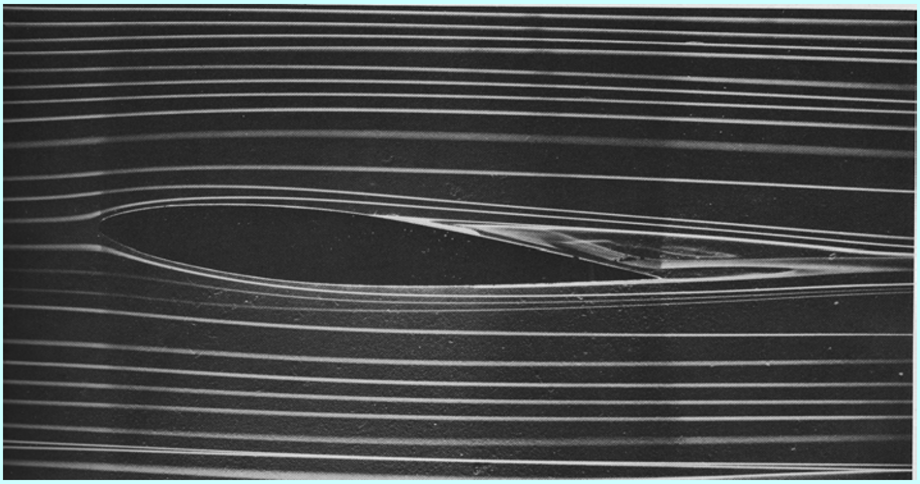
Introduction to Fluid Dynamics



Roger K. Smith

Skript - auf englisch!

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Aim

- To provide a survey of basic concepts in fluid dynamics as a preliminary to the study of dynamical meteorology.
- Based on a more extensive course of lectures prepared by Professor B. R. Morton of Monash University, Australia.
- A useful reference is: D. J. Acheson

Description of fluid flow

- The description of a fluid flow requires a specification or determination of the velocity field, i.e. a specification of the fluid velocity at every point in the region.

$$\mathbf{u} = \mathbf{u}(x, t)$$

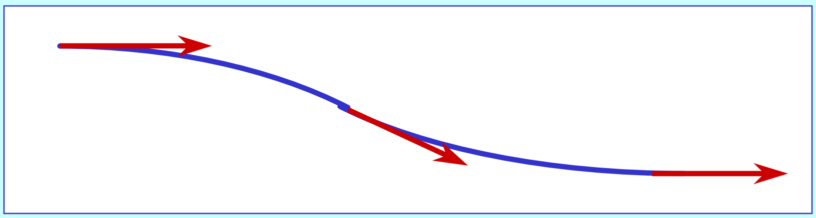
- In general, this will define a vector field of position and time.
- Steady flow occurs when **u** is independent of time - i.e.

$$\partial \mathbf{u} / \partial t \equiv 0$$

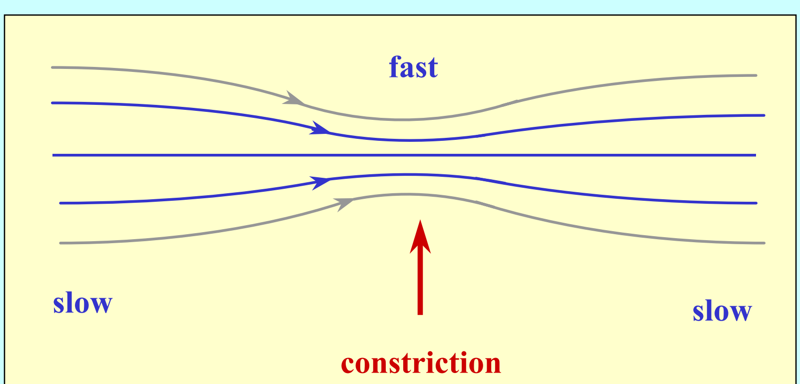
- Otherwise the flow is unsteady.

Streamlines

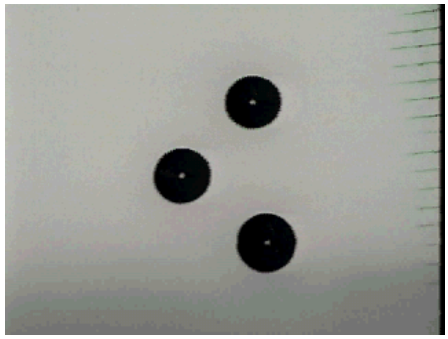
- **Streamlines - lines which at a given instant are everywhere in the direction of the velocity (analogous to electric or magnetic field lines).**



- **In steady flow the streamlines are independent of time, but the velocity can vary in magnitude along a streamline (as in flow through a constriction in a pipe).**



Streamlines



Particle paths

- **Particle paths** are lines traced out by "marked" particles as time evolves.
- In **steady flow particle paths are identical to streamlines.**
- In **unsteady flow they are different**, and sometimes very different.
- Particle paths are visualized in the laboratory using small floating particles of the same density as the fluid.
- Sometimes they are referred to as **trajectories**.

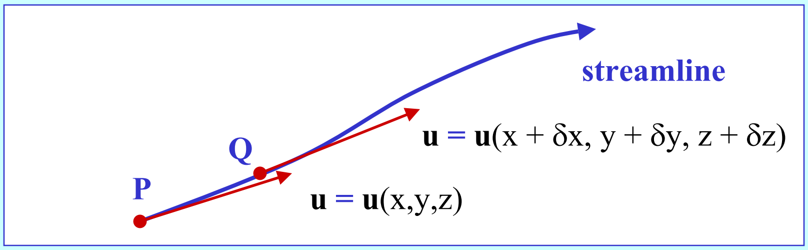
Filament lines or streaklines

- Filament lines or streaklines are traced out over time by all particles passing through a given point.
- They may be visualized, for example, using a hypodermic needle and releasing a slow stream of dye.
- In steady flow these are streamlines.
- In unsteady flow they are neither streamlines nor particle paths.

Streaklines

Equations for streamlines

- The streamline through the point P, say (x,y,z), has the direction of $\mathbf{u} = \mathbf{u}(x,y,z)$.



$$\delta x = u\,\delta t\,,\quad \delta y = v\,\delta t\,,\quad \delta z = w\,\delta t$$

$$\Rightarrow \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Parametric representation of the streamlines

$$\int \frac{dx}{u} = \int dt, \quad \int \frac{dy}{v} = \int dt, \quad \int \frac{dz}{w} = \int dt$$

Example: Find the streamlines for the velocity field **u** = (–Ωy, Ωx, 0), where Ω is a constant

Solution

$$\frac{dx}{-\Omega y} = \frac{dy}{\Omega x} = \frac{dz}{0}$$

$$\int \Omega(xdx + ydy) = 0$$

$$x^2 + y^2 = \Gamma(z)$$

Γ an arbitrary function

$$\int dz = 0$$

z = constant

Streamlines are circles $x^2 + y^2 = c^2$, on planes z = constant

Fluids as continuous media

- Fluids are molecular in nature, but they can be treated as **continuous media** for most practical purposes.
- The exception is rarefied gases.

Compressibility

- Real fluids generally show some **compressibility** defined as

$$\kappa = \frac{1}{\rho}\frac{d\rho}{dp} = \frac{\text{changes in density per unit change in pressure}}{\text{density}}$$

- At normal atmospheric flow speeds, the compressibility of air is a relative small effect and for liquids it is generally negligible.
- The exception is rarefied gases.
- Note that sound waves owe their existence to compressibility effects as do "supersonic bangs", produced by aircraft flying faster than sound.

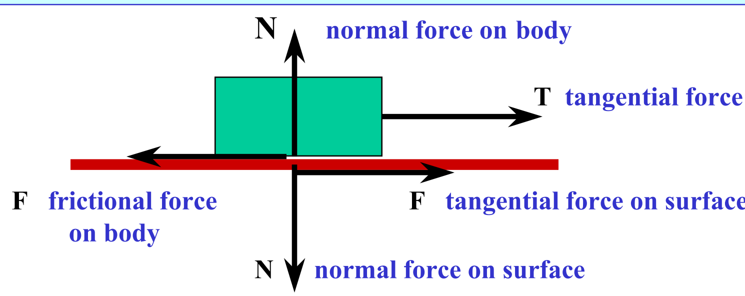
Incompressible fluids

- For many purposes it is accurate to assume our fluids are **incompressible**, i.e. they suffer no change in density with pressure
- For the present we shall assume also that they are **homogeneous**

→ density ρ = constant

Friction in solids

- When one solid body slides over another, frictional forces act between them to reduce the relative motion.
- Friction acts also when layers of fluid flow over one another.
- When two solid bodies are in contact (more precisely when there is a normal force acting between them) at rest, there is a threshold tangential force *below which* relative motion will not occur. It is called the **limiting friction.**
- **Example**: a solid body resting on a flat surface under the action of gravity.



- As T is increased from zero, F = T until T = μN.
- The coefficient of limiting friction, μ depends on the degree of roughness between the surface.
- For T > μN, the body will overcome the frictional force and accelerate.

Fluids compared with solids

- A distinguishing characteristic of most fluids in their inability to support tangential stresses between layers without motion occurring; i.e. there is no analogue of limiting friction.
- Exception: certain types of so-called visco-elastic fluids e.g. paint.

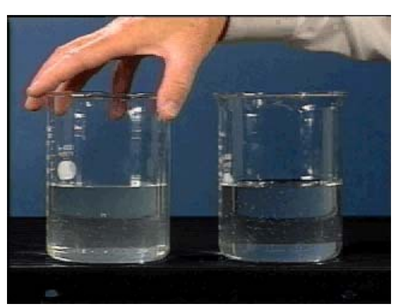
A visco-elastic fluid

A mixture of water and corn starch, when placed on a flat surface, flows as a thick, viscous fluid. However, when the mixture is rapidly disturbed, it appears to fracture and behave more like a solid.

Friction in fluids

- **Fluid friction** is characterized by **viscosity** which is a measure of the magnitude of tangential frictional forces in flows with velocity gradients.
- Viscous forces are important in many flows, but least important in flow past "**streamlined**" bodies.
- We shall be concerned mainly with inviscid flows where friction is not important.
- It is essential to acquire some idea of the sort of flow in which friction may be neglected without completely misrepresenting the behaviour. **Its neglect is risky!**

Viscosity

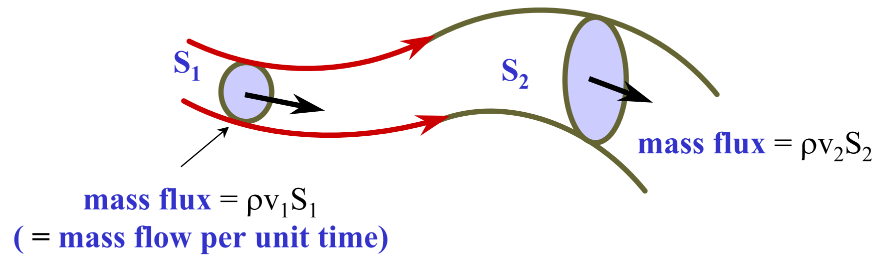


Water

Silicone oil
10,000 times more viscous

Incompressible flow

- Consider a **stream tube** - an element of fluid bounded by a "tube of streamlines".



→ vS = constant

- In **steady flow**, no fluid can cross the walls of the tube (they are everywhere in the direction of flow).

In the limit, for stream tubes of small cross-section,

vS = constant along an elementary stream tube.

Where streamlines contract the velocity increases, where they expand it decreases. →

The streamline pattern contains a great deal of information about the velocity distribution.

All vector fields with the property that

(vector magnitude) * (area of tube)

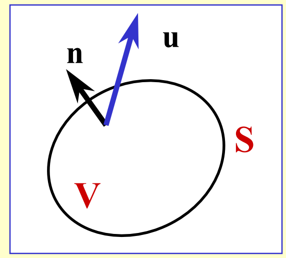
remains constant along a tube are called **solenoidal**.

The velocity field for an incompressible fluid is solenoidal.

Conservation of mass - the continuity equation

Apply the divergence theorem

$$\int_V \nabla \cdot \mathbf{u} \, dV = \int_S \mathbf{u} \cdot \mathbf{n} \, ds$$



to an arbitrarily chosen volume V with closed surface S.

Assume: incompressible fluid and no mass sources or sinks within S.

There can be neither continuing accumulation of fluid within V nor continuing loss.

The net flux of fluid across the surface S must be zero, i.e.,

$$\int_S \mathbf{u} \cdot \mathbf{n} \, dS = 0$$

$$\int_V \nabla \cdot \mathbf{u} \, dV = 0$$

This holds for an arbitrary volume V

$$\nabla \cdot \mathbf{u} = 0$$

This is the continuity equation for a homogeneous, incompressible fluid.

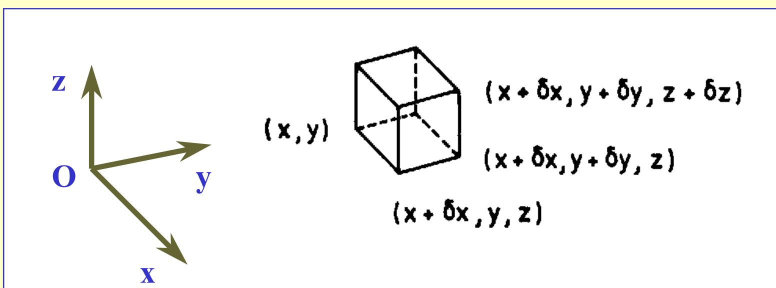
Equation of motion for an inviscid homogeneous fluid

- The equation of motion is an expression of Newton´s second law of motion:

mass × acceleration = force

- To apply this law we must focus our attention on a particular element of fluid.

We consider a small rectangular element which at time t has vertex at P [= (x, y, z)] and edges of length δx, δy, δz.

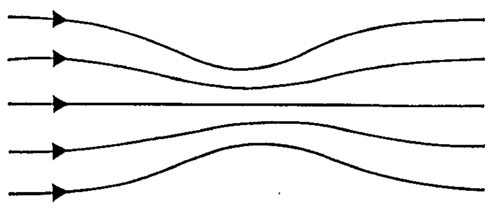


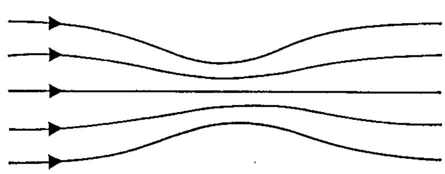
The mass of this element is $\rho\, \delta x\, \delta y\, \delta z$.

Assume ρ constant

ρ is the fluid **density** = mass per unit volume

- The velocity in the fluid, $\mathbf{u} = \mathbf{u}(x, y, z, t)$ is a function both of position (x, y, z) and time t.
- From this we must derive a formula for the acceleration of the element of fluid which is changing its position with time.
- **Example:** Consider steady flow through a constriction in a pipe.





- Fluid elements must accelerate into the constriction as the streamlines close in and decelerate beyond as they open out again.
- Thus, in general, the acceleration of an element (i.e., the rate-of-change of $\mathbf{u}$ with time for that element) includes a rate-of-change at a fixed position $\partial \mathbf{u}/\partial t$ plus a change associated with its change of position with time.
- I will derive an expression for the total rate-of-change shortly.

Forces acting on the fluid element

The forces acting on the elements δx, δy, δz **consist of:**

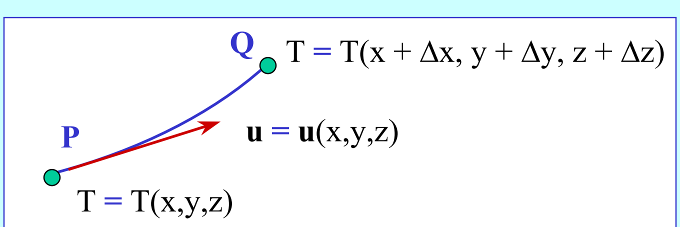
(i) **body forces, which are forces per unit mass acting throughout the fluid because of external causes, such as the gravitational weight, and**

(ii) **contact forces acting across the surface of the element from adjacent elements.**

These are discussed further below.

Rate-of-change moving with the fluid

- **Consider first the rate-of-change of a scalar property, for example the temperature of a fluid,** $T = T(x, y, z, t)$, **following a fluid element.**
- **Suppose that an element of fluid moves from** P [$= (x, y, z)$] **at time** t **to** Q [$= (x + \Delta x, y + \Delta y, z + \Delta z)$] **at time** $t + \Delta t$.



- If we stay at a particular point (x_0, y_0, z_0), then T is effectively a function of t only.
- If we move with the fluid, T(x,y,z,t) is a function both of position (x,y,z) and time t.

The total change in T between P and Q in time Δt is

$$\Delta T = T_Q - T_P = T(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - T(x, y, z, t)$$

The total rate of change of T moving with the fluid is

$$\lim_{\Delta t \to 0} \frac{\Delta T}{\Delta t} = \lim_{\Delta t \to 0} \frac{T(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - T(x, y, z, t)}{\Delta t}$$

For small increments x, y, z, t we may use a Taylor expansion

$$T(x + \Delta x, y + \Delta y, z + +\Delta z, t + \Delta t) - T(x, y, z, t) + \left[\frac{\partial T}{\partial t}\right]_P \Delta t +$$

$$\left[\frac{\partial T}{\partial x}\right]_P \Delta x + \left[\frac{\partial T}{\partial y}\right]_P \Delta y + \left[\frac{\partial T}{\partial z}\right]_P \Delta z + \text{higher order terms in } \Delta x, \Delta y, \Delta z, \Delta t\,.$$

The rate-of-change moving with the fluid element

$$= \lim_{\Delta t \to 0} \left[\frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z\right] / \Delta t$$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z},$$

since higher order terms $\to 0$.

Note that $u = dx/dt$, $v = dy/dt$, $w = dz/dt$, where $\mathbf{r} = \mathbf{r}(t)$ is the coordinate vector of the moving fluid element.

Total rate-of-change moving with the fluid

To emphasize that we mean the **total rate of change moving with the fluid** we write

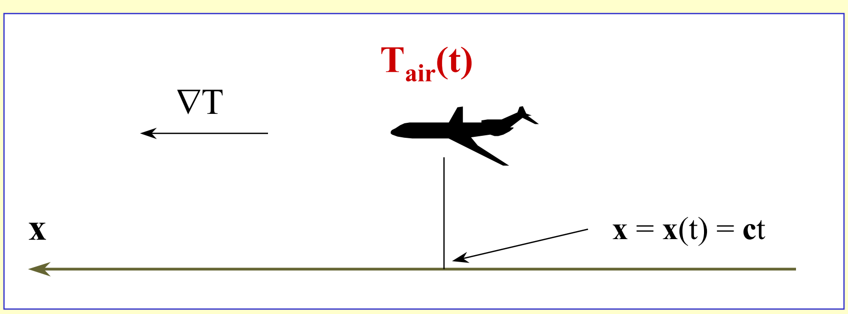
$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

the **local rate-of-change** with time at a fixed position

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \mathbf{u}\cdot\nabla T$$

is the **advective rate-of-change** associated with the movement of the fluid element

Imagine that one is flying in an aeroplane that is moving with velocity $c(t) = (dx/dt, dy/dt, dz/dt)$ and that one is measuring the air temperature with a thermometer mounted on the aeroplane.



If the air temperature changes both with space and time, the rate-of-change of temperature that we would measure from the aeroplane would be

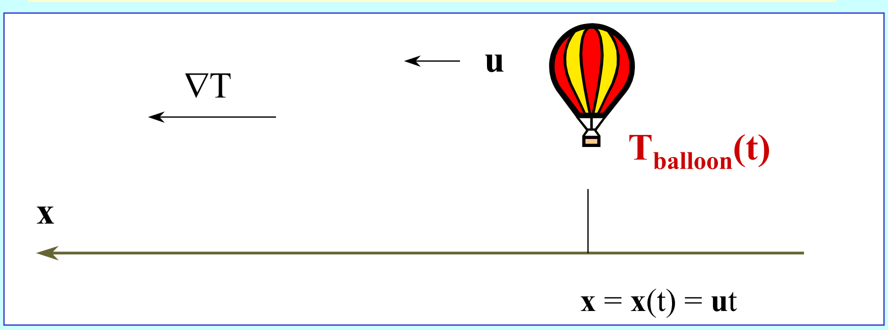
$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{c}\cdot\nabla T$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{c} \cdot \nabla T$$

The rate at which the temperature is varying locally; i.e., at a fixed point in space.

The rate-of-change that we observe on account of our motion through a spatially-varying temperature field.

Suppose that we move through the air at a speed exactly equal to the local flow speed u, i.e., we move with an air parcel as in a balloon.



The rate-of-change of any quantity related to the air parcel, for example its temperature or its x-component of velocity is given by

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

operating on the quantity in question.

The total derivative

We call $\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ the total derivative and often use the notation D/Dt for it.

Thus the x-component of acceleration of the fluid parcel is

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u$$

while the rate at which its potential temperature changes is expressed by

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta$$

Case of potential temperature

In many situations, θ is conserved following a fluid parcel, i.e.,

$$\frac{D\theta}{Dt} = 0$$

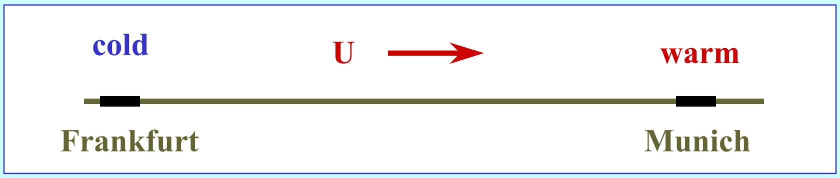
$$\frac{\partial \theta}{\partial t} = -\mathbf{u} \cdot \nabla \theta$$

The rate-of-change of potential temperature at a point is due entirely to advection.

The change occurs solely because fluid parcels arriving at the point come from a place where the potential temperature is different.

Example

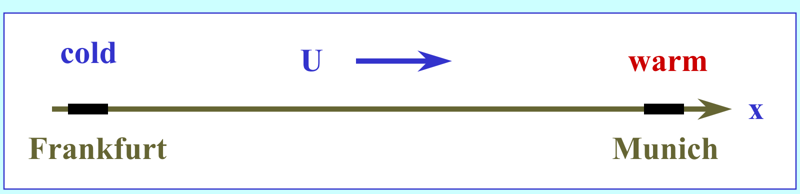
Suppose that there is a uniform potential temperature gradient of -1°C/100 km between Munich and Frankfurt, i.e. the air temperature in Frankfurt is cooler.



- **If the wind blows directly from Frankfurt to Munich, the potential temperature in Munich will fall steadily at a rate proportional to the wind speed and to the temperature gradient.**
- **If the air temperature in Frankfurt is higher than in Munich, then the potential temperature in Munich will rise.**

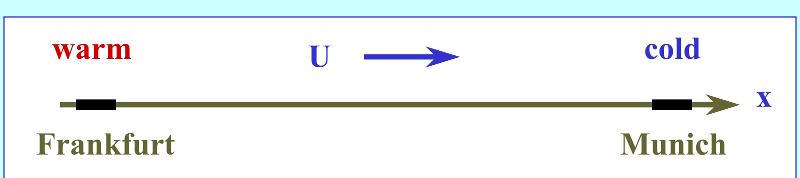
cold air advection

$$\frac{\partial \theta}{\partial t} = -U \frac{\partial \theta}{\partial x} < 0$$



warm air advection

$$\frac{\partial \theta}{\partial t} = -U \frac{\partial \theta}{\partial x} > 0$$



Example

Show that $$\frac{DF}{Dt} = \frac{\partial \mathbf{F}}{\partial T} + (\mathbf{u} \cdot \nabla)\mathbf{F}$$

represents the total rate-of-change of any vector field F moving with the fluid velocity (velocity field u), and in particular that the acceleration (or total change in u moving with the fluid) is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

Solution

The previous result for the rate-of-change of a scalar field can be applied to each of the component of F, or to each of the velocity components (u,v,w) **and these results follow at once.**

Example

Show that $$\frac{D\mathbf{r}}{Dt} = \mathbf{u}$$

Solution

$$\frac{D\underline{r}}{Dt} = \frac{\partial \mathbf{r}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{r}$$

$$= 0 + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(x, y, z)$$

$$= (u, v, w)$$

as x, y, z, t **are independent variables.**

Question

Are the two x-components in rectangular Cartesian coordinates,

$$(\mathbf{u}\cdot\nabla\mathbf{u})_x \qquad \textbf{and} \qquad \left(\nabla\tfrac{1}{2}\mathbf{u}^2\right)_x$$

the same or different?

Note that $\quad (\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\tfrac{1}{2}\mathbf{u}^2\right) - \mathbf{u}\wedge\omega$