/2}$ the equatorial radius of deformation defined by the equatorial beta parameter β and the gravity-wave speed c_g. Subscript n is added to the meridional wind v in order to indicate that this is a particular solution with the nth Hermite mode. Subscript n is also added to all the subsequent formulas obtained with $v = v_n$. In the following derivation, the recursive relations

$$\frac{\partial}{\partial \xi} v_n = n v_{n-1} - \frac{1}{2} v_{n+1}$$

$$\xi v_n = n v_{n-1} + \frac{1}{2} v_{n+1}$$

are used.

Substitution of the solution v_n into the linear shallow-water equation provides an expression for the zonal wind

$$u_n = i\beta\lambda_\beta \left[\frac{n v_{n-1}}{kc_g + \nu} - \frac{v_{n+1}}{2(kc_g - \nu)} \right].$$

By further substituting these expressions for u_n and v_n into the definitions of the divergence and vorticity, $\delta_n = iku_n + \partial v_n/\partial y$ and $\zeta_n = ikv_n - \partial u_n/\partial y$, respectively, we obtain

$$\delta_n = \left(\frac{1}{\lambda_\beta}\right) \left[\frac{n}{kc_g/\nu + 1} v_{n-1} + \frac{1}{2(kc_g/\nu - 1)} v_{n+1} \right]$$

$$\zeta_n = \frac{i\beta}{\nu} \left\{ -\frac{n(n-1)}{kc_g/\nu + 1} v_{n-2} + \left[\frac{k\nu}{\beta} + \frac{n}{2(kc_g/\nu + 1)} + \frac{n+1}{2(kc_g/\nu - 1)} \right] v_n - \frac{1}{4(kc_g/\nu - 1)} v_{n+2} \right\}.$$

Multiplying by complex conjugates $\delta^\dagger$ and $\zeta^\dagger$, and integrating them over the latitudes, we obtain

$$\langle \delta\delta^\dagger \rangle_n = \left(\frac{1}{\lambda_\beta}\right) \left[\frac{n^2}{(kc_g/\nu + 1)^2} h_{n-1} + \frac{1}{4(kc_g/\nu - 1)^2} h_{n+1} \right] \quad \text{(A1a)}$$

$$\langle \zeta\zeta^\dagger \rangle_n = \lambda_\beta \left(\frac{\beta}{\nu}\right)^2 \left\{ \left[\frac{n(n-1)}{kc_g/\nu + 1} \right]^2 h_{n-2} + \left[\frac{k\nu}{\beta} + \frac{n}{2(kc_g/\nu + 1)} + \frac{n+1}{2(kc_g/\nu - 1)} \right]^2 h_n + \frac{1}{16(kc_g/\nu - 1)^2} h_{n+2} \right\}, \quad \text{(A1b)}$$

where $\langle \quad \rangle = \int_{-\infty}^{+\infty} (\quad) \mathrm{d}y$ designates an integral over latitude, $h_n = \sqrt{\pi} 2^n n!$ is a normalization factor for the nth Hermite polynomial.

In calculating the RMS for the divergence and the vorticity from eqs (A1a) and (A1b) for the free waves, a frequency $\nu(k, n)$ for a given type of equatorial waves with a given wavenumber k and latitudinal mode n is substituted into ν in the formula. Formulas for the wave dispersions $\nu(k, n)$ are given in section 11.4 of Holton (2004) for example.

References

Bender, C. M. and Orszag, S. A. 1978. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, New York, 593 pp.

Butchart, N., Haines, K. and Marshall, J. C. 1989. A theoretical and diagnostic study of solitary waves and atmospheric blocking. *J. Atmos. Sci.* **46**, 2063–2078.

Charney, J. G. 1963. A note on large-scale motions in the tropics. *J. Atmos. Sci.* **20**, 607–609.

Emanuel, K. A. 1987. An air-sea interaction model of intraseasonal oscillations in the tropics. *J. Atmos. Sci.* **44**, 2324–2340.

Flierl, G. R. 1987. Isolated eddy models in geophysics. *Ann. Rev. Fluid Mech.* **19**, 493–530.

Flierl, G. R., Larichev, V. D., McWilliams, J. C. and Reznik, G. M. 1980. The dynamics of baroclinic and barotropic solitary eddies. *Dyn. Atmos. Ocean* **5**, 1–41.

Fulton, S. R. and Schubert, W. H. 1985. Vertical normal mode transforms: theory and application. *Mon. Wea. Rev.* **113**, 647–658.

Gill, A. E. 1980. Some simple solutions for heat-induced tropical circulation. *Quart. J. Roy. Meteor. Soc.* **106**, 447–462.

Gill, A. E. and Phlips, P. J. 1986. Nonlinear effects on heat-induced circulation of the tropical atmosphere. *Quart. J. Roy. Meteor. Soc.* **112**, 69–91.

Hayashi, Y. 1970. A theory of large-scale equatorial waves generated by condensation heat and accelerating the zonal wind. *J. Met. Soc. Japan* **48**, 140–160.

Holton, J. R. 2004. *An Introduction to Dynamic Meteorology*, 4th Ed. (International Geophysical Series, 88) Academic Press, San Diego.

Kasahara, A. and Puri, K. 1981. Spectral representation of three-dimensional global data by expansion in normal mode functions. *Mon. Wea. Rev.* **109**, 37–51.

Lau, K.-M. and Lim, H. 1982. Thermally driven motions in an equatorial β-plane: Hadley and Walker circulations during the winter monsoon. *Mon. Wea. Rev.* **110**, 336–353.

Lighthill, M. J. 1969. Dynamic response of the Indian Ocean to onset of the southwest monsoon. *Phil. Trans. Roy. Soc. London* **265**, 45–92.

Lindzen, R. 1974. Wave-CISK in the tropics. *J. Atmos. Sci.* **31**, 156–179.

Matsuno, T. 1966. Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Jpn.* **44**, 25–43.

Milliff, R. F. and Madden, R. A. 1996. The existence and vertical structure of fast, eastward-moving disturbances in the equatorial troposphere. *J. Atmos. Sci.* **53**, 596–597.

Neelin, J. D., Held, I. M. and Cook, K. H. 1987. Evaporation-wind feedback and low-frequency variability in the tropical atmosphere. *J. Atmos. Sci.* **44**, 2341–2348.

Stevens, D. E., Kuo, H.-C., Schubert, W. H. and Ciesielski, P. E. 1990. Quasi-balanced dynamics in the tropics. *J. Atmos. Sci.* **47**, 2262–2273.

Van Tuyl, A. H. 1986. Advective influences on forced tropical motions. *J. Atmos. Sci.* **43**, 141–161.

Van Tuyl, A. H. 1987. Nonlinearities in low-frequency equatorial waves. *J. Atmos. Sci.* **44**, 2478–2492.

Verkley, W. T. M. 2001. Salmon's Hamiltonian approach to balanced flow applied to a one-layer isentropic model of the atmosphere. *Quart. J. Roy. Meteor. Soc.* **126**, 263–274.

Verkley, W. T. M. 2009. A balanced approximation of the one-layer shallow-water equations on a sphere. *J. Atmos. Sci.* **66**, 1735–1748.

Webster, P. J. 1972. Response of the tropical atmosphere to local steady forcing. *Mon. Wea. Rev.* **100**, 518–541.

Wheeler, M. and Kiladis, G. N. 1999. Convectively coupled equatorial waves: analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.* **56**, 374–399.

Yano, J. I. and Emanuel, K. A. 1991. An improved model of the equatorial troposphere and its coupling with the stratosphere. *J. Atmos. Sci.* **48**, 377–389.

Yano, J.-I. and Bonazzola, M. 2009. Scale analysis for large-scale tropical atmospheric dynamics. *J. Atmos. Sci.* **66**, 159–172.

Yano, J.-I., Mulet, S. and Bonazzola, M. 2009. Tropical large-scale circulations: asymptotically non-divergent? *Tellus* **61A**, 417–427.