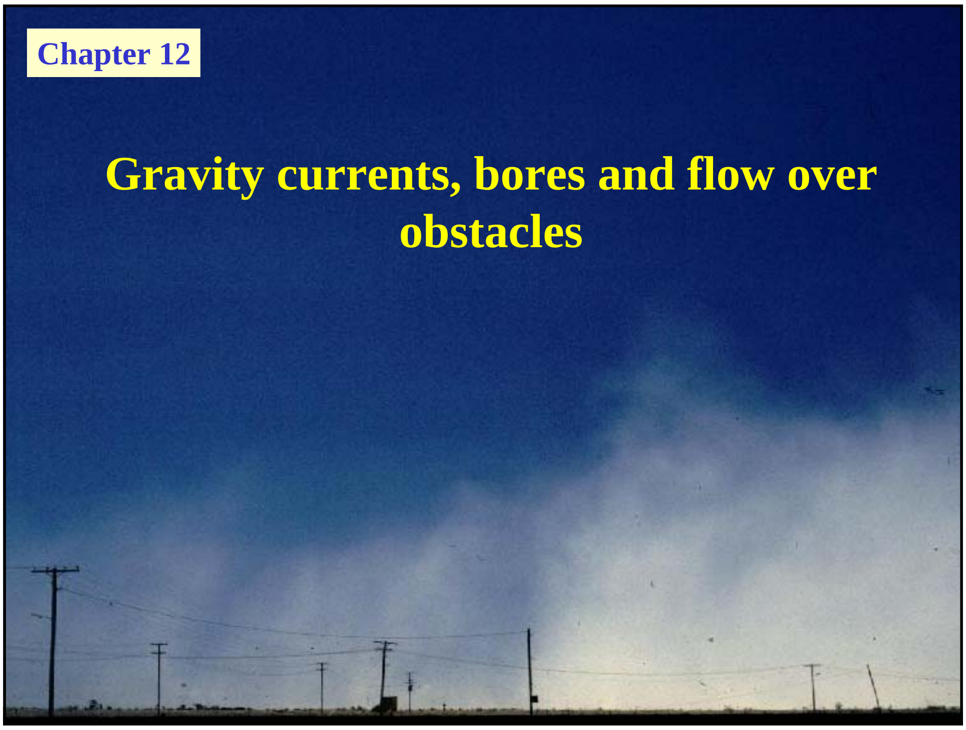
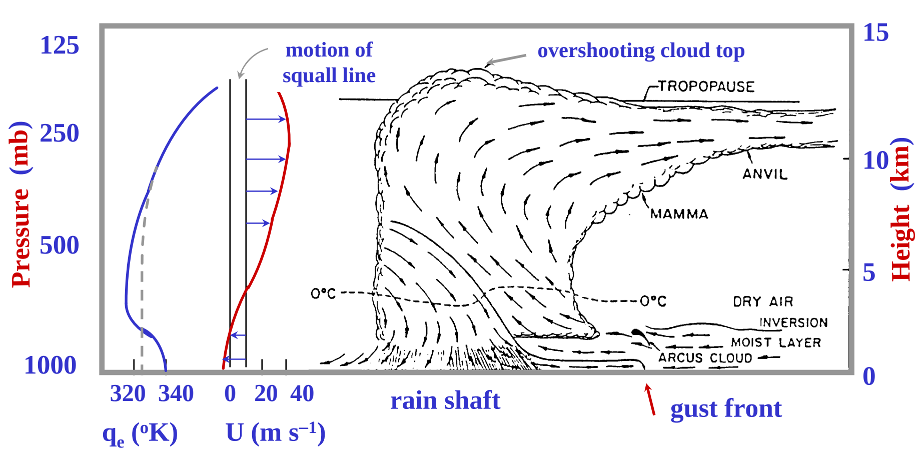


Hydraulic theory

- Now we examine some simple techniques from the theory of hydraulics to study a range of small scale atmospheric flows, including gravity currents, bores (hydraulic jumps) and flow over orography.

Gravity currents

- A **gravity current** is produced when a relatively dense fluid moves quasi-horizontally into a lighter fluid. **Examples**: =>
- **Sea breezes**, produced when air over the land is heated during the daytime relative to that over the sea;
- **Katabatic** (or **drainage**) **winds**, produced on slopes or in mountain valleys when air adjacent to the slope cools relative to that at the same height, but further from the slope;
- **Thunderstorm outflows**, produced beneath large storms as air below cloud base is cooled by the evaporation of precipitation into it and spreads horizontally, sometimes with a strong gust front at its leading edge.
- A concise **review** is given by **Simpson (1987)**.



Schematic diagram of a squall-line thunderstorm showing the low-level cold air outflow

Bores

- The more familiar examples of bores occur on water surfaces.
- Examples are:

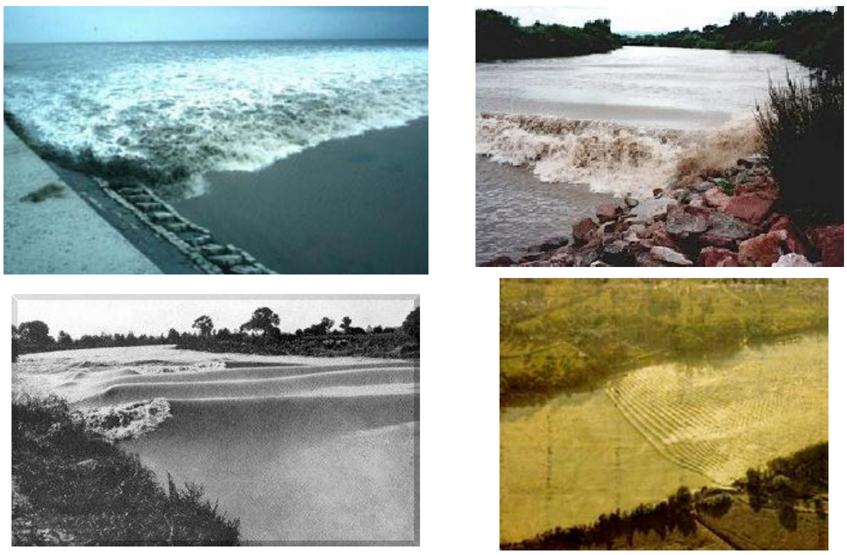
  - bores on tidal rivers,

  - quasi-stationary bores produced downstream of a weir, and

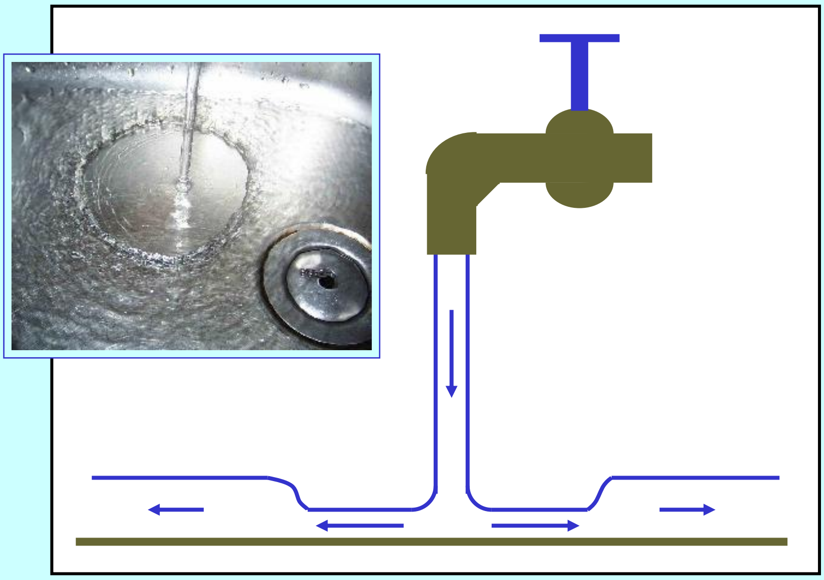
  - the bore produced in a wash basin when the tap is turned on and a laminar stream of water impinges on the bottom of the basin.

- Perhaps the best known of atmospheric bores is the so-called 'morning glory' of northern Australia.

Bores on rivers

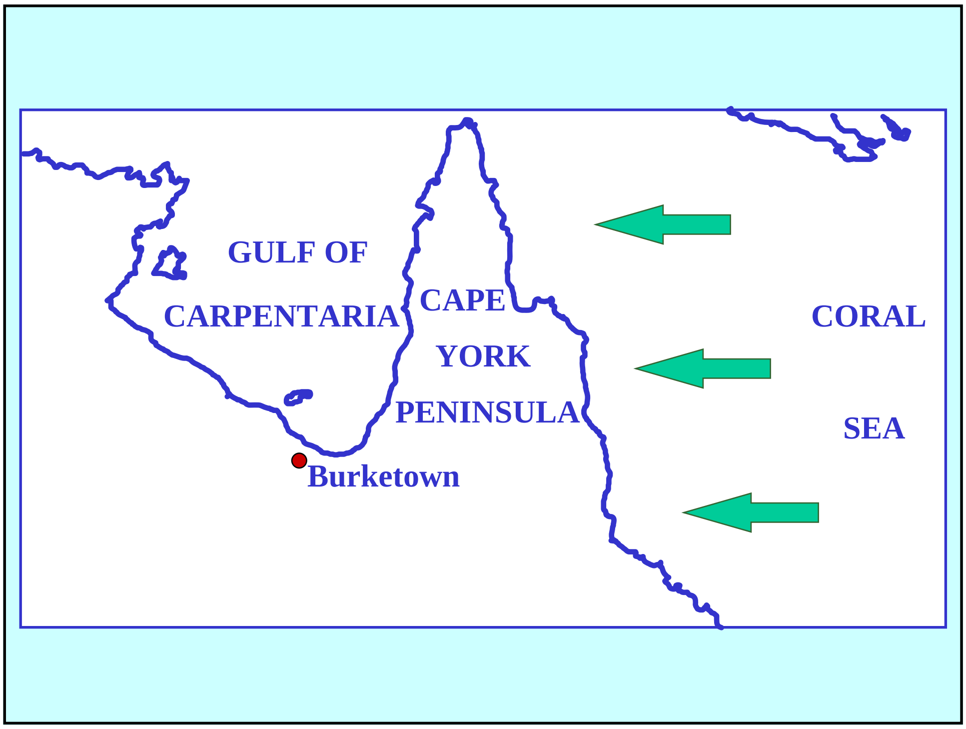


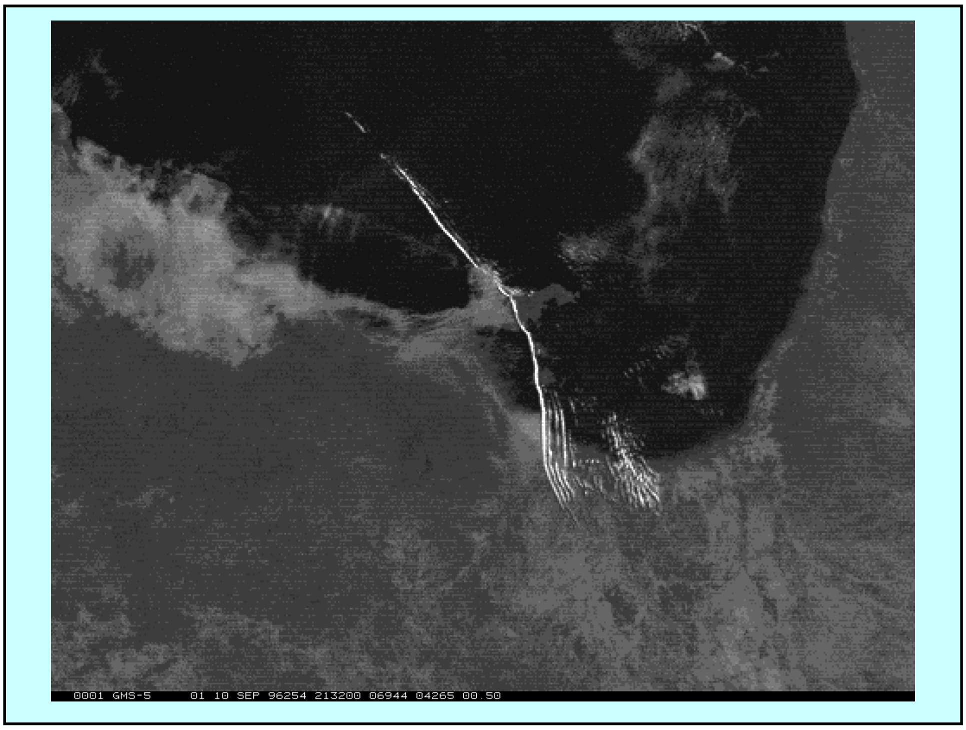
Mascaret bore, France



- The morning glory is produced by the collision of two sea breezes over Cape York Peninsula and is formed on a low-level stable layer, typically 500 m deep.
- The bore is regularly accompanied by spectacular roll clouds.
- Similar phenomena occur elsewhere, but not with such regularity in any one place.
- Another atmospheric example is when a stratified airstream flows over a mountain ridge.
- Under certain conditions a phenomenon akin to a bore, or hydraulic jump, may occur in the lee of the ridge.

The "Morning Glory"


GULF OF
CARPENTARIA
CAPE
YORK
PENINSULA
CORAL
SEA
Burketown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Bernoulli's theorem

- Euler's equation for an inviscid rotating flow on an f-plane is

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + f\mathbf{k}\wedge\mathbf{u} = -(1/\rho)\nabla p_T - \mathbf{g}$$

Use the vector identity

$\boldsymbol{\omega}$ = curl $\mathbf{u}$

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(\tfrac{1}{2}\mathbf{u}^2) + \boldsymbol{\omega}\wedge\mathbf{u}$$

$$\partial_t \mathbf{u} + \nabla(\tfrac{1}{2}\mathbf{u}^2 + gz) + (1/\rho)\nabla p_T + (f\mathbf{k} + \omega)\wedge\mathbf{u} = 0$$

Assume steady flow ($\partial_t \mathbf{u} = \mathbf{0}$) and a homogeneous fluid (ρ = constant): then

$$\mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}^2 + p_T/\rho + gz) = 0$$

$$\mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}^2 + p_T/\rho + gz) = 0$$

For the steady flow of homogeneous, inviscid fluid, the quantity H given by

$$H = \tfrac{1}{2}\mathbf{u}^2 + p_T/\rho + gz$$

is a constant along a streamline.

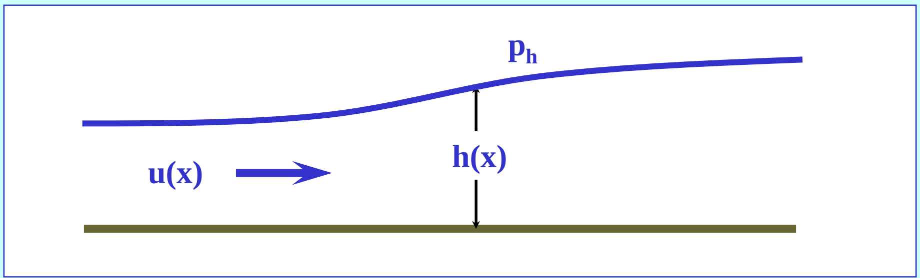
- This is Bernoulli's theorem.
- The quantity H is called the total head along the streamline and is a measure of the total energy per unit volume on that streamline.
- Note: it may be that a flow in which we are interested is unsteady, but can be made steady by a Galilean coordinate transformation.
- Then Bernoulli's theorem can be applied in the transformed frame.

Flow force

- **Consider steady motion of an inviscid rotating fluid in two dimensions.**
- **In flux form the x-momentum equation is**

$$\partial_x(\rho u^2) + \partial_z(\rho uw) - \rho f v = -\partial_x p_T$$

Consider the motion of a layer of fluid of variable depth $h(x)$**; see next figure: =>**



Integrating $\partial_x(\rho u^2) + \partial_z(\rho uw) - \rho f v = -\partial_x p_T$ **with respect to** z

$$\int_0^h \partial_x[\rho u^2 + p_T]dz = -[\rho uw]_0^h + \int_0^h f\rho v\,dz$$

$$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz - [\rho u^2 + p_T]_h \frac{\partial h}{\partial x}$$

$$= -(\rho u^2)_h \frac{\partial h}{\partial x} + f\int_0^h \rho v\,dz .$$

$$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]\,dz - [\rho u^2 + p_T]_h \frac{\partial h}{\partial x}$$

$$= -(\rho u^2)_h \frac{\partial h}{\partial x} + f\int_0^h \rho v\,dz\,.$$

In particular, if $f = 0$**, then**

$$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]\,dz = p_h \frac{\partial h}{\partial x}$$

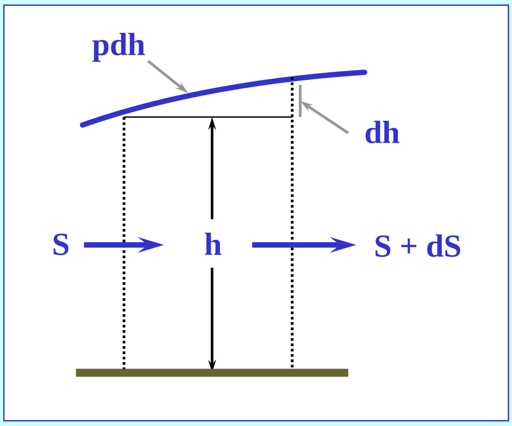
Define

$$S = \int_0^h [\rho u^2 + p_T]\,dz$$

then

$$\frac{\partial S}{\partial x} = p_h \frac{\partial h}{\partial x}$$

Considering the control volume =>



$$\frac{\partial S}{\partial x} = p_h \frac{\partial h}{\partial x} \quad \Rightarrow \quad dS = p_h\,dh$$

$$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz = p_h \frac{\partial h}{\partial x} \quad \textbf{and} \quad S = \int_0^h [\rho u^2 + p_T]dz$$

Include a frictional force, $-\rho D$**, per unit volume**

then $$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz = p_h \frac{\partial h}{\partial x} - \underbrace{\int_0^h \rho D\, dz}_{\text{call } D^*}$$

$$\Rightarrow \quad dS = p_h dh - D^* dx$$

Two useful deductions of $\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz = p_h \frac{\partial h}{\partial x}$ **are:**

1. **If** $h = \text{constant}$**,** $\int_0^h [\rho u^2 + p_T]dz = \text{constant}$

2. **If** $p_h = \text{constant}$**,** $\int_0^h [\rho u^2 + p_T]dz - p_h h = \text{constant}$

Summary of important results

Bernoulli's theorem

For the steady flow of homogeneous, inviscid fluid

$$H = \tfrac{1}{2}\mathbf{u}^2 + p_T/\rho + gz$$

is a constant along a streamline.

Flow force

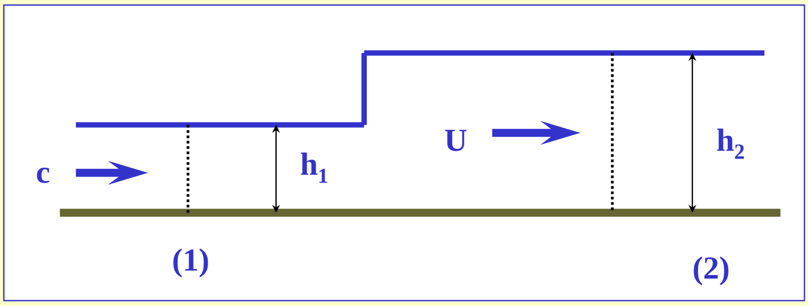
$$S = \int_0^h [\rho u^2 + p_T]dz$$

1. **If** $h = \text{constant}$**,** $\int_0^h [\rho u^2 + p_T]dz = \text{constant}$

2. **If** $p_h = \text{constant}$**,** $\int_0^h [\rho u^2 + p_T]dz - p_h h = \text{constant}$

Theory of hydraulic jumps, or bores

- **We idealize a jump by an abrupt transition in fluid depth.**



Schematic diagram of a hydraulic jump, or bore

- **Express mathematically in terms of the Heaviside step function**

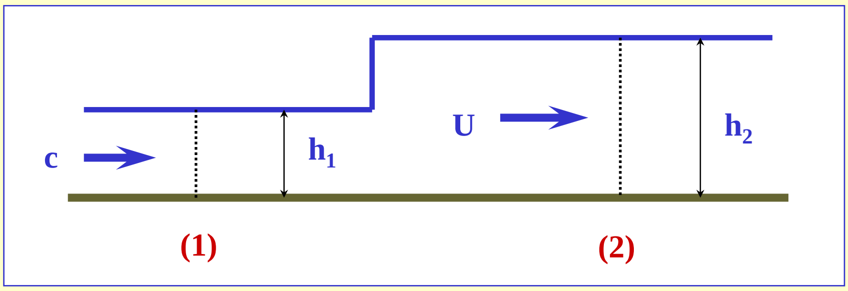
$$h(x) = h_1 + (h_2 - h_1)H(x)$$

$$\frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz = p_h \frac{\partial h}{\partial x}$$

atmospheric pressure

$$\Rightarrow \quad \frac{\partial}{\partial x}\int_0^h [\rho u^2 + p_T]dz = p_a(h_2 - h_1)\delta(x)$$

Dirac delta function



Integrate with respect to x between (1) and (2) =>

$$\int_0^{h_2} [\rho u^2 + p_T]_2\, dz = \int_0^{h_1} [\rho u^2 + p_T]_1\, dz + p_a(h_2 - h_1)$$

$$\int_0^{h_2}[\rho u^2 + p_T]_2\,dz = \int_0^{h_1}[\rho u^2 + p_T]_1\,dz + p_a(h_2 - h_1)$$

$$U^2 h_2 + \tfrac{1}{2} g h_2^2 = c^2 h_1 + \tfrac{1}{2} g h_1^2$$

We have used the fact that the flow at positions (1) and (2) is horizontal and therefore the pressure is hydrostatic.

Continuity of mass (and hence volume) gives

$$ch_1 = Uh_2$$

Solve for c^2 and U^2 in terms of h_1 and h_2

$$c = [\tfrac{1}{2} g(h_1 + h_2) h_2 / h_1]^{1/2}$$

and $$U = [\tfrac{1}{2} g(h_1 + h_2) h_1 / h_2]^{1/2}$$

Use Bernoulli's theorem =>

The change in total head along the surface streamline is

$$\delta H = p_a + \tfrac{1}{2}\rho U^2 + \rho g h_2 - p_a - \tfrac{1}{2} p c^2 - \rho g h_1,$$

$$= \tfrac{1}{2}\rho[\tfrac{1}{2} g(h_1 + h_2)\left[\tfrac{h_1}{h_2} - \tfrac{h_2}{h_1}\right] + 2g(h_2 - h_1)],$$

$$= \tfrac{1}{4}\rho g\left[\frac{(h_1{}^2 - h_2{}^2)(h_2 + h_1)}{h_1 h_2} + 4(h_2 - h_1)\right],$$

$$= \tfrac{1}{4}\rho g(h_1 - h_2)^3 \cdot \frac{1}{h_1 h_2}$$

$$< 0 \text{ if } h_1 < h_2 .$$

Energy is lost at the jump

Energy loss

- The energy lost supplies the source for the turbulent motion at the jump that occurs in many cases.
- For weaker bores, the jump may be accomplished by a series of smooth waves.
- Such bores are termed **undular**.
- In these cases the energy loss is radiated away by the waves. See Lighthill, 1978, §2.12.

- It follows from

$$\delta H = \tfrac{1}{4}\rho g(h_1 - h_2)^3 \cdot \frac{1}{h_1 h_2} < 0 \text{ if } h_1 < h_2 \ .$$

that the depth of fluid must **increase**, since a decrease would require an energy supply.

Then $c = [\tfrac{1}{2} g(h_1 + h_2) h_2 / h_1]^{1/2}$ and

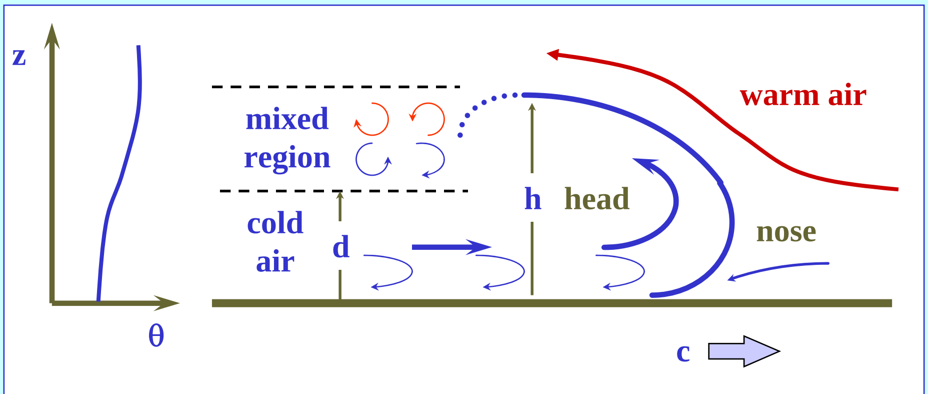
$$U = [\tfrac{1}{2} g(h_1 + h_2) h_1 / h_2]^{1/2}$$

$\Rightarrow$ $c > \sqrt{gh_1}$ and $U < \sqrt{gh_2}$

$$c > \sqrt{gh_1} \quad \textbf{and} \quad U < \sqrt{gh_2}$$

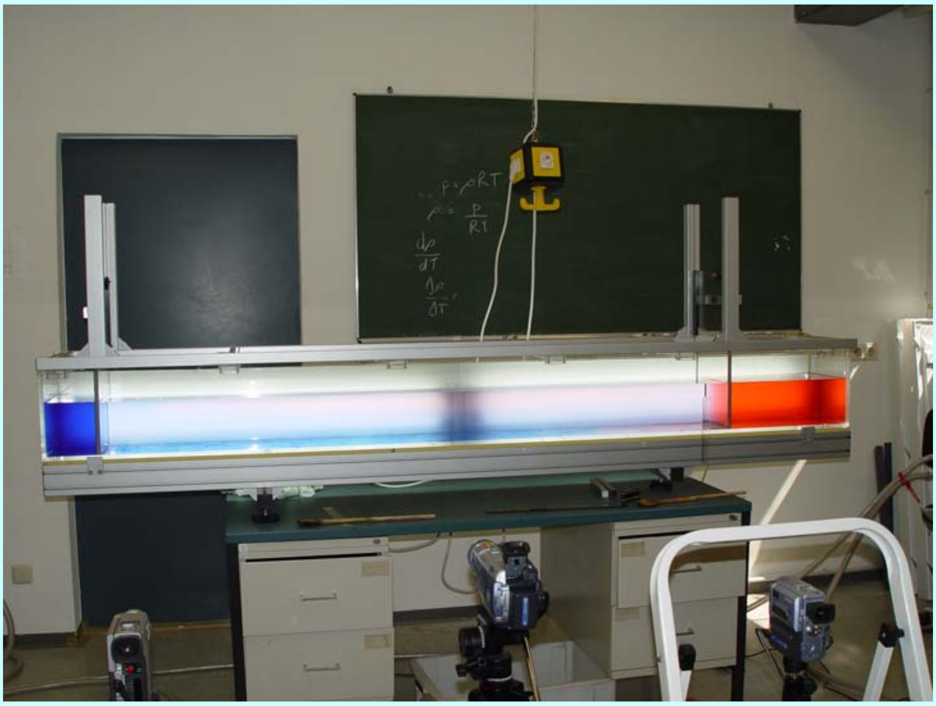
- Recall that $\sqrt{gh}$ is the phase speed of long gravity waves on a layer of fluid of depth h.
- On the upstream side of the bore, gravity waves cannot propagate against the stream whereas, on the downstream side they can.
- Accordingly we refer to the flow upstream as **supercritical** and that downstream as **subcritical**.
- These terms are analogous to **supersonic** and **subsonic** in the theory of gas dynamics.

Theory of gravity currents



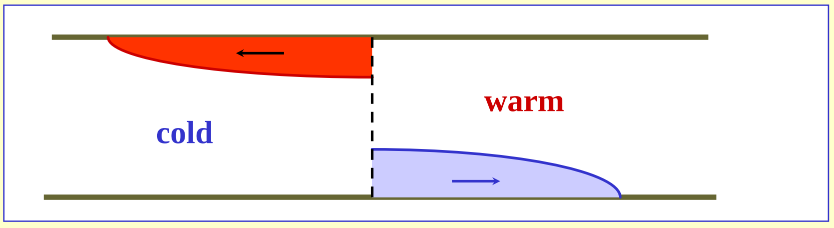
Schematic diagram of a steady gravity current

Haboob



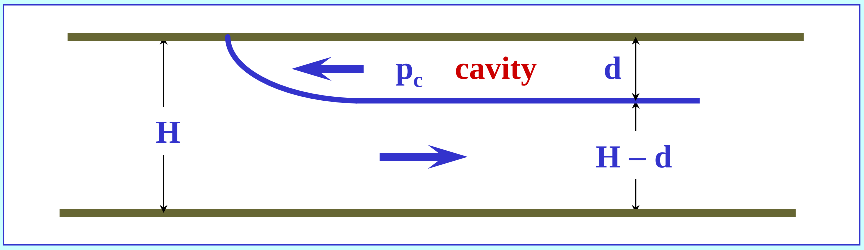
Show movies

- There is a certain symmetry between a gravity current of dense fluid that moves along the lower boundary in a lighter fluid, and a gravity current of light fluid that moves along the upper boundary of a denser fluid.



- The latter type occurs, for example, in a cold room when the door to a warmer room is opened.
- Then, a warm gravity current runs along the ceiling of the cold room and a cold gravity current runs along the floor of the warm room.

Cavity flow



- The simplest flow configuration of these types is the flow of an **air cavity** into a long closed channel of fluid.
- In this case we can neglect the motion of the air in the cavity to a good first approximation.
- In practice the cavity will move steadily along the tube with speed c, say.

Summary of important results

Bernoulli's theorem

For the **steady flow** of a **homogeneous, inviscid** fluid

$$H = \tfrac{1}{2}\mathbf{u}^2 + p_T/\rho + gz$$

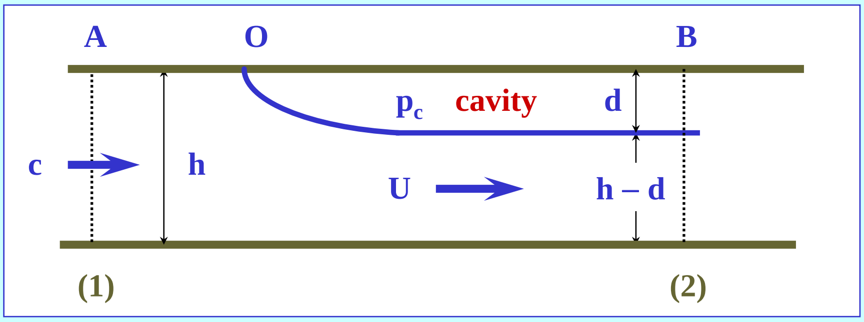
is a constant along a streamline.

Flow force

$$S = \int_0^h [\rho u^2 + p_T]\,dz$$

1. If $h = \text{constant}$, $\int_0^h [\rho u^2 + p_T]\,dz = \text{constant}$

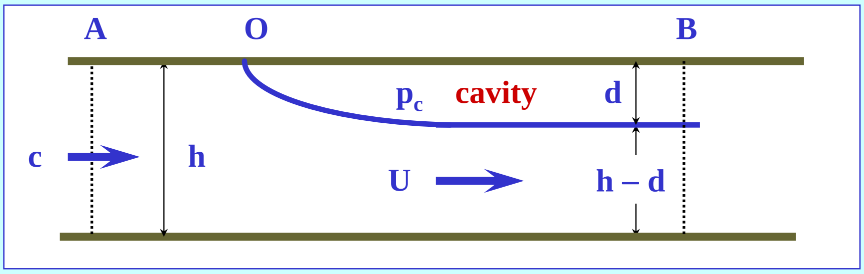
2. If $p_h = \text{constant}$, $\int_0^h [\rho u^2 + p_T]\,dz - p_h h = \text{constant}$



- Choose a frame of reference in which the cavity is stationary => the fluid upstream of the cavity moves towards the cavity with speed c.
- Apply Bernoulli's theorem along the streamline from A to O, => since z = h = constant,

$$p_A + \tfrac{1}{2}\rho c^2 = p_c$$

Note: O is a stagnation point => the pressure there is equal to the cavity pressure.

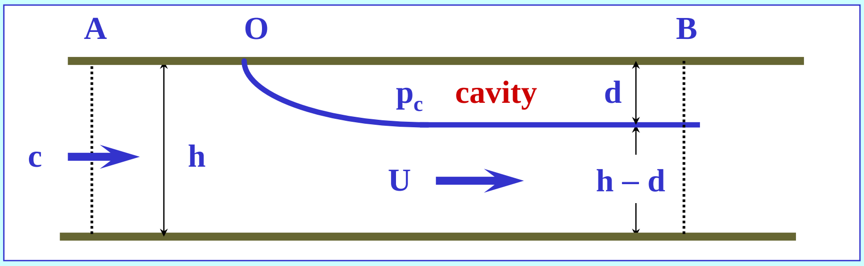


- Along the section between A and O

$$\int_0^h [\rho u^2 + p_T]_A \, dz = \int_0^h [\rho u^2 + p_T]_o \, dz$$

- Along the section between O and B

$$\int_0^h [\rho u^2 + p_T]_o \, dz - p_c h = \int_0^{h-d} [\rho u^2 + p_T]_B \, dz - p_c (h-d)$$



$$\int_0^h [\rho u^2 + p_T]_A \, dz = \int_0^{h-d} [\rho u^2 + p_T]_B \, dz + p_c d$$

At A and B where the flow is parallel (i.e. w = 0**), the pressure is hydrostatic**

$$\int_0^h p_T \, dz = p_h h + \tfrac{1}{2}\rho g h^2$$

u **independent of** z **and** ρ = constant,

$$\int_0^h \rho u^2 dz = \rho u^2 h$$

Using $\int_0^h p_T \, dz = p_h h + \tfrac{1}{2}\rho g h^2$ **and** $\int_0^h \rho u^2 dz = \rho u^2 h$

$$\int_0^h [\rho u^2 + p_T]_A \, dz = \int_0^{h-d} [\rho u^2 + p_T]_B \, dz + p_c d$$

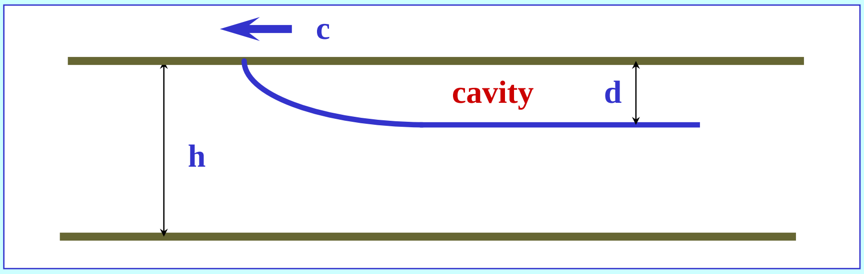
$$p c^2 h + \tfrac{1}{2}\rho g h^2 + \rho_A h = \rho U^2 (h-d) + \tfrac{1}{2}\rho g (h-d)^2 + p_c h$$

Continuity of mass (volume) implies that: $ch = U(h - d)$

Recall that $p_A + \tfrac{1}{2}\rho c^2 = p_c$

Then

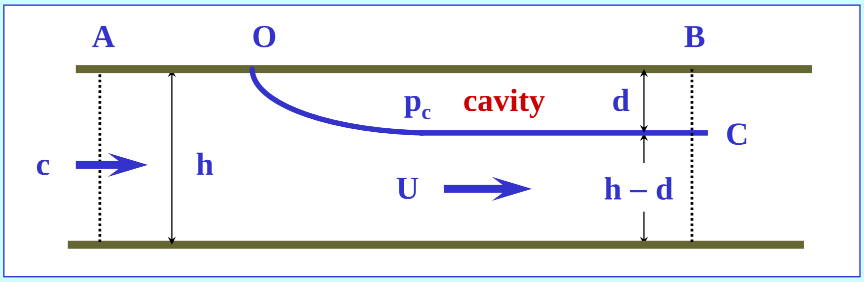
$$c^2 = gd\left[\frac{2h-d}{h}\right]\left[\frac{h-d}{h+d}\right] \quad \textbf{and} \quad U^2 = gd\left[\frac{2h-d}{h^2-d^2}\right]h$$



- **For a channel depth** h**, a cavity of depth** d **advances with speed c given by**

$$c^2 = gd\left[\frac{2h-d}{h}\right]\left[\frac{h-d}{h+d}\right]$$

- **Note that, as** $d/h \to 0$, $c^2/(gd) \to 2$**, appropriate to the case of a shallow cavity.**



- **Suppose that the flow behind the cavity is energy conserving.**
- **Then we can apply Bernoulli's theorem along the free streamline from O to C, whereupon**

$$p_c + \rho g h = p_c + \tfrac{1}{2}\rho U^2 + \rho g (h - d)$$

$$\Rightarrow \quad U^2 = 2gd$$

$$U^2 = 2gd \quad \text{and} \quad U^2 = gd\left[\frac{2h-d}{h^2-d^2}\right]h$$

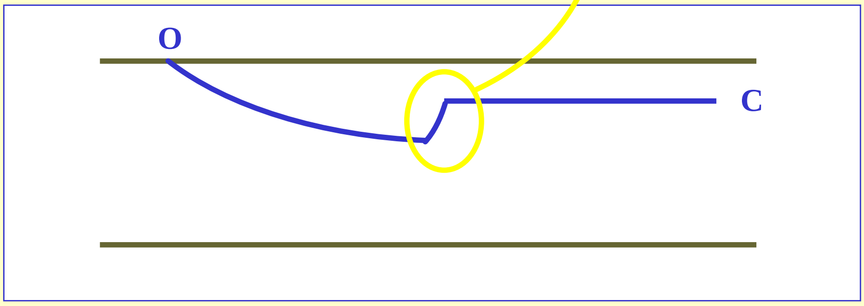
$$\Rightarrow \quad d = \tfrac{1}{2}H$$

Then $$c^2 = gd\left[\frac{2h-d}{h}\right]\left[\frac{h-d}{h+d}\right] \quad \Rightarrow \quad c^2 = \tfrac{1}{2}gd$$

$\Rightarrow$

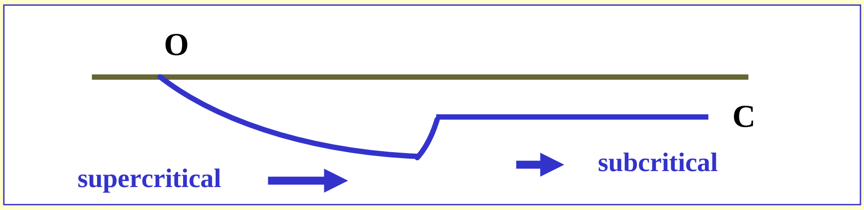
In an energy conserving flow, the cavity has a depth far downstream equal to one half the channel depth.

- **If the flow is not energy conserving, there must be a jump in the stream depth behind the cavity.**



Cavity flow with hydraulic jump

- According to the hydraulic jump theory, energy loss occurs at the jump and there must be a **loss of total head**, say, along the streamline **O** to **C**.



Then $$H_O = p_c + \rho g h \qquad H_C = p_c + \tfrac{1}{2}\rho U^2 + \rho g(h-d)$$

$$H_1 - H_2 = \rho\chi$$

$$\Rightarrow \quad p_c + \rho g h = p_c + \tfrac{1}{2}\rho U^2 + \rho g(h-d) + \rho\chi,$$

or $$-U^2 + gd = \chi.$$

$$U^2 = gd\left[\frac{2h-d}{h^2-d^2}\right]h \quad \Rightarrow$$

$$\frac{d(h-2d)}{h^2-d^2} = \frac{2\chi}{gd} > 0$$

$$\Rightarrow \quad d < \tfrac{1}{2}h$$ as expected

When the **cavity flow** is turned upside down, it begins to look like the **gravity-current configuration** - the jump and corresponding energy loss is analogous to the turbulent mixing region behind the gravity-current head.

- The foregoing theory can be applied to a gravity current of heavy fluid of density ρ_2 moving into lighter fluid of density ρ_1 if we neglect the motion within the heavier fluid.
- Then, g must be replaced by the reduced gravity

$$g' = g\frac{(\rho_2 - \rho_1)}{\rho_1}$$

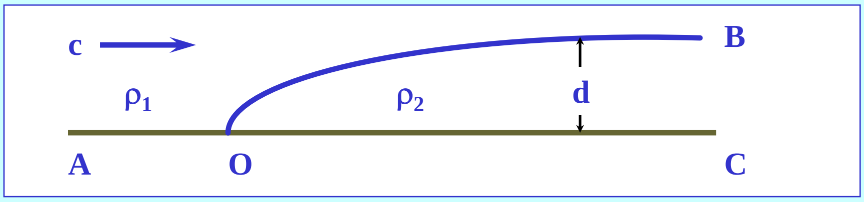
The deep fluid case

- The case of a shallow gravity current moving in a deep layer of lighter fluid cannot be obtained simply by taking the limit as $d/h \to 0$.
- This would imply an infinite energy loss according to the foregoing theory.
- Von-Kàrmàn considered this case and obtained the same speed c that would have been obtained by taking the limit $d/h \to 0$;

$$\Rightarrow \quad c/\sqrt{g'd} = \sqrt{2}$$

- Although this result is correct, von-Kàrmàn's derivation was incorrect as pointed out by Benjamin (1968).
- I will consider von-Kàrmàn's method before Benjamin's.

Deep fluid gravity current



- **Assumptions:**
  - there is no flow in the dense fluid
  - the pressure is hydrostatic and horizontally uniform

$$p_O = p_C = p_B + g\rho_2 d$$

- **Von-Kàrmàn applied Bernoulli's theorem between O and B (equivalent to the assumption of energy conservation) =>**

$$p_O = p_B + \tfrac{1}{2}\rho_1 c^2 + g\rho_1 d$$

Eliminate the pressure difference $p_O - p_B$ **using**

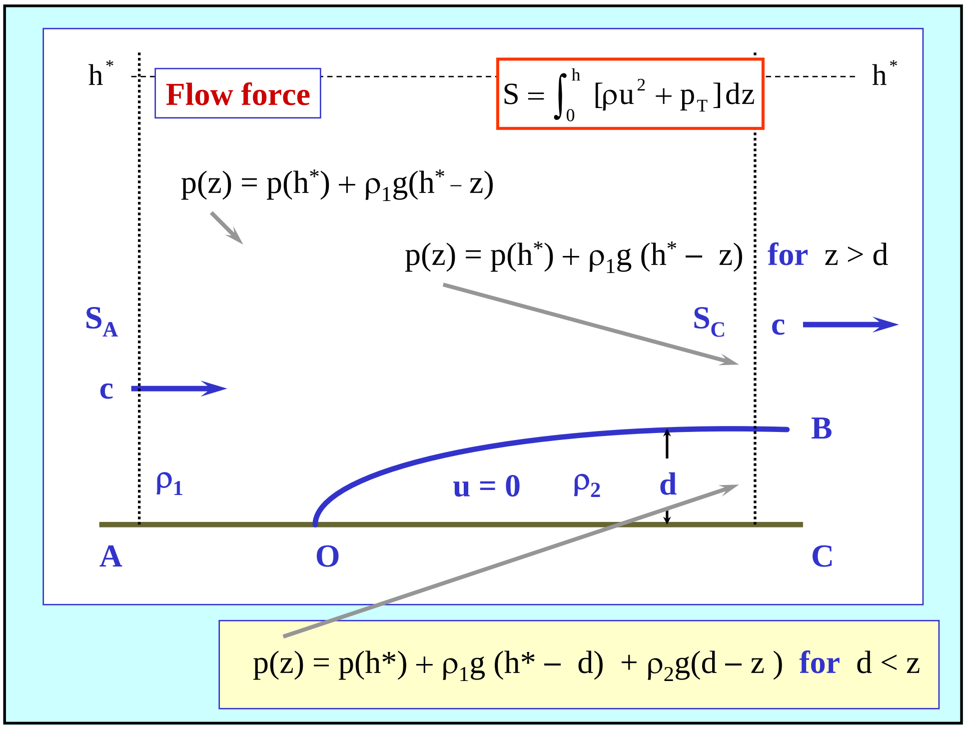
$$p_O = p_C = p_B + g\rho_2 d$$

$$p_O = p_B + \tfrac{1}{2}\rho_1 c^2 + g\rho_1 d$$

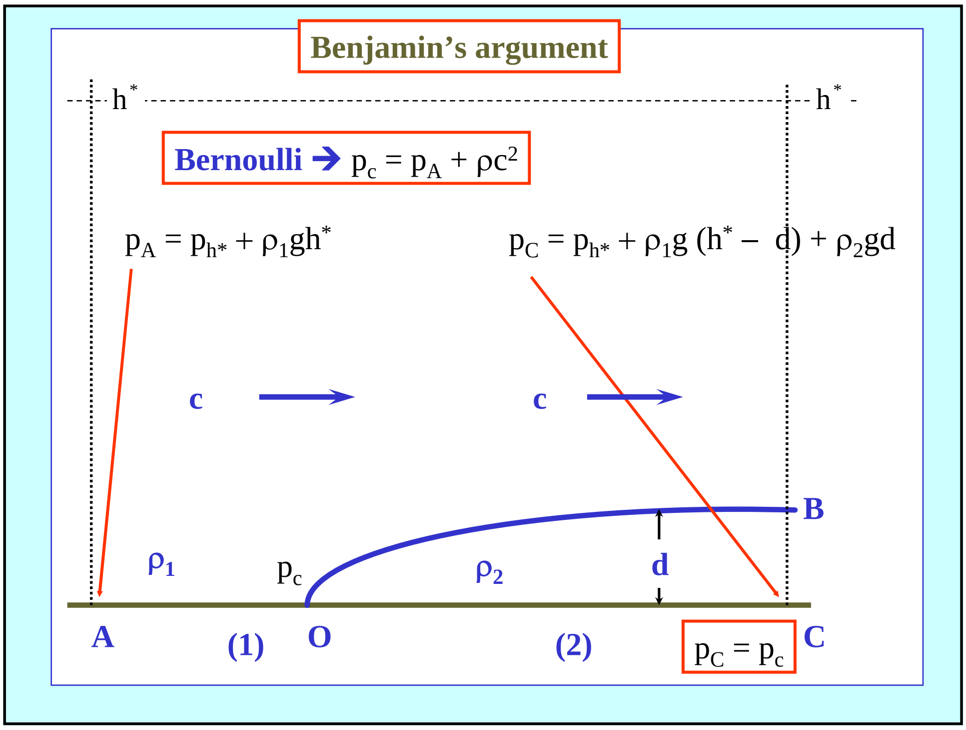
$$\Rightarrow \quad c^2 = 2gd\,\frac{(\rho_2 - \rho_1)}{\rho_1}$$

- **Benjamin (1968) pointed out that the assumption of energy conservation is inconsistent with that of steady flow in this problem, because there is a net force on any control volume enclosing the point O and extending vertically to infinity.**
- **The net force is associated with the horizontal pressure gradient that results from the higher density on the right of the control volume.**

The idea …



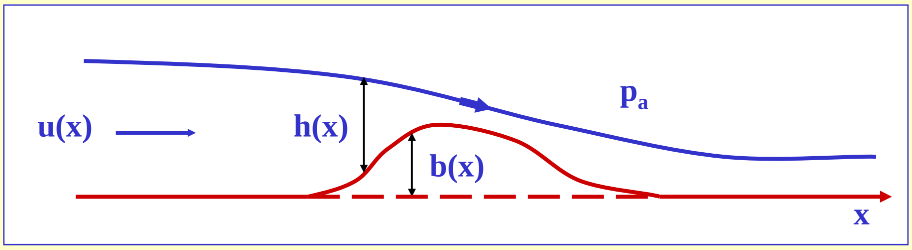
Flow force
$S = \int_0^h [\rho u^2 + p_T] dz$
h*
h*
$p(z) = p(h^*) + \rho_1 g(h^* - z)$
$p(z) = p(h^*) + \rho_1 g (h^* - z)$ for $z > d$
S_A
S_C
c
c
B
ρ_1
u = 0
ρ_2
d
A
O
C
$p(z) = p(h^*) + \rho_1 g (h^* - d) + \rho_2 g(d - z)$ for $d < z$



Benjamin's argument
h*
h*
Bernoulli → $p_c = p_A + \rho c^2$
$p_A = p_{h*} + \rho_1 g h^*$
$p_C = p_{h*} + \rho_1 g (h^* - d) + \rho_2 g d$
c
c
B
ρ_1
p_c
ρ_2
d
A
(1)
O
(2)
$p_C = p_c$
C

Flow over orography

- Consider the steady flow of a layer of non-rotating, homogeneous liquid over an obstacle



- Assume that the streamline slopes are small enough to neglect the vertical velocity component in comparison with the horizontal component =>
- Bernoulli's theorem gives for the free surface streamline

$$p_a + \tfrac{1}{2}\rho u^2 + \rho g (h + b) = \text{constant}$$

$$p_a + \tfrac{1}{2}\rho u^2 + \rho g [h + b(x)] = \text{constant}$$

➡ $$e(x) = \frac{u^2}{2g} + h = -b(x) + \text{constant}$$

Defines the **specific energy**

Continuity => uh = Q = constant

the volume flux per unit span

Can express e as a function of h

➡ $$e = e(h) = \frac{Q^2}{2gh^2} + h$$

A graph of this function is shown in the next picture ➡

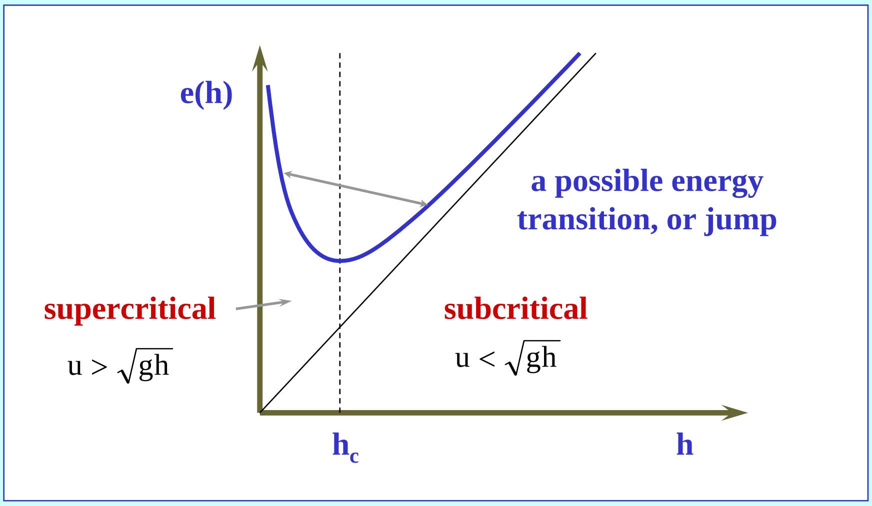
Differentiating $e = \frac{Q^2}{2gh^2} + h$ ⇒ $\frac{de}{dh} = 1 - \frac{Q^2}{gh^3}$

$\frac{de}{dh} = 0$ **when** $Q^2 = gh_c^3$ ⇒ $u^2 = gh_c$

For a given energy $e(h) > e(h_c)$**, there are two possible values for** h**, one** $> h_c$ **and one** $< h_c$**.**

- **Given the flow speed** U **and fluid depth** H **far upstream where** $b(x) = 0$**,** $Q = UH$ **and**

$$e = \frac{Q^2}{2gh^2} + h \quad \Rightarrow \quad \frac{Q^2}{2g}\left[\frac{1}{h^2} - \frac{1}{H^2}\right] + h - H = -b(x)$$



Regime diagram for flow over an obstacle

$$\frac{Q^2}{2g}\left[\frac{1}{h^2} - \frac{1}{H^2}\right] + h - H = -b(x)$$

- **This may be solved for** $h(x)$ **given** $b(x)$ **as long as there are no jumps in the flow =>**
- **e.g. if** $h(x) > h_c$ **for all values of** x**, in other words if the flow remains subcritical.**
- **If the flow is anywhere supercritical, there arises the possibility that hydraulic jumps will occur, leading to an abrupt transition to a subcritical state.**
- **The possibilities were considered in a series of laboratory experiments by Long (1953). See also Baines (1987).**

The End