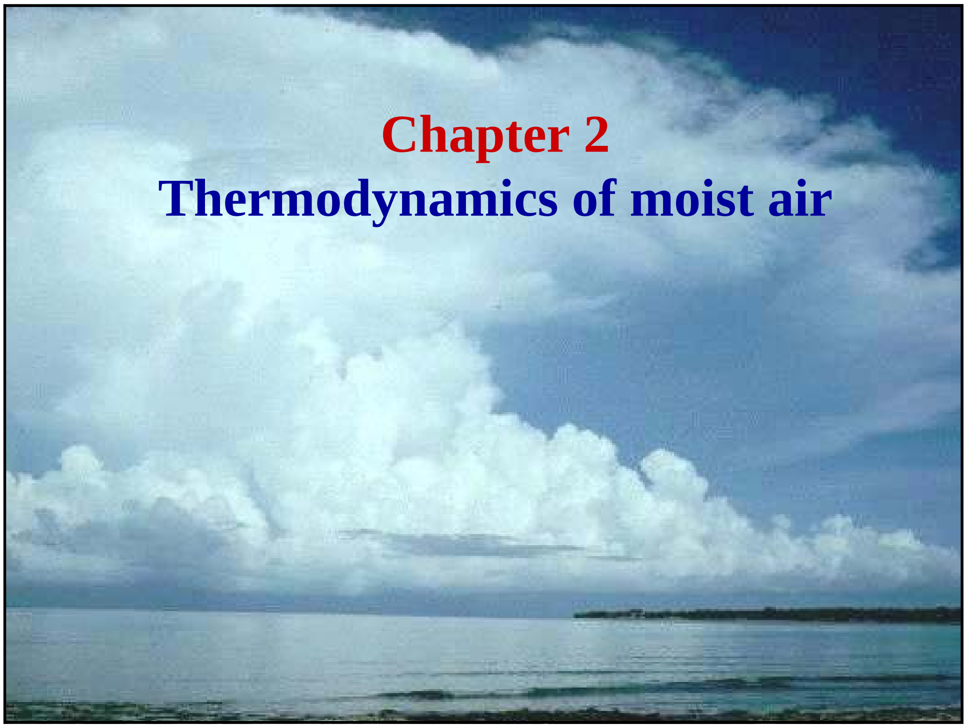
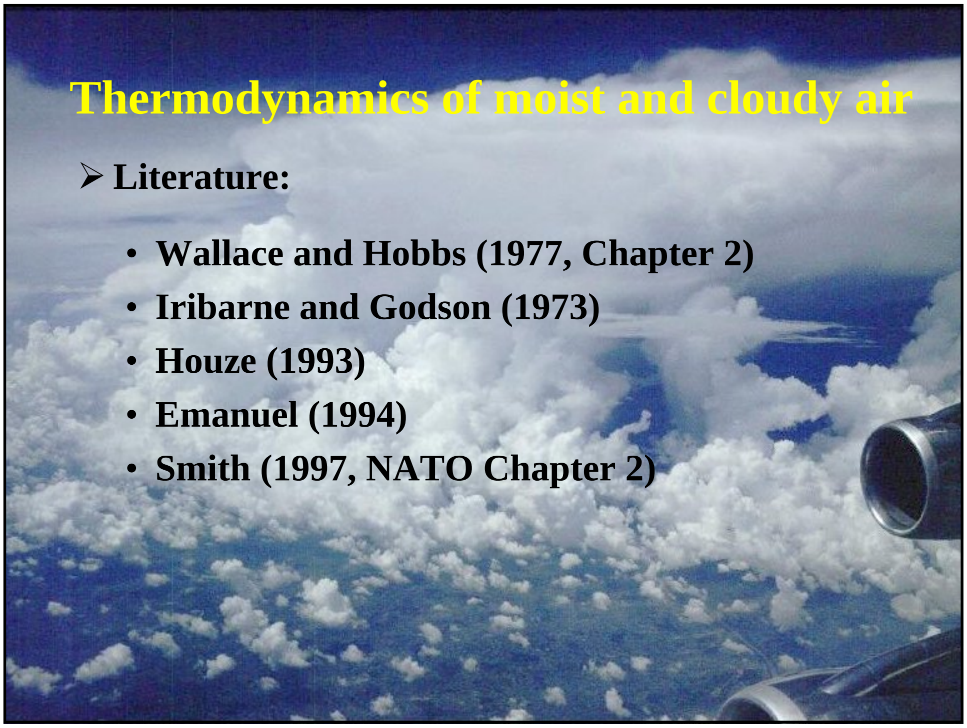


Chapter 2
Thermodynamics of moist air



Thermodynamics of moist and cloudy air

➢ **Literature:**

- **Wallace and Hobbs (1977, Chapter 2)**
- **Iribarne and Godson (1973)**
- **Houze (1993)**
- **Emanuel (1994)**
- **Smith (1997, NATO Chapter 2)**

The equation of state for moist unsaturated air

- **The state of a sample of moist air is characterized by its:**
  - **Pressure, p**
  - **Absolute temperature, T**
  - **Density, ρ (or specific volume α = 1/ρ), and**
  - **Some measure of its moisture content, e.g.**
  - **The water vapour mixing ratio, r, defined as the mass of water vapour in the sample per unit mass of dry air.**
- **These quantities are connected by the equation of state.**

Equation of state:

specific gas constant for dry air

$$p\alpha = R_d T(1 + r/\varepsilon)/(1+\varepsilon) \approx R_d T(1 + 0.61r)$$

$\varepsilon = R_d/R_v = 0.662$

specific gas constant water vapour

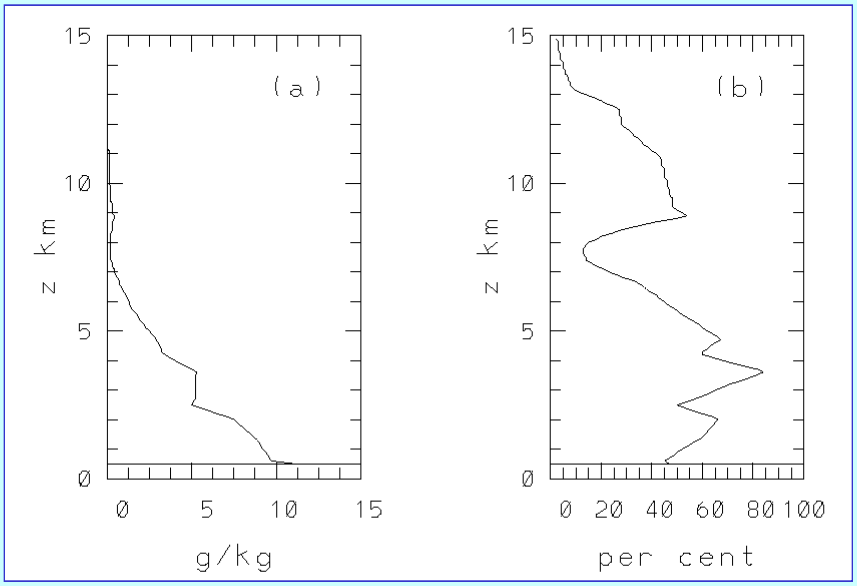
r is normally expressed in g/kg, but must be in kg/kg in any formula!

Other moisture variables

- **The vapour pressure,** $e = rp/(\varepsilon + r)$ **which is the partial pressure of water vapour.**
- **The relative humidity,** $RH = 100 \times e/e^*(T)$.
  - $e^* = e^*(T)$ **is the saturation vapour pressure - the maximum vapour that an air parcel can hold without condensation taking place**
- **The specific humidity,** $q = r/(1 + r)$**, which is the mass of water vapour per unit mass of moist air.**

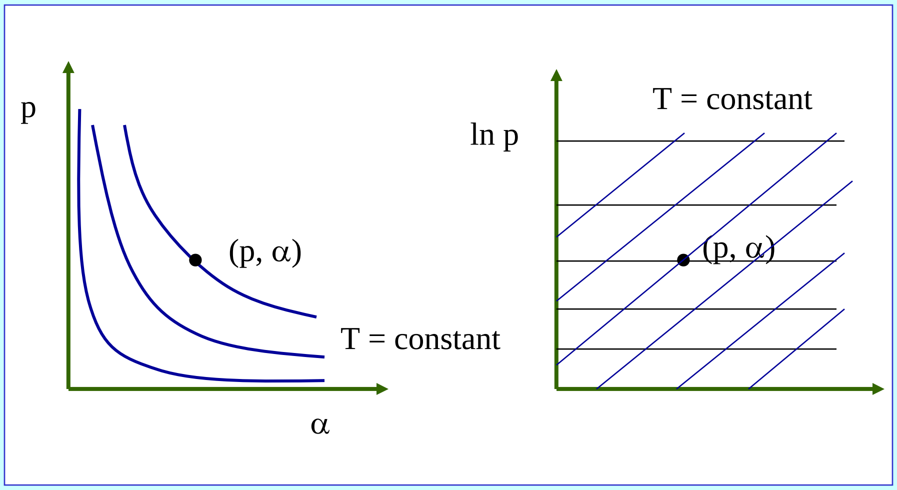
More moisture variables

- **The dew-point temperature,** T_d**, which is the temperature at which an air parcel first becomes saturated as it is cooled isobarically.**
- **The wet-bulb temperature,** T_w**, which is the temperature at which an air parcel becomes saturated when it is cooled isobarically by evaporating water into it. The latent heat of evaporation is extracted from the air parcel.**

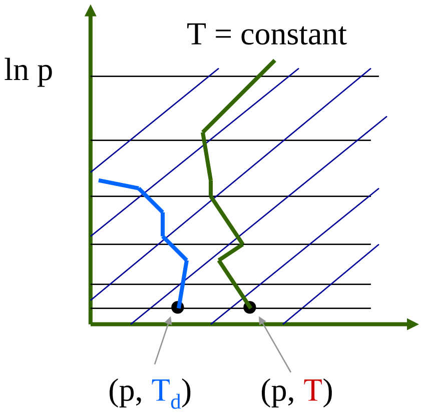


The vertical distribution of r and RH obtained from a radiosonde sounding on a humid summers day in central Europe.

Aerological (or thermodynamic) diagrams



Aerological diagram with plotted sounding



The effect of the water vapour on density is often taken into account in the equation of state through the definition of the **virtual temperature:**

$$T_v = T(1 + r/\varepsilon)/(1+\varepsilon) \approx T(1 + 0.61r)$$

Then, **the density of a sample of moist air** is characterized by its **pressure** and its **virtual temperature**, i.e.

$$\rho = \frac{p}{RT_v}$$

Moist air (r > 0) has a larger virtual temperature than dry air (r = 0) => the presence of **moisture decreases the density of air** --- important when considering the buoyancy of an air parcel!

The equation of state for cloudy air

Consider cloudy air as a single, heterogeneous system

specific volume = (total volume)/(total mass)

$$\alpha = \left(V_a + V_l + V_i\right) / \left(M_d + M_v + M_l + M_i\right)$$

total mixing ratio of water substance

Divide by M_d

$$\alpha = \alpha_d \left(1 + r_l(\alpha_l / \alpha_d) + r_i(\alpha_i / \alpha_d)\right) / \left(1 + r_T\right)$$

$$\alpha = \frac{R_d T}{p_d}\frac{1}{1+r_T} = \frac{R_d T}{p}\frac{p_d + e}{p_d}\frac{1}{1+r_T} = \frac{R_d T}{p}\frac{1 + r/\varepsilon}{1+r_T}$$

$\varepsilon = R_d/R_v = 0.622$

defines the density temperature for cloudy air:

$$\boxed{T_\rho = T(1 + r/\varepsilon)/(1 + r_T)}$$

specific heat of water vapour

The first law of thermodynamics

The first law of thermodynamics for a sample of moist air may be written alternatively as

$$dq = c'_v dT + p d\alpha, \quad = c'_p dT - \alpha dp,$$

incremental changes in temperature, pressure and specific volume

the heat supplied per unit mass to the air sample

specific heats of the air sample

$$c'_v = c_{vd}(1 + 0.94r) \qquad c'_p = c_{pd}(1 + 0.85r)$$

The quantities c_{vd} (≈ 1410 J kg^{-1} K^{-1}) and c_{pd} (≈ 1870 J kg^{-1} K^{-1}) are the corresponding values for dry air.

Adiabatic processes

An **adiabatic process** is one in which there is zero heat input (dq = 0); in particular, heat generated by frictional dissipation is ignored.

$$d\ln T = \left(R'/c'_p\right)d\ln p$$

$$R' = R_d(1 + r/\varepsilon)/(1 + r)$$

If the process is also reversible and unsaturated, r is a constant.

The equation can be integrated exactly (ignoring the small temperature dependence of c'_p) to give:

$$\ln T = (R'/c'_p)\ln p + \ln A$$

a constant

Define A such that, when p equals some standard pressure, p_o, usually taken to be 1000 mb, T = θ.

The potential temperature

The quantity θ is called the **potential temperature** and is given by:

$$\theta = T\left(\frac{p_o}{p}\right)^{\frac{R'}{c'_p}} = T\left(\frac{p_o}{p}\right)^{\frac{R_d}{c_{pd}}\frac{1 + r/\varepsilon}{1 + rc_{pv}/c_{pd}}} \approx T\left(\frac{p_o}{p}\right)^{\kappa'}$$

$$\kappa' = \kappa(1 - 0.24r)$$

The variation in κ is < 1 % and is usually ignored

$$\kappa = R_d / c_{pd}$$

We define the **virtual potential temperature**, θ_v by

$$\theta_v = T_v\left(\frac{p_o}{p}\right)^{\kappa}$$

take the value of κ for dry air.

Enthalpy

The first law of thermodynamics can be expressed as

$$dq = d(u + p\alpha) - \alpha dp = dk - \alpha dp$$

$k = u + p\alpha$ **is called the specific enthalpy**.

The enthalpy is a measure of heat content at constant pressure.

For an ideal gas, $k = c_pT$.

Water substance and phase changes

The latent heat associated with a phase transition of a substance is defined as the difference between the heat contents, or enthalpies, of the two phases; i.e.

$$L_{i,ii} = k_{ii} - k_i,$$

subscripts refer to the two phases

The dependence of $L_{i,ii}$ **on** T **and** p **may be obtained by differentiation using** $k = u + p\alpha$ **(for details see e.g. E94, p114).**

It turns out that: $L_{i,ii} = L_{i,iio} + (c_{pii} - c_{pi})(T - 273.16\text{ K})$

$L_{i,iio}$ **is the latent heat at the so-called triple point (**$T = 273.16$ K $= 0.01^{o}$C, and $e = 6.112$ mb**) where all three phases of water substance are in equilibrium.**

e **is the water vapour pressure**

The Clausius-Clapeyron equation

The pressure and temperature at which two phases are in equilibrium are governed by the Clausius-Clapeyron equation:

$$\left(\frac{dp}{dT}\right)_{ii,i} = \frac{L_{i,ii}}{T(\alpha_{ii} - \alpha_i)}.$$

For liquid-vapour equilibrium, $\alpha_l << \alpha_v$**, and using the ideal gas law for vapour, we obtain an equation for the saturation vapour pressure,** $e^*(T)$**:**

$$\frac{de^*}{dT} = \frac{L_v e^*}{R_v T^2}$$

L_v **= the latent heat of vaporization.**

$$\frac{de^*}{dT} = \frac{L_v e^*}{R_v T^2}$$

This equation may be integrated for $e^*(T)$**.**

A more accurate empirical formula is (see E94, p117):

$$\ln e^* = 53.67957 - 6743.769 / T - 4.8451 \ln T$$

e^* **in mb and** T **in K**

A corresponding expression for ice-vapour equilibrium is:

$$\ln e^* = 23.33086 - 6111.72784 / T + 0.15215 \ln T$$

These formulae are used to calculate the water vapour content of a sample of air. If the air sample is unsaturated, the dew point temperature (or ice point temperature) must be used.

Moist enthalpy

- Define the **moist enthalpy** k of a sample of cloudy air by the formula:

$$M_d k = M_d k_d + M_v k_v + M_L k_L$$

- Then $k = k_d + r k_v + r_L k_L$, expressed per unit mass of dry air.
- But $L_v(T) = k_v - k_L$, => $k = k_d + L_v r + r_T k_L$
- $r_T = r + r_L$.
- Since $k_d = c_{pd}T$ and $k_L = c_L T$,

$$k = (c_{pd} + r_T c_L)T + L_v r$$

c_L is the specific heat of liquid water

The moist enthalpy is conserved in an isobaric process as long as $dq = 0$ and $dr_T = 0$.

Entropy

An excellent reference is Chapter 4 of the book:

C. F. Bohren & B. A. Albrecht

Atmospheric Thermodynamics

Oxford University Press

The moist entropy and related quantities

- **In analogy with the definition of k, we define the total specific entropy per unit mass of dry air as:**

$$s = s_d + rs_v + r_L s_L$$

- **s_d, s_v, and s_L are the specific entropies of dry air, water vapour and liquid water, respectively.**
- **The equality of the Gibbs free energy in phase equilibrium between water vapour and liquid water leads to the expression:**

$$L_v = T(s_v^* - s_L)$$

the saturation specific entropy of water vapour

Some algebra

$$s = s_d + rs_v + r_L s_L$$

$$L_v = T(s_v^* - s_L)$$

$$s = s_d + r_T s_L + \frac{L_v r}{T} + r(s_v - s_v^*)$$

Substitute

$$s_d = c_{pd} \ln T - R_d \ln p_d$$

$$s_L = c_L \ln T$$

$$s_v = c_{pv} \ln T - R_v \ln e$$

$$s_v^* = c_{pv} \ln T - R_v \ln e^*$$

$$\Rightarrow \quad s = (c_{pd} + r_T c_L) \ln T - R_d \ln p_d + \frac{L_v r}{T} - rR_v \ln(RH)$$

Conservation of entropy

$$s = (c_{pd} + r_T c_L)\ln T - R_d \ln p_d + \frac{L_v r}{T} - rR_v \ln(RH)$$

- s **is conserved under reversible moist adiabatic transformations**

- **It is not conserved when**
  - $dq \neq 0$**: radiative heating or cooling, or conduction**
  - $dr \neq 0$**: by external evaporation, precipitation loss**
  - **by evaporation of rain into unsaturated air**

Equivalent potential temperature

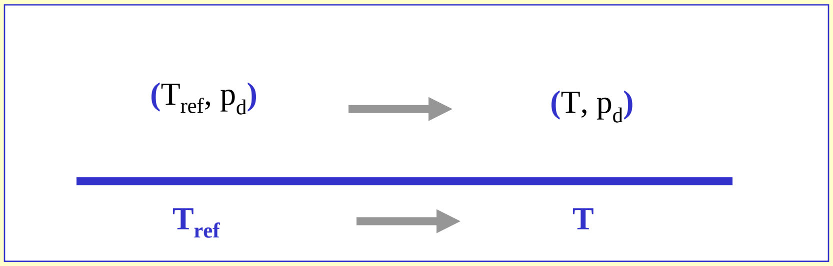
Put $$s = \left(c_{pd} + r_T c_L\right)\ln \theta_e - R_d \ln p_o$$

This defines the equivalent potential temperature, θ_e

$$\theta_e = T\left(\frac{p_o}{p_d}\right)^{R_d/(c_{pd}+r_T c_L)} (RH)^{-rR_v/(c_{pd}+r_T c_L)} \exp\left[\frac{L_v r}{(c_{pd}+r_T c_L)T}\right]$$

Entropy

- **Entropy** s **is defined by a process in which heat and water substance are added to air which is kept just saturated by reversible evaporation from a planar water surface as its temperature is raised from a reference temperature and partial pressure** p_d **to the state** (T, p_d)**.**
- s **is conserved under reversible moist adiabatic transformations.**



$(T_{ref}, p_d) \longrightarrow (T, p_d)$

$\mathbf{T_{ref}} \longrightarrow \mathbf{T}$

Equivalent potential temperature

$$\theta_e = T\left(\frac{p_o}{\underset{\nwarrow\ = \mathbf{p}}{p_d}}\right)^{R_d/(c_{pd}+\cancel{r_T c_L})} \underbrace{(RH)^{-rR_v/(c_{pd}+r_T c_L)}}_{=\mathbf{1}} \underbrace{\exp\left[\frac{L_v r}{(c_{pd}+r_T c_L)T}\right]}_{=\mathbf{1}}$$

- **For dry air (**$r = 0$, $r_L = 0$, $r_T = 0$**),** θ_e **reduces to** θ**.**
- **Note that** θ_e **is not a state variable** → **isopleths of constant** θ_e **cannot be plotted on an aerological diagram.**

Differential form for s

- **The differential form of** s **for (saturated) cloudy air** (RH = 1, $r_L \geq 0$) **is**

$$ds = (c_{pd} + r_T c_{pv})\frac{dT}{T} - R_d \frac{dp_d}{p_d} + d\left(\frac{L_v r^*}{T}\right)$$

- **An alternative form is, using** $0 = dr^* + dr_L$

$$ds = (c_{pd} + r_T c_{pv})\frac{dT}{T} - r_T R_v \frac{de^*}{e^*} - R_d \frac{dp_d}{p_d} - d\left(\frac{L_v r_L}{T}\right)$$

Alternative forms

- **Betts showed that the two forms can be written more symmetrically as:**

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_e^*}{\theta_e^*} = c_{pm}\frac{dT}{T} + \frac{L_v}{T}dq^* - R_m\frac{dp}{p}$$

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_l}{\theta_l} = c_{pm}\frac{dT}{T} - \frac{L_v}{T}dq_l - R_m\frac{dp}{p}$$

where

$(1 + r_T)\, c_{pm} = (c_{pd} + r^* c_{pv} + r_L c_l)$

$(1 + r_T)\, R_m = (R_d + R_v)$

$q^* = r^* /(1 + r_T)$

$q_l = r_L /(1 + r_T)$

The equations ⇩

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_e^*}{\theta_e^*} = c_{pm}\frac{dT}{T} + \frac{L_v}{T}dq^* - R_m\frac{dp}{p}$$

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_l}{\theta_l} = c_{pm}\frac{dT}{T} - \frac{L_v}{T}dq_l - R_m\frac{dp}{p}$$

define ⇧

- **the saturation equivalent potential temperature,** θ_e*
- **the liquid-water potential temperature,** θ_l

Approximate forms of θ_e* and θ_l

Let $c_{pm} \approx c_{pd}$, $R_m \approx R_d$ **then**

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_e^*}{\theta_e^*} = c_{pm}\frac{dT}{T} + \frac{L_v}{T}dq^* - R_m\frac{dp}{p}$$

$$\Rightarrow \quad c_{pd}\frac{d\theta_e^*}{\theta_e^*} = c_{pd}\frac{d\theta}{\theta} + \frac{L_v}{T}dr^*$$

and

$$\frac{ds}{1+r_T} = c_{pm}\frac{d\theta_l}{\theta_l} = c_{pm}\frac{dT}{T} - \frac{L_v}{T}dq_l - R_m\frac{dp}{p}$$

$$\Rightarrow \quad c_{pd}\frac{d\theta_L}{\theta_L} = c_{pd}\frac{d\theta}{\theta} - \frac{L_v}{T}dr_L$$

Assume that

$(L_v/T)dr^* \approx d(L_v r^*/T)$ **and** $(L_v/T)dr_L \approx d(L_v r_L/T)$

Then

$$c_{pd}\frac{d\theta_e^*}{\theta_e^*} = c_{pd}\frac{d\theta}{\theta} + \frac{L_v}{T}dr^*$$

⇒

$$\theta_e^* \approx \theta \exp\left(\frac{L_v r^*}{c_{pd}T}\right)$$

c/f

$$\theta_e^* = T\left(\frac{p_o}{p_d}\right)^{R_d/(c_{pd}+r_T c_L)} \exp\left[\frac{L_v r}{(c_{pd}+r_T c_L)T}\right]$$

$$c_{pd}\frac{d\theta_L}{\theta_L} = c_{pd}\frac{d\theta}{\theta} - \frac{L_v}{T}dr_L$$

⇒

$$\theta_L \approx \theta \exp(-L_v r_L/c_{pd}T)$$

Note that when $r_L = 0$, $\theta_L = \theta$

→ **the liquid water potential temperature is the potential temperature attained by the evaporation of all liquid water in an air parcel through reversible moist descent.**

$$\theta_L \approx \theta \exp\left(-L_v r_L / c_{pd} T\right)$$

An unapproximated expression for θ_l is:

$$\theta_L \equiv T\left(\frac{p_o}{p}\right)^{\chi}\left(1-\frac{r_L}{\varepsilon+r_T}\right)^{\chi}\left(1-\frac{r_L}{r_T}\right)^{-\gamma}\exp\left[\frac{-L_v r_L}{(c_{pd}+r_T c_{pv})T}\right]$$

where $\chi \equiv (R_d + r_T R_v)/\left(c_{pd} + r_T c_{pv}\right)$

$$\gamma \equiv r_T R_v / \left(c_{pd} + r_T c_{pv}\right)$$

The liquid-water virtual potential temperature

The liquid-water virtual potential temperature θ_{lv} is defined by:

$$\theta_{Lv} \equiv T_v\left(\frac{p_o}{p}\right)^{\chi}\left(1-\frac{r_L}{1+r_T}\right)\left(1-\frac{r_L}{\varepsilon+r_T}\right)^{\chi-1}\left(1-\frac{r_L}{r_T}\right)^{\chi-1}\exp\left[\frac{-L_v r_L}{(c_{pd}+r_T c_{pv})T}\right]$$

- **Note that $\theta_{Lv} = \theta_v$ when $r_L = 0$**

Some notes

- θ_e, θ_L, **and** θ_{Lv} **are conserved in reversible adiabatic processes involving changes in state of unsaturated or cloudy air.**
- θ_e, θ_L, **and** θ_{Lv} **are not functions of state - they depend on** p, T, r **and** r_L
- → **Curves representing reversible, adiabatic processes cannot be plotted in an aerological diagram**
- **In a saturated process,** $r = r^*(p, T)$

The pseudo-equivalent potential temperature

- **Approximation: neglect the liquid water content (set** $r_L = 0$**).**
- **This makes the approximate forms of** θ_e, θ_L, **and** θ_{Lv} **functions of state which can be plotted in an aerological diagram.**
- **Adiabatic processes where the liquid water is ignored are called pseudo-adiabatic processes. They are not reversible!**
- **The formula** $\theta_e^* \approx \theta \exp(L_v r^*/c_{pd}T)$ **is an approximation for the pseudo-equivalent potential temperature** θ_{ep} **. A more accurate formula is:**

$$\theta_{ep} = T\left(\frac{p_o}{p}\right)^{0.2854/(1-0.28r)} \exp\left[r(1+0.81r)\left(\frac{3376}{T_{LCL}} - 2.54\right)\right]$$

Temperature at the LCL

The Lifting Condensation Level Temperature

- T_{LCL} **is given accurately (within** 0.1°C**) by the empirical formula**:

$$T_{LCL} = \left[\frac{1}{T_d - 56} + \frac{\ln(T_K / T_d)}{800}\right]^{-1} + 56$$

T_K **and** T_d **in degrees K**

[Formula due to Bolton, MWR, 1980]

- **If an air parcel is lifted pseudo-adiabatically to the high atmosphere until all the water vapour has condensed out,** $\theta_{ep} = \theta$.
- **The pseudo-equivalent potential temperature is the potential temperature that an air parcel would attain if raised pseudo-adiabatically to a level at which all the water vapour were condensed out.**
- **The isopleths of** θ_{ep} **can be plotted on an aerological diagram. These are sometimes labelled by their temperature at** 1000 mb **which is called the wet-bulb potential temperature,** θ_w.

The adiabatic lapse rate

- **Lift a parcel of unsaturated air adiabatically**
- **it expands and cools, conserving its** θ_v
- **its** T **decreases with height at the dry adiabatic lapse rate,** Γ_d :

$$\Gamma_d = -\left(\frac{dT}{dz}\right)_{dq=0} = \frac{g}{c_{pd}}\,\frac{1+r}{1+r(c_{pv}/c_{pd})}$$

- **Note that** r **is conserved, but** r* **decreases because** $e^*(T)$ **decreases more rapidly than** p.
- **Saturation occurs at the lifting condensation level (LCL) when** $T = T_{LCL}$ **and** $r = r^*(T_{LCL}, p)$.

Above the LCL, the rate of which its temperature falls, Γ_m**, is less than** Γ_d **because condensation releases latent heat.**

For reversible ascent:

$$\Gamma_m \equiv -\left(\frac{dT}{dz}\right)_s = \frac{g}{c_{pd}}\,\frac{1+r_T}{1+r\dfrac{c_{pv}}{c_{pd}}} \times \left[\frac{1+\dfrac{L_v r}{R_d T}}{1+r_L\dfrac{c_L}{c_{pd}+rc_{pv}}+\dfrac{L_v^2(1+r/\varepsilon)r}{R_v T^2(c_{pd}+rc_{pv})}}\right]$$

When r_T **is small, the ratio** Γ_m/Γ_d **is only slightly less than unity, but when the atmosphere is very moist, it may be appreciably less than unity.**

The moist static energy and related quantities

The first law gives

$$dq = dk - \alpha_d dp$$

where dq **is expressed per unit mass of dry air.**

Adiabatic process (dq = 0) $\rightarrow$ $dk - \alpha_d dp = 0$

$$\alpha_d = \alpha(1 + r_T)$$

For a hydrostatic pressure change, $\alpha dp = -gdz$.

Under these conditions:

$$dh \equiv (c_{pd} + r_T c_L)dT + d(L_v r) + (1 + r_T)gdz = 0$$

Some notes

If r_T is conserved, we can integrate

$$h = (c_{pd} + r_T c_l)T + L_v r + (1 + r_T)gz = \text{constant}.$$

- **The quantity** h **is called the moist static energy.**
- h **is conserved for adiabatic, saturated or unsaturated transformations in which mass is conserved and in which the pressure change is strictly hydrostatic.**
- h **is a measure of the total energy:**

(internal + latent + potential)

The dry static energy

- **Define the dry static energy,** h_d.
- **Put** $r_T = r$ **=>**

$$h_d = (c_{pd} + rc_L)T + (1 + r)gz.$$

- **This is conserved in hydrostatic unsaturated transformations.**
- h **and** h_d **are very closely related to** θ_e **and** θ**.**

The (virtual) liquid water static energy

- **Define two forms of static energy related to** θ_L **and** θ_{Lv}**.**
- **These are the liquid water static energy:**

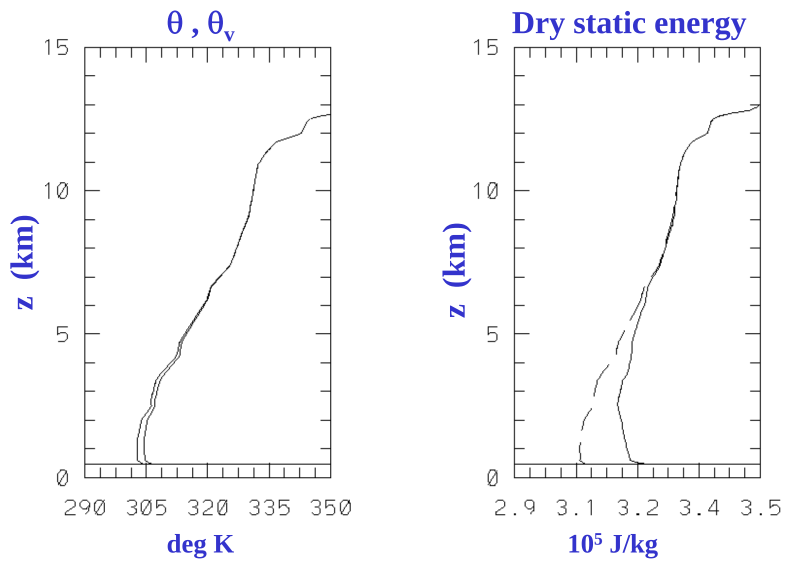
$$h_w = (c_{pd} + r_T c_{pv})T - L_v r_L + (1 + r_T)gz$$

and the virtual liquid water static energy

$$h_{Lv} = c_{pd}\left(\frac{\varepsilon + r_T}{\varepsilon + r_T - r_L}\right)T_\rho - \frac{L_v r_L}{1 + r_T} + gz$$

- h_{Lv} **is almost precisely conserved following slow adiabatic displacements.**
- **If** $r_L = 0$, $h_{Lv} = c_{pd}T_v + gz$ **(just as** θ_L **reduces to** θ**).**

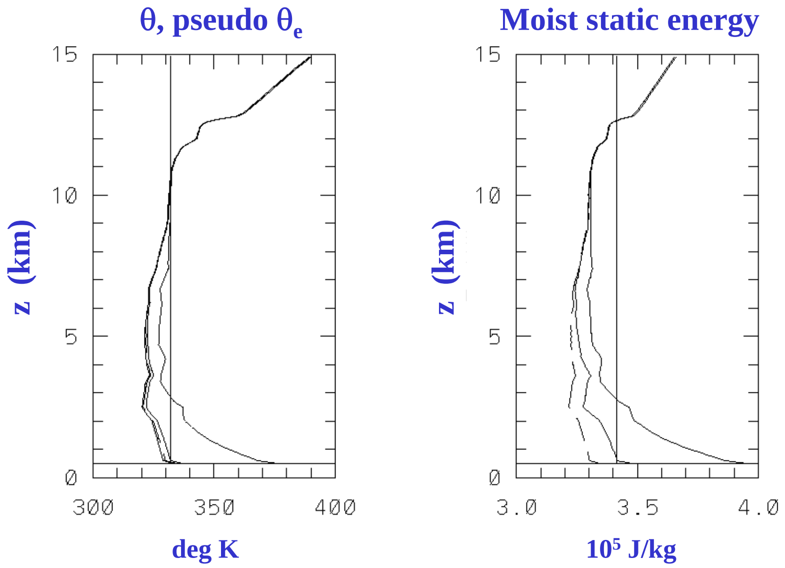
Vertical profiles of dry conserved variables



θ , θ_v

Dry static energy

Vertical profiles of moist conserved variables



θ, pseudo θ_e

Moist static energy

The stability of the atmosphere

- Consider the vertical displacement of an air parcel from its equilibrium position
- Calculate the buoyancy force at its new position
- Consider first an infinitesimal displacement ξ; later we consider finite-amplitude displacements
- Parcel motion is governed by the vertical momentum equation

$$\frac{d^2\xi}{dt^2} = b$$

$$b(\xi) = -g\left(\frac{\rho_p - \rho_a}{\rho_p}\right) = g\left(\frac{\alpha_p - \alpha_a}{\alpha_a}\right)$$

is the buoyancy force per unit mass

Newton's law for an air parcel:

$$\frac{d^2\xi}{dt^2} = \frac{\text{buoyancy force}}{\text{unit mass}}$$

$$\frac{\text{buoyancy force}}{\text{unit mass}}: \quad b(\xi) = \left.\frac{\partial b}{\partial z}\right|_{z=0} \xi +$$

$$\frac{d^2\xi}{dt^2} - \left.\frac{\partial b}{\partial z}\right|_{z=0} \xi = 0$$

The motion equation for small displacements is:

$$\frac{d^2\xi}{dt^2} + N^2\xi = 0 \qquad \text{where} \qquad N^2 = -\frac{\partial b}{\partial z}$$

The motion equation for small displacements is:

$$\frac{d^2\xi}{dt^2} + N^2\xi = 0$$

where $$N^2 = -\frac{\partial b}{\partial z}$$

For an unsaturated displacement, θ_{vp} **is conserved and we can write**

$$b(\xi) = g\left(\frac{T_{vp} - T_{va}}{T_{va}}\right) = g\left(\frac{\theta_{vp} - \theta_{va}}{\theta_{va}}\right)$$

Since θ_{vp} = **constant** $\cong \theta_{va}$**,**

$$N^2 = -\frac{\partial b}{\partial z} = g\frac{\theta_{vp}}{\theta_{va}^2}\frac{\partial\theta_{va}}{\partial z} \cong \frac{g}{\theta_{va}}\frac{\partial\theta_{va}}{\partial z}$$

Stability criteria

- **Parcel displacement is:**
  - **stable if** $\partial\theta_{va}/\partial z > 0$
  - **unstable if** $\partial\theta_{va}/\partial z < 0$
  - **neutrally-stable if** $\partial\theta_{va}/\partial z = 0$
- **A layer of air is stable, unstable, or neutrally-stable if these criteria are satisfied in the layer.**

- In a saturated (cloudy) layer of air, the appropriate conserved quantity is the moist entropy s (or the equivalent potential temperature, θ_e)
- Must use the density temperature to calculate b.
- Replace α_p in b by the moist entropy, s.
- In this case

$$N^2 = \frac{1}{1+r_T}\left[\Gamma_m \frac{\partial s}{\partial z} - \left(c_L \Gamma_m \ln T + g\right)\frac{\partial r_T}{\partial z}\right]$$

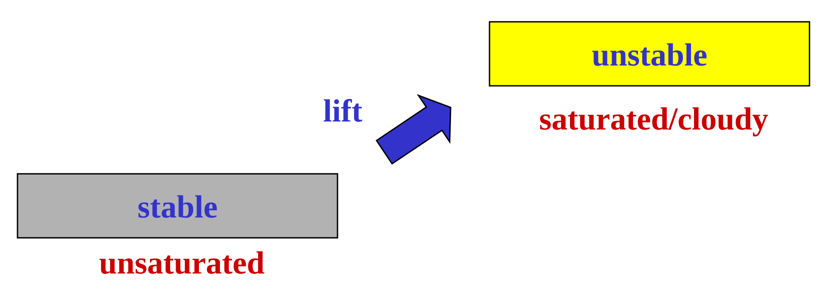
- A layer of cloudy air is stable to infinitesimal parcel displacements if s (or θ_e) increases upwards and the total water (r_T) decreases upwards. It is unstable if θ_e decreases upwards and r_T increases upwards.

Some notes

- The stability criterion does not tell us anything about the finite-amplitude instability of an unsaturated layer of air that leads to clouds.
- Parcel method okay, but must consider finite displacements of parcels originating from the unsaturated layer.

Potential Instability

- A layer of air may be stable if it remains dry, but unstable if lifted sufficiently to become saturated.
- Such a layer is referred to as potentially unstable.
- The criterion for instability is that $d\theta_e/dz < 0$.



unstable

saturated/cloudy

lift

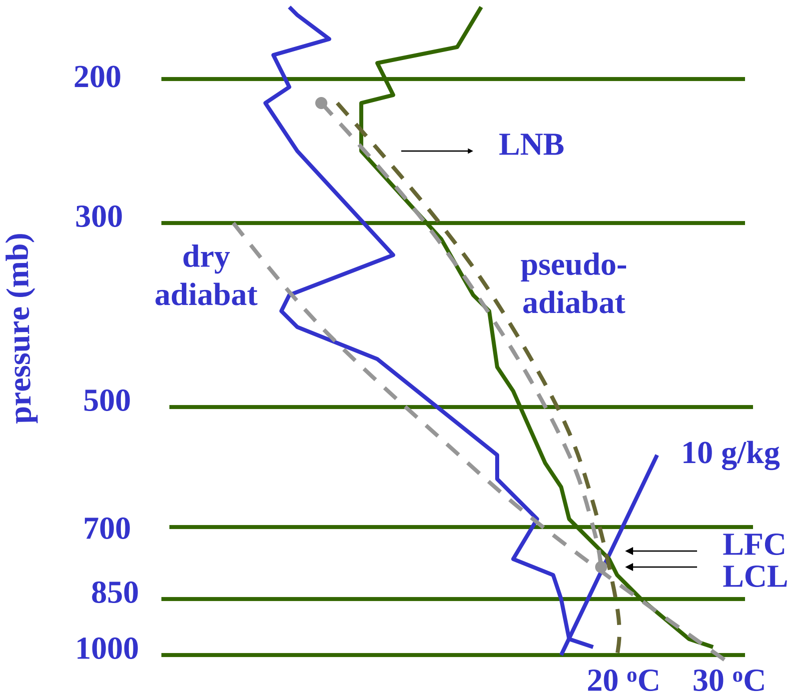
stable

unsaturated

Conditional Instability

- The typical situation is that in which a displacement is stable provided the parcel remains unsaturated, but which ultimately becomes unstable if saturation occurs.
- This situation is referred to as conditional instability.
- To check for conditional instability, we examine the buoyancy of an initially-unsaturated parcel as a function of height as the parcel is lifted through the troposphere, assuming some thermodynamic process (e.g. reversible moist adiabatic ascent, or pseudo-adiabatic ascent).
- If there is some height at which the buoyancy is positive, we say that the displacement is conditionally-unstable.

- If some parcels in an unsaturated atmosphere are conditionally-unstable, we say that the atmosphere is conditionally-unstable.
- Conditional instability is the mechanism responsible for the formation of deep cumulus clouds.
- Whether or not the instability is released depends on whether or not the parcel is lifted high enough.
- Put another way, the release of conditional instability requires a finite-amplitude trigger.
- The conventional way to investigate the presence of conditional instability is through the use of an aerological diagram.



Positive and Negative Area
Convective Inhibition (CIN)

The positive area (PA)

$$PA = \tfrac{1}{2}u_{LNB}^2 - \tfrac{1}{2}u_{LFC}^2 = \int_{p_{LNB}}^{p_{LFC}} \left(T_{vp} - T_{va}\right) R_d \, d\ln p$$

The negative area (NA) or convective inhibition (CIN)

$$NA = CIN = \int_{p_{LFC}}^{p_{parcel}} \left(T_{vp} - T_{va}\right) R_d \, d\ln p$$

Convective Available Potential Energy - CAPE

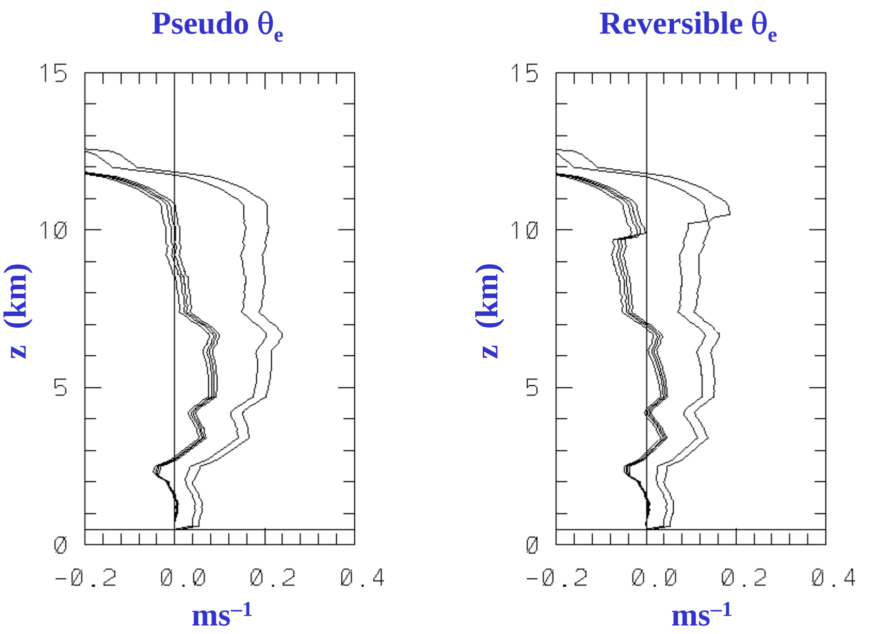
The convective available potential energy or CAPE is the net amount of energy that can be released by lifting the parcel from its original level to its LNB.

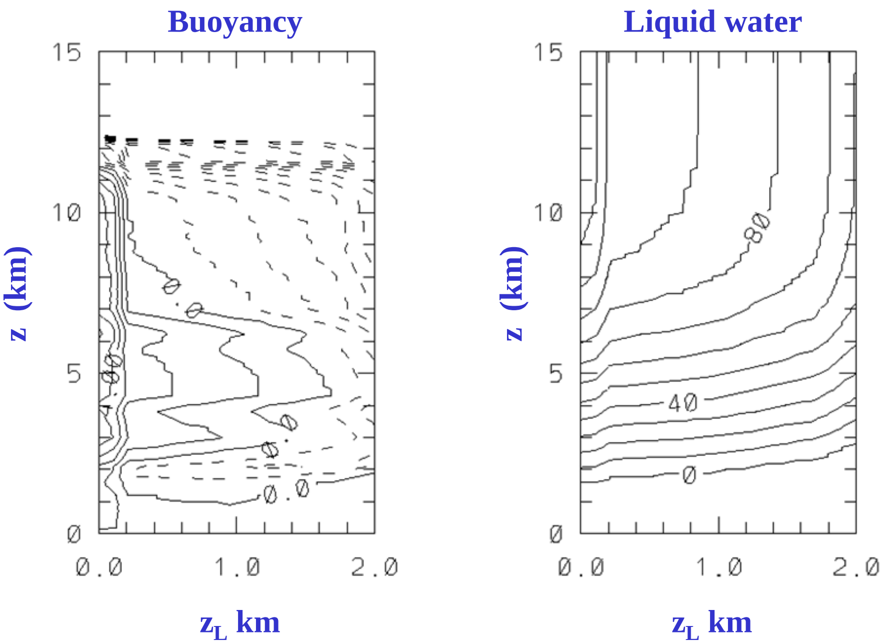
$$CAPE = PA - NA$$

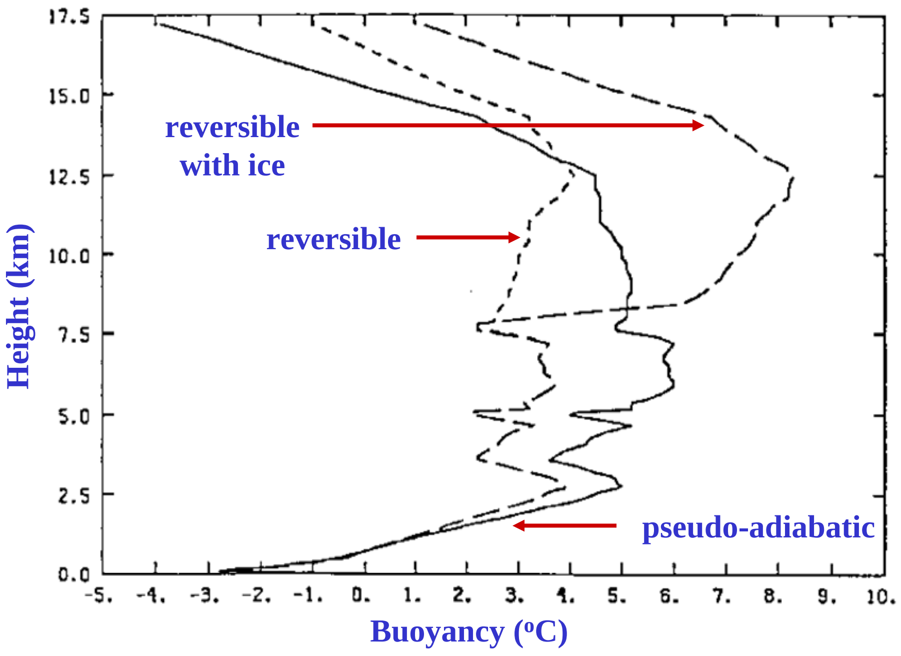
We can define also the downdraught convective available potential energy (DCAPE)

$$DCAPE_i = \int_{p_i}^{p_o} R_d (T_{\rho a} - T_{\rho p}) d\ln p$$

The integrated CAPE (ICAPE) is the vertical mass-weighted integral of CAPE for all parcels with CAPE in a column.

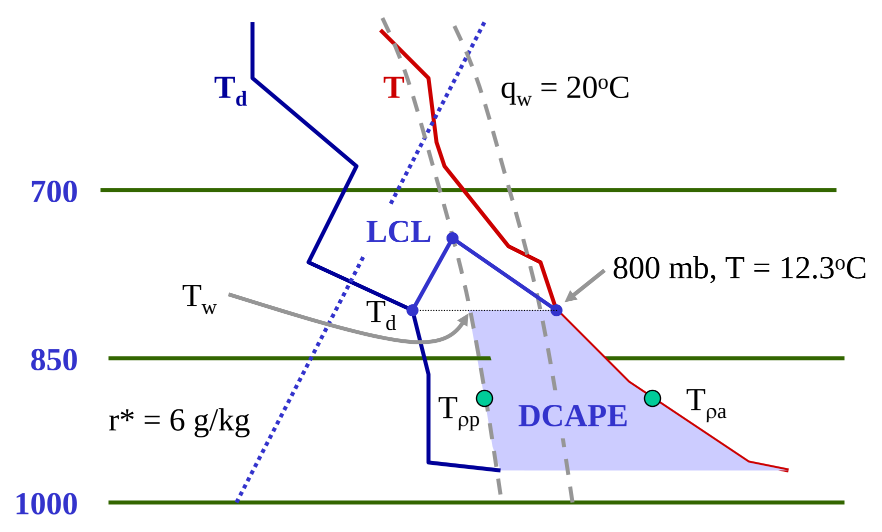

Pseudo θ_e
Reversible θ_e
15
10
5
0
z (km)
-0.2
0.0
0.2
0.4
ms^{-1}


Buoyancy
Liquid water
15
10
5
0
z (km)
0.0
1.0
2.0
z_L km
0
0.0
80
40
0

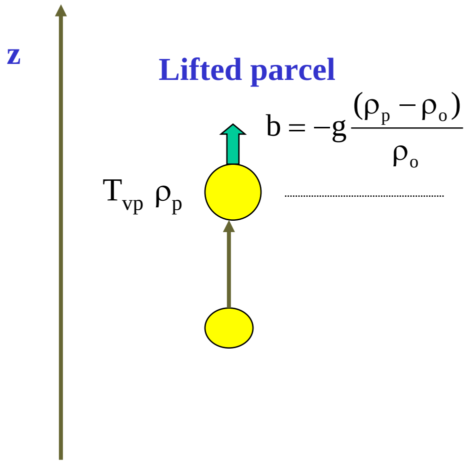


Downdraught convective available potential energy (DCAPE)

$$DCAPE_i = \int_{p_i}^{p_o} R_d (T_{\rho a} - T_{\rho p}) d\ln p$$



Buoyancy and θ_e



Lifted parcel

Environment

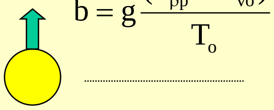
$$b = -g\frac{(\rho_p - \rho_o)}{\rho_o}$$

$T_{vp}\ \rho_p$

$T_o(z), r_o(z), p(z), \rho_o(z)$

or

$$b = g\frac{(T_{vp} - T_{vo})}{T_o}$$



Lifted parcel

Environment

$$b = g\frac{(T_{\rho p} - T_{vo})}{T_o}$$

$T_{\rho p}\ \rho_p$

$T_o(z), r_o(z), p(z), \rho_o(z)$

Below the LCL ($T_{\rho p} = T_{vp}$)

$$\text{sgn}\,(b) = \text{sgn}\,\{\,T_p(1 + \varepsilon r_p) - T_o(1 + \varepsilon r_o) = T_p - T_o + \varepsilon[T_p r_p - T_o r_o]\}$$

$\varepsilon = 0.61$

At the LCL ($T_{\rho p} = T_{vp}$)

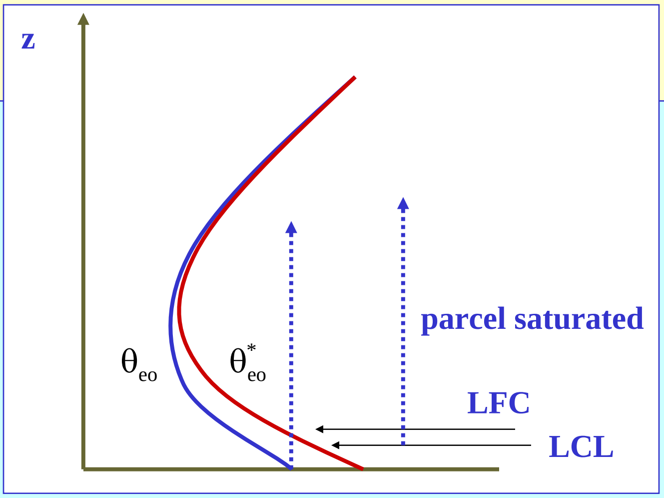
$$\text{sgn}\,(b) = \text{sgn}\,[T_p(1 + \varepsilon r^*(p,T_p)) - T_o(1 + \varepsilon r_o)]$$

$$= \text{sgn}\,[T_p(1 + \varepsilon r^*(p,T_p) - T_o(1 + \varepsilon r^*(p,T_o)) + \varepsilon T_o(r^*(p,T_o) - r_o)]$$

$$\mathrm{sgn}\,(b) = \mathrm{sgn}\,[T_p(1 + \varepsilon r^*(p,T_p) - T_o(1 + \varepsilon r^*(p,T_o)) + \varepsilon T_o(r^*(p,T_o) - r_o)]$$

Since T_v **is a monotonic function of** θ_e**,** $b \propto (\theta^*_{ep} - \theta^*_{eo} + \Delta)$

small



z

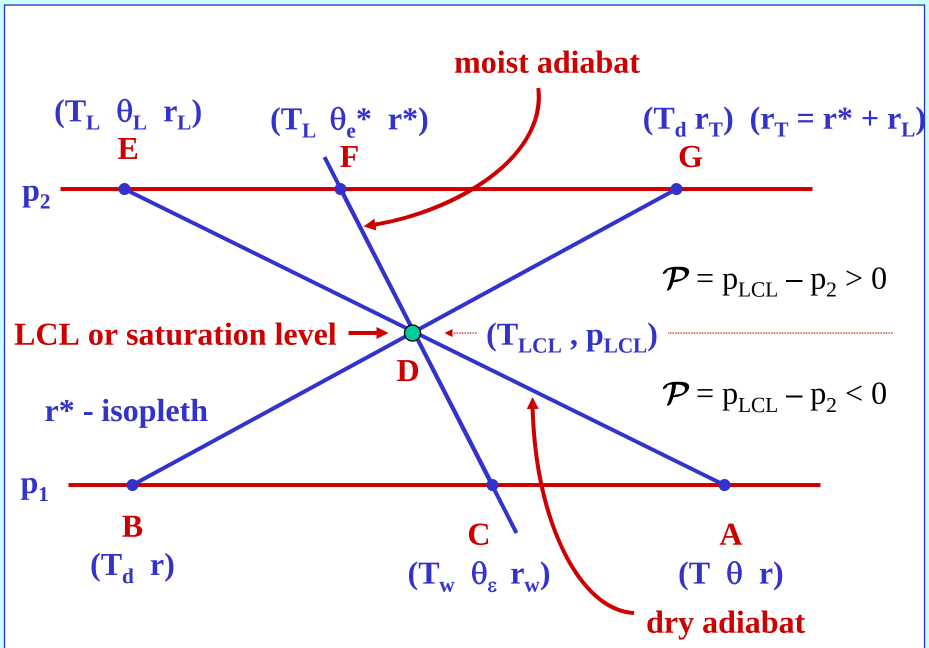
θ_{eo}

θ^*_{eo}

parcel saturated

LFC

LCL



moist adiabat

(T$_L$ θ$_L$ r$_L$)

E

(T$_L$ θ$_e$* r*)

F

(T$_d$ r$_T$) (r$_T$ = r* + r$_L$)

G

p$_2$

$\mathcal{P} = p_{LCL} - p_2 > 0$

LCL or saturation level

(T$_{LCL}$, p$_{LCL}$)

D

$\mathcal{P} = p_{LCL} - p_2 < 0$

r* - isopleth

p$_1$

B

(T$_d$ r)

C

(T$_w$ θ$_\varepsilon$ r$_w$)

A

(T θ r)

dry adiabat

After Betts

The saturation point

- **The point** (T_{LCL} , p_{LCL}) **is referred to as the saturation point.**
- **The saturation point of an air parcel is a conserved quantity in the absence of diabatic or mixing processes.**
- **One can plot the saturation points of parcels from a radiosonde sounding on an aerological diagram.**
- **The state of a parcel of air is characterized by its saturation point and the departure of the actual pressure from the saturation pressure, i.e.** $\mathcal{P} = p_{LCL} - p$.
- **If** $\mathcal{P} > 0$, **the parcel is cloudy, if** $\mathcal{P} < 0$ **it is unsaturated.**

The End