Two dimensional flow of a homogeneous, incompressible, inviscid fluid

In two dimensions , the Euler equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (2)$$

Continuity equation $\quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (3)$

Taking $\frac{\partial}{\partial z}(6.1) - \frac{\partial}{\partial x}(6.2)$ and using the continuity equation

$\Rightarrow \quad \frac{D\eta}{Dt} = 0$ where $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

The vorticity ω has only one non-zero component, the y-component, i.e., $\omega = (0, \eta, 0)$.

The equation $\frac{D\eta}{Dt} = 0$ states that fluid particles conserve their vorticity as they move around.

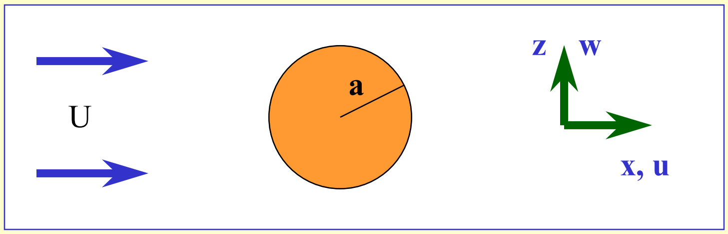
This is a powerful and useful constraint.

In some problems, $\eta = 0$ for all particles. Such flows are called irrotational.

Example $\Rightarrow$

Consider, for example, the problem of a steady, uniform flow U **past a cylinder of radius** a**. All fluid particles originate from far upstream (**$x \to -\infty$**) where** $u = U$, $w = 0$, **and therefore** $\eta = 0$.

It follows that fluid particles have zero vorticity for all time.



The inviscid flow problem can be solved as follows:

Define a streamfunction ψ **by the equations**

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x} \quad \text{to satisfy continuity} \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Substitute $u = \frac{\partial \psi}{\partial z}, \; w = -\frac{\partial \psi}{\partial x}$ **into** $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

$$\Rightarrow \quad \eta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}$$

In the case of irrotational flow, $\eta = 0$ **and** ψ **satisfies Laplace´s equation:**

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

Boundary conditions:

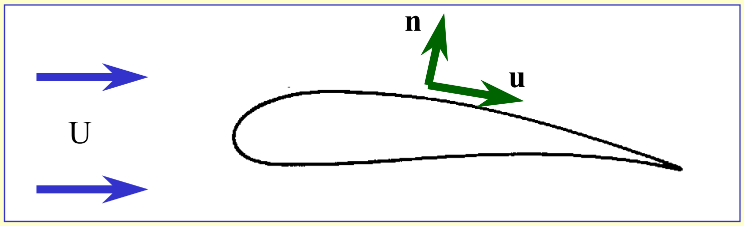
On a solid boundary, the normal velocity must be zero.

$$\Rightarrow \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on the boundary.}$$

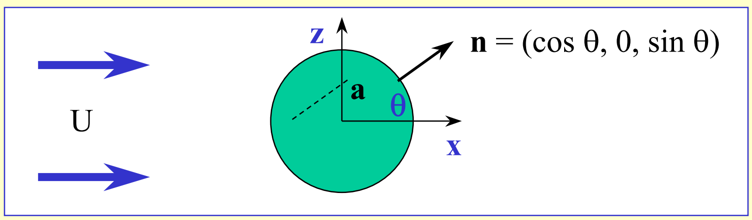
Let $\mathbf{n} = (n_1, 0, n_3)$ $\Rightarrow$ $n_1 \frac{\partial \psi}{\partial z} - n_3 \frac{\partial \psi}{\partial x} = 0$ **or** $\mathbf{n} \wedge \nabla \psi = 0$

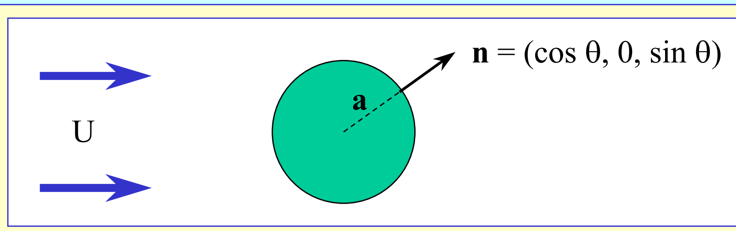
$\mathbf{n} \wedge \nabla\psi = 0$ ⇒ $\nabla\psi$ **is in the direction of n**

⇒ ψ **is a constant on the boundary itself.**



Uniform flow past a cylinder of radius a





The problem is to solve $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial z^2} = 0$ **in the region outside the cylinder** $(r > a)$ **subject to the boundary conditions that**

and
$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad r = a \\ \mathbf{u} = \left(\dfrac{\partial\psi}{\partial z}, 0, -\dfrac{\partial\psi}{\partial x}\right) \rightarrow (U, 0, 0) \quad \text{as} \quad r \rightarrow \infty \end{cases}$$

where $r = \left(x^2 + y^2\right)^{1/2}$.

For this problem it turns out to be easier to work in cylindrical polar coordinates centred on the cylinder.

It is easy to check that the solution of $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0$

satisfying the appropriate boundary conditions is

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

Note that for large r, $\psi \sim Ur\sin\theta = Uz$

$$\Rightarrow \quad u = \partial\psi / \partial z \sim U$$

Now $$\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z}$$

and $z = r\sin\theta \Rightarrow \partial r/\partial z = 1/\sin\theta \quad \Rightarrow \quad 1 = r\cos\theta\,\partial\theta/\partial z$

$$\Rightarrow \quad \frac{\partial\psi}{\partial z} = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial\psi}{\partial\theta}$$

Similarly $$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x}$$

and $x = r\cos\theta \Rightarrow \partial r/\partial x = 1/\cos\theta \quad \Rightarrow \quad 1 = -r\sin\theta\,\partial\theta/\partial x$

$$\Rightarrow \quad \frac{\partial\psi}{\partial x} = \frac{1}{\cos\theta}\frac{\partial\psi}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta}$$

The boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $r = a$ **requires that**

$$n_1 \frac{\partial\psi}{\partial z} - n_3 \frac{\partial\psi}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial\psi}{\partial z}\cos\theta - \frac{\partial\psi}{\partial x}\sin\theta = 0$$

at $r = a$ **for all** θ**.**

substitute

$$\Rightarrow \quad \frac{\partial\psi}{\partial\theta} = 0 \quad \text{at } r = a\,.$$

$\Rightarrow$ ψ **is a constant on the cylinder; i.e., the surface of the cylinder must be a streamline.**

Substitution of $\psi = U\left(r - \dfrac{a^2}{r}\right)\sin\theta$ **into** $\dfrac{\partial\psi}{\partial\theta} = 0$ **confirms that** $\psi = 0$ **on the cylinder.**

It remains to show that ψ **satisfies**

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

To do this one can use

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x} \quad \text{and} \quad \frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z}$$

to transform Laplace's equation to cylindrical polar coordinates:

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} = 0$$

It is now easy to verify that $\psi = U\left(r - \dfrac{a^2}{r}\right)\sin\theta$ **satisfies**

Laplace's equation and it is therefore the solution for steady irrotational flow past a cylinder.

Note that the solution for ψ is unique only to within a constant value.

If we add any constant to it, it will satisfy equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \quad \textbf{or} \quad \frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} = 0$$

but the velocity field would be unchanged.

- **It is important to note that we have obtained a solution without reference to the pressure field, but the pressure distribution determines the force field that drives the flow!**
- **We seem to have by-passed Newton´s second law, and have obviously avoided dealing with the nonlinear nature of the momentum equations.**
- **Looking back we will see that the trick was to use the vorticity equation, a derivative of the momentum equations.**
- **For a homogeneous fluid, the vorticity equation does not involve the pressure since $\nabla \wedge \nabla p \equiv 0$.**

We infer from the vorticity constraint

$$\eta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}$$

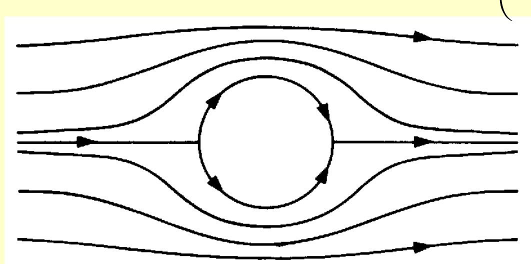
that the flow must be irrotational everywhere

We use this, together with the continuity constraint (which is automatically satisfied when we introduce the streamfunction) to infer the flow field.

If desired, the pressure field can be determined, for example, by integrating the momentum equations, or by using Bernoulli´s equation along streamlines.

What does the solution look like?

The streamline corresponding with $\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$



Note that the streamlines are symmetrical around the cylinder.

Apply Bernoulli´s equation to the streamline around the cylinder.

The pressure distribution is symmetric

The total pressure force on the upstream side of the cylinder is exactly equal to the pressure on the downwind side.

d´Alembert´s Paradox

- **In other words, the net pressure force on the cylinder is zero!**
- **This result, which is a general one for irrotational inviscid flow past a body of any shape, is known as d´Alembert´s Paradox.**
- **It is not in accord with our experience as you know when you try to cycle against a strong wind!**
- **What then is wrong with the theory?**
- **What does the flow round a cylinder look like in reality?**
- **The reasons for the breakdown of the theory help us to understand the limitations of inviscid flow theory in general and help us to see the circumstances under which it may be applied with confidence.**

To answer these questions we must return to viscous theory.

The Navier-Stokes´ equation is the statement of Newton´s second law of motion for a viscous fluid

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

The quantity of ν is called the kinematic viscosity.

For air

$$\nu = 1.5 \times 10^{5}\,\mathrm{m}^2\mathrm{s}^{-1}$$

For water

$$\nu = 1.0 \times 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$$

The relative importance of viscous effect is characterized by the **Reynolds' number** Re, a nondimensional number defined by

$$\mathrm{Re} = \frac{UL}{\nu}$$

where U and L are typical velocity and length scales.

The Reynolds' number is a measure of the ratio of the acceleration term to the viscous term in the Navier-Stokes' equation.

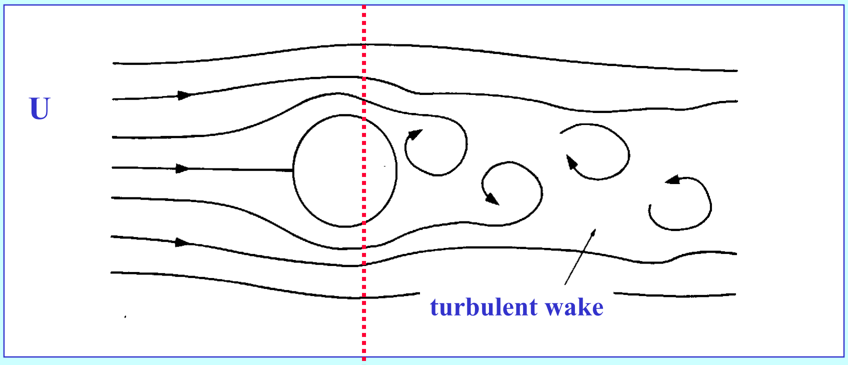
For many flows of interest, Re >> 1 and viscous effects are relatively unimportant.

But - **viscous effects are always important near boundaries**, even if only in a thin "boundary-layer" adjacent to the boundary.

Importance of the boundary layer

- The dynamics of this boundary layer may be crucial to the flow in the main body of fluid under certain circumstances.
- For example, in flow past a circular cylinder it has important consequences for the flow downstream.
- The observed streamline pattern in this case at large Reynolds numbers is sketched in the figure in the **next figure**.

Flow past a cylinder

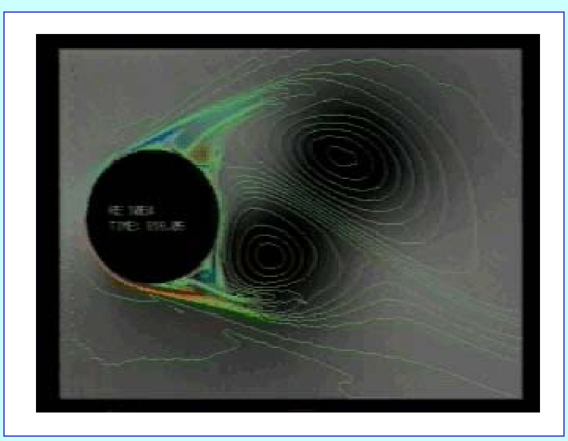


Flow is similar to that predicted by the inviscid theory, except in a thin viscous boundary-layer adjacent to the cylinder.

Flow separates and there is an unsteady turbulent wake behind it.

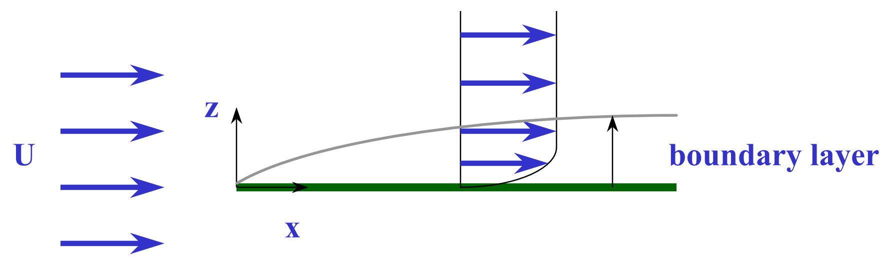
The wake destroys the symmetry predicted by the inviscid theory.

Flow past a cylinder



Boundary layers in nonrotating fluids

Consider steady two-dimensional boundary layer on a flat plate at normal incidence to a uniform stream U.



The Navier Stokes' equations for flow with typical scales written below each component are:

$$u\frac{\partial u}{\partial x} \quad + \quad w\frac{\partial u}{\partial z} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad + \quad \nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{U^2}{L} \qquad \frac{UW}{H} \rightarrow \qquad \frac{\Delta P}{\rho L} \quad \rightarrow \frac{\nu U}{L^2} \rightarrow \frac{\nu U}{H^2}$$

$$u\frac{\partial w}{\partial x} \quad + \quad w\frac{\partial w}{\partial z} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad + \quad \nu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right]$$

$$\frac{UW}{L} \qquad \frac{W^2}{H} \rightarrow \qquad \frac{\Delta P}{\rho H} \quad \rightarrow \frac{\nu W}{L^2} \rightarrow \frac{\nu W}{H^2}$$

Continuity equation $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

$$\frac{U^2}{L} \rightarrow \frac{W}{H}$$

Since $|\partial u / \partial x| = |\partial w / \partial z|$ ⇨ $W \sim UH/L$

⇨ $u\frac{\partial u}{\partial x} \approx w\frac{\partial u}{\partial z} \approx \frac{U^2}{L}$ **in u-equation**

$u\frac{\partial w}{\partial x} \approx w\frac{\partial w}{\partial z} \approx \frac{U^2}{L}\frac{H}{L}$ **in w-equation**

For a thin boundary layer, $H/L \ll 1$ **so that the derivatives** $\partial^2 / \partial x^2$ **can be neglected compared with** $\partial^2 / \partial z^2$.

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\underline{\frac{U^2}{L}} \qquad \underline{\frac{U^2}{L}} \rightarrow \frac{\Delta P}{\rho L} \rightarrow \frac{\nu U}{L^2} \rightarrow \underline{\frac{\nu U}{H^2}}$$

Assuming that the pressure gradient term is not larger than both inertial or friction terms

⇨ $$\underline{\frac{U^2}{L} \sim \frac{\nu U}{H^2}} \geq \frac{\Delta P}{\rho L}$$

$H \sim L\, Re^{-1/2}$

$Re = UL/\nu$ **has the form of a Reynolds' number**

Alternatively, the expression $H \sim L\,Re^{-1/2}$ **implies that the boundary thickness increases downstream like** $x^{1/2}$

[i.e., $H \sim L^{1/2} (\nu/U)^{1/2}$**].**

Then in

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

$$\frac{UW}{L} \qquad \frac{W^2}{H} \rightarrow \qquad \frac{\Delta P}{\rho H} \qquad \rightarrow \frac{\nu W}{L^2} \rightarrow \frac{\nu W}{H^2}$$

$$\frac{\Delta P}{\rho H}\Big/\frac{UW}{L} \sim \frac{\rho U^2}{\rho H}\Big/\frac{U^2 H}{L^2} \sim \frac{L^2}{H^2} >> 1$$

$$\frac{\Delta P}{\rho H}\Big/\frac{\nu W}{H^2} \sim \frac{\rho U^2}{\rho H}\Big/\frac{\nu U}{HL} \sim \frac{UL}{\nu} = Re >> 1$$

If both the inertia terms and friction terms in the w**-equation are much less than the pressure gradient term, the equation must be accurately approximated by**

$$\frac{\partial p}{\partial z} = 0$$

The perturbation pressure is constant across the boundary layer.

The horizontal pressure gradient in the boundary layer is equal to that in free stream.

The boundary-layer equations

An approximate form of the Navier-Stokes' equations for the boundary layer is

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$U = U(x)$ **is the (possible variable) free stream velocity above the boundary layer**.

Blasius´ solution (U = constant)

The boundary-layer momentum equation becomes

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}$$

We look for a solution satisfying the boundary conditions:

- $u = 0,\ w = 0$ **at** $z = 0$
- $u = U$ **at** $x = 0$
- $u \to U$ **as** $z \to \infty$

Introduce a streamfunction ψ **such that** $u = \dfrac{\partial \psi}{\partial z},\ \ w = -\dfrac{\partial \psi}{\partial x}$

ψ **must satisfy the conditions**:

$\psi = \text{constant},\ \ \partial\psi/\partial z = 0$ at $z = 0$,

$\psi \sim Uz$ **as** $z \to \infty$ **and** $\psi = Uz$ **at** $x = 0$.

It is easy to verify that a solution satisfying these conditions is

$$\psi = (2\nu U x)^{1/2} f(\chi)$$

where $\chi = (U / 2\nu x)^{1/2} z$ satisfies the ODE

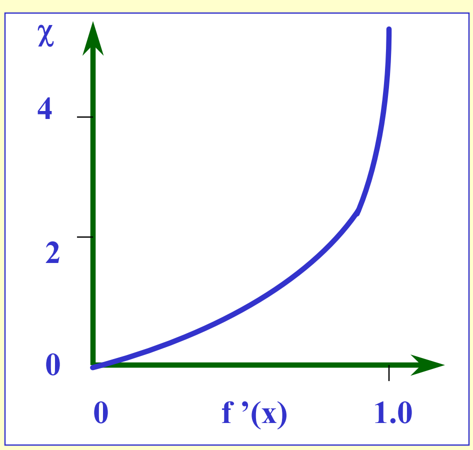
$$f''' + ff'' = 0$$

subject to the boundary conditions: $f(0) = f'(0) = 0, f(\infty) = 1$

A prime denotes differentiation with respect to χ

It is easy to solve this equation numerically:
(see e.g. Rosenhead, 1966,
Laminar Boundary Layers, p. 222-224).

Blasius velocity profile



The profile of f' which characterizes the variation of u across the boundary layer is proportional to χ and we might take $\chi = 4$ as the edge of the boundary layer.

$$\chi = \left(U / 2\nu x\right)^{1/2} z$$

The dimensional boundary thickness $\delta(x)$ is

$$\delta(x) = 4(2\nu x / U)^{1/2}$$

Note that $\delta(x)$ increases like the square root of the distance from the leading edge of the plate.

We can understand the thickening of the boundary layers as due to the progressive retardation of more and more fluid as the fictional force acts over a progressively longer distance downstream.

- **Often the boundary layer is relatively thin. Consider for example the boundary layer in an aeroplane wing.**
- **Assume that the wing has a span of 3 m and that the aeroplane flies at 200 ms^{-1}.**
- **The boundary layer at the trailing edge of the wing (assuming the wing to be a flat plate) would have thickness of**

$$4\left(2 \times 1.5 \times 10^{-5} \times 3 / 200\right)^{1/2} = 2.7 \times 10^{-3} \text{ m}$$

using the value $\nu = 1.5 \times 10^{-5} \, m^2 s^{-1}$ **for the viscosity of air.**

The calculation assumes that the boundary layer remains laminar; if it becomes turbulent, the random eddies in the turbulence have a much larger effect on the lateral momentum transfer than do random molecular motions, increasing the effective value of ν, possibly by an order of magnitude or more, and hence the boundary layer thickness.

Note that the boundary layer is rotational since $\omega = (0, \eta, 0)$, where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$ or approximately just $-\partial u / \partial z$.

Further reading

Acheson, D. J., 1990, *Elementary Fluid Dynamics*, Oxford University Press, pp406.

Morton, B. R., 1984: The generation and decay of vorticity. *Geophys. Astrophys. Fluid Dynamics*, 28, 277-308.

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