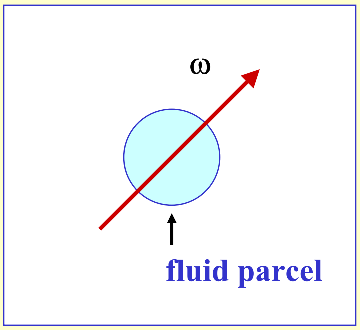
The vorticity field

The vector $\boldsymbol{\omega} = \nabla \wedge \mathbf{u} \equiv \text{curl}\,\mathbf{u}$ is twice the local angular velocity in the flow, and is called the vorticity of the flow (from Latin for a whirlpool).



Vortex lines are everywhere in the direction of the vorticity field (cf. streamlines)

Bundles of vortex lines make up vortex tubes

Thin vortex tubes, with their constituent vortex lines approximately parallel to the tube axis are vortex filaments.

A dust devil

Waterspouts

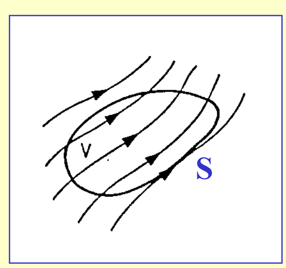
The vorticity field is solenoidal ⟷ $\nabla \cdot \boldsymbol{\omega} = 0$

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u})$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0\,.$$

Divergence theorem, for any volume V with boundary surface S,

$$\int_S \boldsymbol{\omega} \cdot \mathbf{n}\, ds = \int_V \nabla \cdot \boldsymbol{\omega}\, dr = 0$$

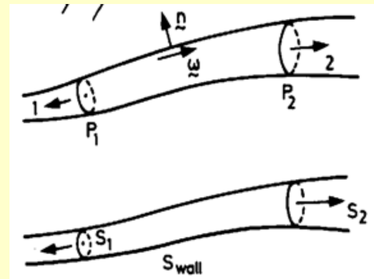


➡ **there is zero net flux of vorticity (or vortex tubes) out of any volume.**

➡ **there can be no sources of vorticity in the interior of a fluid.**

Consider a length P_1P_2 of vortex tube.

Divide the surface of the length P_1 P_2 into cross-sections and the tube wall,



Divergence theorem ➡

$$\int_S \omega \cdot \mathbf{n}\, ds = \int_v \nabla \cdot \omega \; dr = 0$$

$$S = S_1 + S_2 + S_{wall},$$

Then

$$\int_S \omega \cdot \mathbf{n}\, ds = \int_{S_1} \omega \cdot \mathbf{n}\, ds + \int_{S_2} \omega \cdot \mathbf{n}\, ds + \underbrace{\int_{S_{wall}} \omega \cdot \mathbf{n}\, ds}_{=0} = 0$$

➡ $$\int_{S_2} \omega \cdot \mathbf{n}\, ds = \int_{S_1} \omega \cdot (-\mathbf{n})\, ds$$

Note: the positive sense for normals is that of increasing distance along the tube.

➡ $\int_{S_{section}} \omega \cdot \mathbf{n}\, ds$ **measured over a cross-section of the vortex tube with n taken in the same sense is constant.**

This integral defines the the strength of the vortex tube.

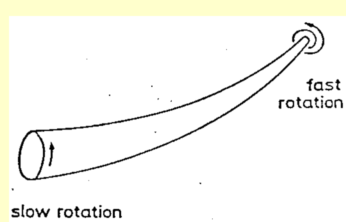
In a thin vortex tube, we have approximately:

$$\int_S \omega \cdot \mathbf{n}\, dS \approx \omega \cdot \mathbf{n} \int_S dS = \omega S$$

and ω * area = constant along tube

a property of all solenoidal fields

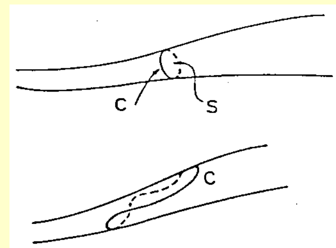
$\omega = |\omega|$



Circulation $\oint_C \mathbf{u} \cdot d\mathbf{r}$

Stokes' theorem is $$\int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, ds = \oint_C \mathbf{u} \cdot d\mathbf{r}$$

The line integral of the velocity field in any circuit C that passes once round a vortex tube is equal to the total vorticity cutting any cap S on C, and is therefore equal to the strength of the vortex tube.



We measure the strength of a vortex tube by calculating $\oint \mathbf{u} \cdot d\mathbf{r}$ around any circuit C enclosing the tube once only.

The quantity $\oint \mathbf{u} \cdot d\mathbf{r}$ is termed the circulation.

Vorticity may be regarded as circulation per unit area, and the component in any direction of ω is

$$\lim_{S \to 0} \frac{1}{S} \oint_c \mathbf{u} \cdot d\mathbf{r}$$

where C is a loop of area S perpendicular to the direction specified.

Bernoulli's theorem applies also to a vortex line!

$\frac{1}{2}\mathbf{u}^2 + p/\rho + \Omega =$ constant along a vortex line for steady, incompressible, inviscid flow under conservative external forces.

The Helmholtz equation for vorticity

From Euler´s equation for a homogeneous fluid in a conservative force field

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Omega$$

or

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u} \times \omega = -\nabla\left(\frac{p}{\rho} + \Omega\right)$$

Taking the curl

$$\nabla \times \frac{\partial \mathbf{u}}{\partial t} - \nabla \times (\mathbf{u} \times \omega) + \nabla \times \left[\nabla\left(\frac{1}{2}\mathbf{u}^2 + \frac{p}{\rho} + \Omega\right)\right] = 0$$

Now $\nabla \times (\nabla \phi) \equiv \text{for all}\ \phi$

ω is solenoidal

and $\nabla \times (\mathbf{u} \times \omega) = \mathbf{u}(\nabla \cdot \omega) - \omega(\nabla \cdot \mathbf{u}) + (\omega \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\omega$

for an incompressible fluid u is solenoidal

$$\nabla \times (\mathbf{u} \times \omega) = (\omega \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\omega$$

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u}$$

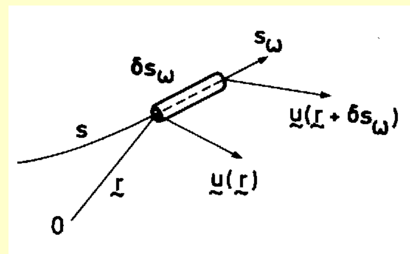
the Helmholtz vorticity equation.

Physical significance of the term $(\omega \cdot \nabla)\mathbf{u}$

We can understand the significance of the term $(\omega \cdot \nabla)\mathbf{u}$ in the Helmholtz equation by recalling that $\omega \cdot \nabla$ is a directional derivative and is proportional to the derivative in the direction of ω along the vortex line (see example 7).

$$\frac{D\mathbf{u}}{Dt} = (\omega \cdot \nabla)\mathbf{u} = |\omega|\,\hat{\omega} \cdot \nabla \mathbf{u} = \omega \frac{\partial \mathbf{u}}{\partial s_\omega}$$

δs_ω is the length of an element of vortex tube



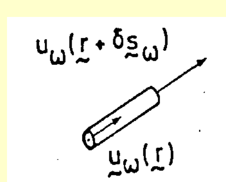
Resolve $\mathbf{u}$ into components $\mathbf{u}_\omega$ parallel to ω and $\mathbf{u}_\perp$ at right angles to ω and hence to $\delta \mathbf{s}_\omega$

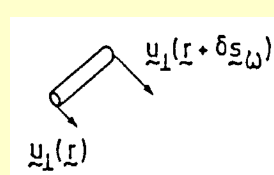
$$\frac{\delta s_\omega}{\omega}\frac{D\omega}{dt} = \frac{\partial}{\partial s_\omega}\left(\mathbf{u}_\omega + \mathbf{u}_\perp\right)\delta s_\omega =$$

$$= \frac{\partial \mathbf{u}_\omega}{\partial s_\omega}\delta s_\omega + \frac{\partial \mathbf{u}_\perp}{\partial s_\omega}\delta s_\omega$$

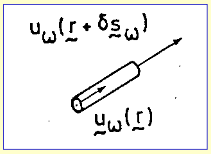
$$\approx \underbrace{\left[\mathbf{u}_\omega(\mathbf{r} + \delta \mathbf{s}_\omega) - \mathbf{u}_\omega(\mathbf{r})\right]}_{} + \underbrace{\left[\mathbf{u}_\perp(\mathbf{r} + \delta \mathbf{s}_\omega) - \mathbf{u}_\perp(\mathbf{r})\right]}_{}$$

rate of stretching of element	rate of turning of element



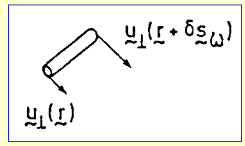


Stretching



Stretching along the length of the filament causes relative amplification of the vorticity field

Turning



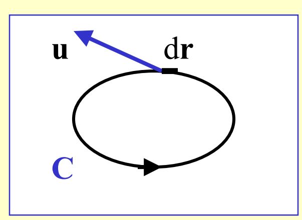
Turning away from the line of the filament causes a reduction of the vorticity in that direction, but an increase in the new direction.

Kelvin´s Theorem

The ideas of vorticity and circulation are important because of the permanence of circulation under deformation of the flow due to pressure forces.

Consider the rate of change of circulation round a circuit moving with an incompressible, inviscid fluid:

$$\frac{D}{Dt}\oint \mathbf{u}\cdot d\mathbf{r} = \oint \frac{D}{Dt}(\mathbf{u}\cdot d\mathbf{r})$$
$$= \oint \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{r} + \oint \mathbf{u}\cdot\frac{D}{Dt}d\mathbf{r}.$$



$$\oint \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{r} = \oint \left(-\frac{1}{\rho}\nabla p - \nabla\Omega\right)\cdot d\mathbf{r}, \quad \text{and} \quad \underbrace{\oint \mathbf{u}\cdot\frac{D}{Dt}d\mathbf{r} = \oint \mathbf{u}\cdot d\mathbf{u}}_{\text{See later}}$$

$$\frac{D}{Dt}\oint \mathbf{u}\cdot d\mathbf{r} = -\oint \nabla\left(\frac{p}{\rho}+\Omega\right)\cdot d\mathbf{r} + \oint \mathbf{u}\cdot d\mathbf{u}$$

$$= \oint \left[-d\left(\frac{p}{\rho}+\Omega\right) + d\left(\frac{1}{2}\mathbf{u}^2\right)\right]$$

$$= \oint d\left(-\frac{p}{\rho} - \Omega + \frac{1}{2}\mathbf{u}^2\right)$$

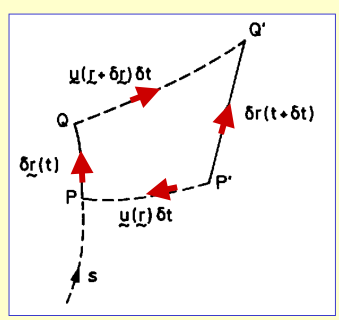
$$= 0$$

As $-p/\rho - \Omega + \frac{1}{2}\mathbf{u}^2$ is a single valued function it returns to its initial value after one circuit since it.

Example 7

Show that $\oint \mathbf{u}\cdot \frac{D}{Dt} d\mathbf{r} = \oint \mathbf{u}\cdot d\mathbf{u}$

Solution



Suppose that the elementary vector $\overline{P}\vec{Q} = \delta\mathbf{r}$ at time t is advected with the flow to $\overline{P'}\vec{Q'} = \delta\mathbf{r}(t+\delta t)$ at time $t+\delta t$.

Then

$$\delta\mathbf{r}(t+\delta t) \approx -\mathbf{u}(r)\delta t + \delta\mathbf{r}(t) + \mathbf{u}(\mathbf{r}+\delta\mathbf{r})\delta t,$$

$$\Rightarrow \quad \delta\mathbf{r}(t+\delta t) - \delta\mathbf{r}(t) \approx \mathbf{u}(\mathbf{r}+\delta \underset{\sim}{r})\delta t - \mathbf{u}(\mathbf{r})\delta t,$$

$$\delta\mathbf{r}(t+\delta t) - \delta\mathbf{r}(t) \approx \mathbf{u}(\mathbf{r}+\delta\underline{r})\delta t - \mathbf{u}(\mathbf{r})\delta t,$$

$$\lim_{\delta t\to 0}\frac{\delta\mathbf{r}(t+\delta t)-\delta\mathbf{r}(t)}{\delta t} = \lim_{\delta s\to 0}\frac{\mathbf{u}(\mathbf{r}+\delta\mathbf{r})-\mathbf{u}(\mathbf{r})}{\delta s}\delta s$$

$$\frac{D}{Dt}(\delta\mathbf{r}) \approx \frac{\partial \mathbf{u}}{\partial s}\delta s \approx \delta\mathbf{u}$$

$|\delta\mathbf{r}| \to \delta s$ **and** s **is arc length along the path P.**

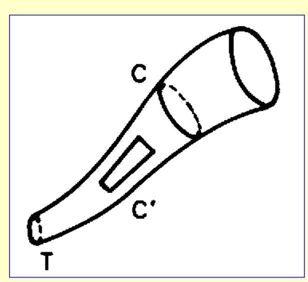
In the limit as $\delta\mathbf{r} \to d\mathbf{r}$, $\delta\mathbf{u} \to d\mathbf{u}$

$$\frac{D}{Dt}(d\mathbf{r}) = d\mathbf{u}$$

Results following from Kelvin´s Theorem

Helmholtz theorem: vortex lines move with the fluid

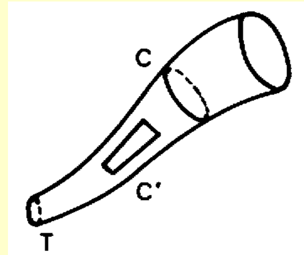
Consider a tube of particles T, which at time t **forms a vortex tube of strength** k**.**



At time t **the circulation round any circuit C´ lying in the tube wall, but not linking (i.e. embracing) the tube is zero, while that in an circuit C linking the tube *once* is** k**.**

Results following from Kelvin´s Theorem 2

These circulations suffer no change moving with the fluid: hence the circulation in C´ remains zero and that in C remains k.



⇒ the fluid comprising the vortex tube at T continues to comprise a vortex tube (as the vorticity component normal to the tube wall - measured in C´- is always zero), **and** the strength of the vortex remains constant.

A vortex line is the limiting case of a small vortex tube ⇒ vortex lines move with (or are frozen into) inviscid fluids.

Results following from Kelvin´s Theorem 3

A flow which is initially irrotational remains irrotational

Circulation is advected with the fluid in inviscid flows, and vorticity is "circulation per unit area".

If initially

$$\frac{D}{Dt}\oint \mathbf{u}\cdot d\mathbf{r} = \oint \frac{D}{Dt}(\mathbf{u}\cdot d\mathbf{r}) = \oint \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{r} + \oint \mathbf{u}\cdot\frac{D}{Dt}d\mathbf{r}\,.$$

for all closed circuits in some region of flow, it must remain so for all subsequent times.

Motion started from rest is initially irrotational (free from vorticity) and will therefore remain irrotational provided that it is inviscid.

Results following from Kelvin´s Theorem 4

The direction of the vorticity turns as the vortex line turns, and its magnitude increases as the vortex line is stretched.

The circulation round a thin vortex tube remains the same; as it stretches the area of section decreases and

$$\frac{\text{circulation}}{\text{area}} = \text{vorticity}$$

increases in proportion to the stretch.

Rotational and irrotational flow

Flow in which the vorticity is everywhere zero ($\nabla \times \omega = \mathbf{0}$) is called irrotational.

Other terms in use are vortex free; ideal; perfect.

Much of fluid dynamics used to be concerned with analyzing irrotational flows and deciding where these give a good representation of real flows, and where they are quite wrong.

We have neglected compressibility and viscosity.

It can be shown that the neglect of compressibility is not very serious even at moderately high speeds, but the effect of neglecting viscosity can be disastrous.

Viscosity diffuses the vorticity (much as conductivity diffuses heat) and progressively blurs the results derived above, the errors increasing with time.

There is no term in the Helmholtz equation

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u}$$

corresponding to the generation of vorticity.

The term $\omega \cdot \nabla \mathbf{u}$ represents processing by stretching and turning of vorticity already present).

In homogeneous fluids all vorticity must be generated at boundaries.

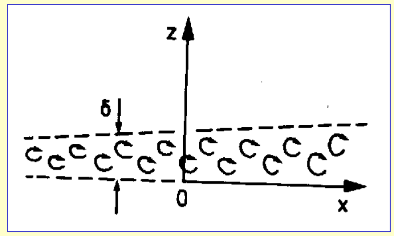
- In real (viscous) fluids, the vorticity is carried away from the boundary by diffusion and is then advected into the body of the flow.
- But in inviscid flow vorticity cannot leave the surface by diffusion, nor can it leave by advection with the fluid as no fluid particles can leave the surface.
- It is this inability of inviscid flows to model the diffusion/advection of vorticity generated at boundaries out into the body of the flow that causes most of the failures of the model.
- In inviscid flows we are left with a free slip velocity at the boundaries which we may interpret as a thin vortex sheet wrapped around the boundary.

Vortex sheets

Consider a thin layer of thickness δ in which the vorticity is large and is directed along the layer (parallel to Oy). The vorticity is

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

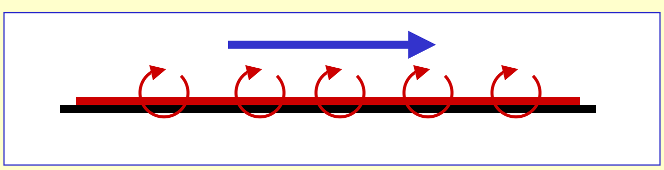
where $\partial u/\partial z$ is large.



We can suppose that within the vortex layer $u = u_o + \omega z$, changing from u_0 to $u_0 + \omega\delta$ between $z = 0$ and $z = \delta$, with mean vorticity

$$\overline{\eta} = \frac{(u_0 + \omega\delta) - u_0}{\delta} = \omega$$

The vortex layer provides a sort of roller action, though it is not of course rigid, and it also suffers high rate-of-strain.

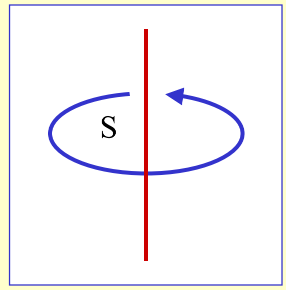


If we idealize this vortex layer by taking the limit $\delta \to 0$, $\omega \to \infty$, with $\omega\delta$ remaining finite, we obtain a vortex sheet, which is manifest only through the free slip velocity.

Such vortex sheets follow the contours of the boundary and clearly may be curved. They are infinitely thin sheets of vorticity with infinite magnitude across which there is finite difference in tangential velocity.

Line vortices

We can represent approximately also strong thin vortex tubes (e.g. tornadoes, waterspouts, draining vortices) by vortex lines without thickness.



The circulation in a circuit round the tube tends to a definite non-zero limit as the circuit area, $S \to 0$.

If the flow outside the vortex is irrotational then all circuits round the vortex have the same circulation, the strength κ of the vortex:

$$\oint \mathbf{u} \cdot d\mathbf{r} \to \kappa \quad \text{as} \quad C \to 0$$

$$\oint \mathbf{u} \cdot d\mathbf{r} \to \kappa \quad \text{as} \quad C \to 0$$

The velocity $\to \infty$ as the line vortex is approached, like $\kappa \times (\text{radial distance})^{-1}$

Example: Suppose that C is a circle of radius r and $\mathbf{u}$ is in the tangent direction with speed $v(r)$. Then

$$v(r) = \frac{\kappa}{2\pi r}$$

Vorticity, viscosity and boundary layers

- The effect of viscosity is to thicken vortex sheets and line vortices by diffusion
- However, the effect of diffusion is often slow relative to that of advection by the flow, and as a result large regions of flow will often remain free from vorticity.
- Vortex sheets at surfaces diffuse to form boundary layers in contact with the surfaces; or if free they often break up into line vortices.
- Boundary layers on bluff bodies often separate or break away from the body, forming a wake of rotational, retarded flow behind the body, and it is these wakes that are associated with the drag on the body.

Motion started from rest impulsively

- Viscosity (which is really just distributed internal fluid friction) is responsible for retarding or damping forces which cannot begin to act until the motion has started; i.e. take time to act.
- Hence any flow will be initially irrotational everywhere except at actual boundaries.
- Within increasing time, vorticity will be diffused form boundaries and advected and diffused out into the flow.
- Motion started from rest by an instantaneous impulse must be irrotational.

Proof

Integrate the Euler equation over the time interval $(t, t+\delta t)$

$$\int_t^{t+\delta t} \frac{D\mathbf{u}}{Dt}\,dt = \int_t^{t+\delta t} \mathbf{F}\,dt - \int_t^{t+\delta t} \frac{1}{\rho}\nabla p\,dt$$

or

$$[\mathbf{u}]^{\delta t} = \int_t^{t+\delta t} \mathbf{F}\,dt - \frac{1}{\rho}\nabla \int_t^{t+\delta t} p\,dt$$

In the limit $\delta t \to 0$ **for start-up by an instantaneous impulse, the impulse of the body force** $\to 0$ **(as the body force is unaffected by the impulsive nature of the start), whereupon**

$$[\mathbf{u}]^{\delta t} = \mathbf{u} - \mathbf{u}_0 = -\frac{1}{\rho}\nabla P$$

$$[\mathbf{u}]^{\delta t} = \mathbf{u} - \mathbf{u}_0 = -\frac{1}{\rho}\nabla P$$

The fluid responds instantaneously with the impulsive pressure field

$$P = \int^{\delta t} p\,dt$$

The impulse on a fluid element is $-P$ **per unit volume, and this produces a velocity from rest (if** $\mathbf{u}_o = \mathbf{0}$**) of**

$$\mathbf{u} = -\frac{1}{\rho}\nabla P$$

This is irrotational as $\nabla \times \mathbf{u} = -\frac{1}{\rho}\nabla \times (\nabla P) \equiv 0$

The End