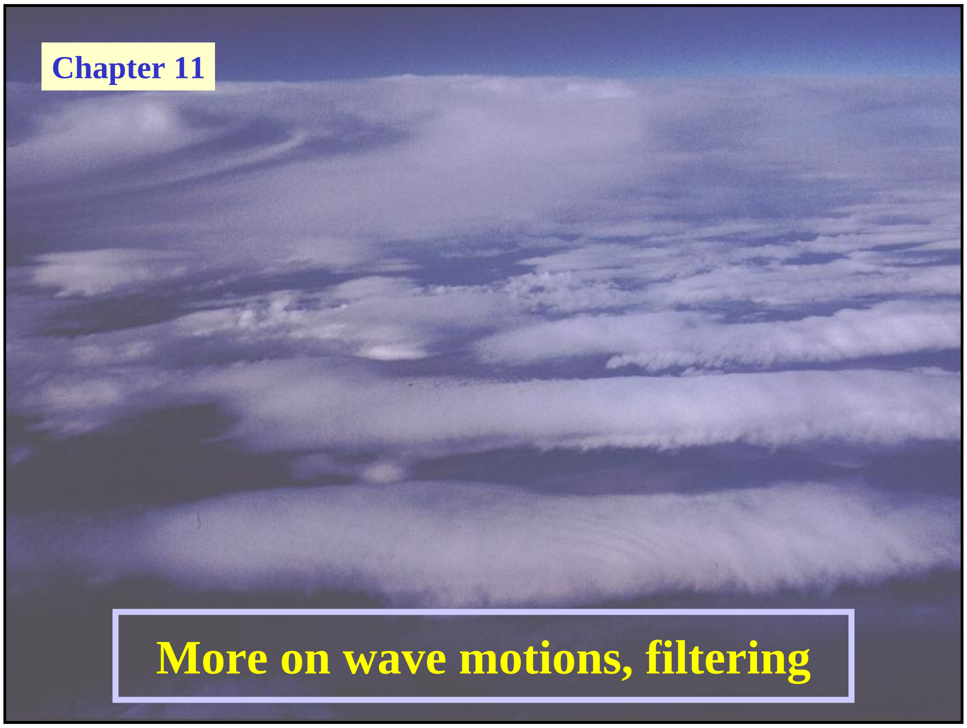
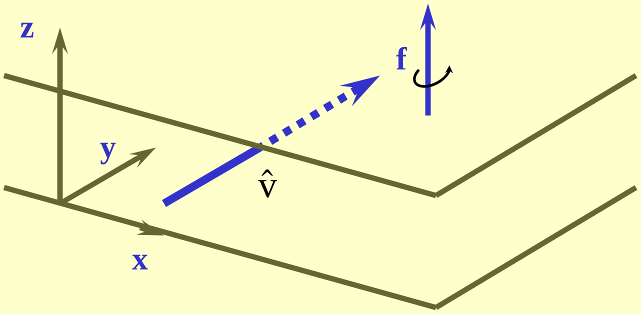


Chapter 11

More on wave motions, filtering

Inertial waves

- Consider a homogeneous layer of inviscid fluid on an f-plane confined between rigid horizontal boundaries.
- Suppose that the entire layer is set **impulsively** in motion in the y-direction with the constant velocity $v = \hat{v}$ at $t = 0$.



Equations

$$\begin{array}{l}\dfrac{\partial u}{\partial t} - fv = 0 \\ \dfrac{\partial v}{\partial t} + fu = 0\end{array} \Longrightarrow \frac{\partial^2 v}{\partial t^2} + f^2 v = 0$$

The same equation is satisfied also by u.

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = 0 \Longrightarrow v = \underset{\text{constant}}{\hat{v}} \cos \underset{\omega = \pm f}{\omega} t$$

The full solution is: $(u, v) = \hat{v}(\sin\ ft,\ \cos\ ft)$

$$(u, v) = \hat{v}(\sin\ ft,\ \cos\ ft)$$

The vector velocity has magnitude V **where**

$$V^2 = u^2 + v^2 = \hat{v} = \hat{v}^2 = \text{cons tan t}$$

The direction changes periodically with time with period $2\pi/f$.

called the inertial period

f **is sometimes referred to as the inertial frequency.**

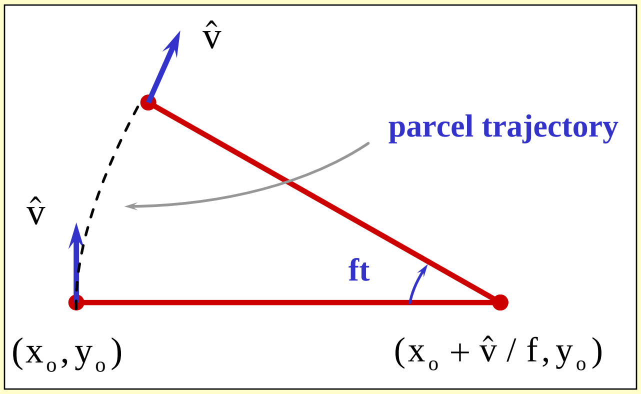
Some notes

- The perturbation velocity is independent of spatial position.
- All fluid parcels move with the same velocity V at any instant --- assumes that the flow domain is infinite and unconstrained by lateral boundaries.
- => at each instant, the layer moves as would a solid block.
- Consider a fluid parcel initially at the point (x_0, y_0) and suppose that it is at the point (x,y) at time t.
- Integrating the velocity =>

$$(x - x_0,\ y - y_0) = \hat{v}\int_0^t (\sin ft\ ,\ \cos ft)\ dt = \frac{\hat{v}}{f}[1 - \cos ft,\ \sin ft]$$

$$\boxed{(x - x_0 - \hat{v}/f)^2 + (y - y_0)^2 = (\hat{v}/f)^2}$$

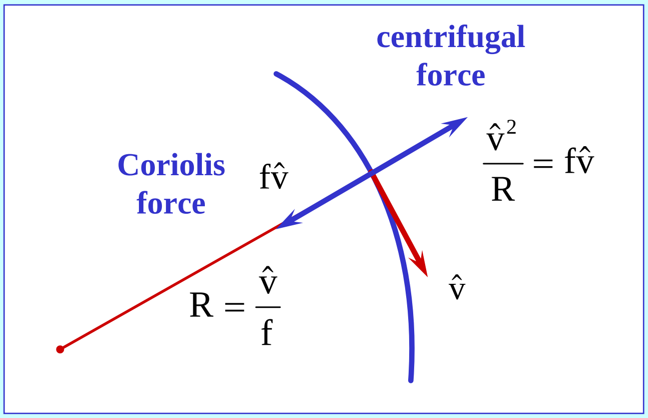
The parcel executes a circular path, an inertia circle, with centre at $(x_0 + \hat{v}/f, y_0)$ and radius $\hat{v}/f$.



The motion is anticyclonic in sense.

The period of motion 2π/ω = 2π/f = half a pendulum day, => the time for a Focault pendulum to turn through 180°.

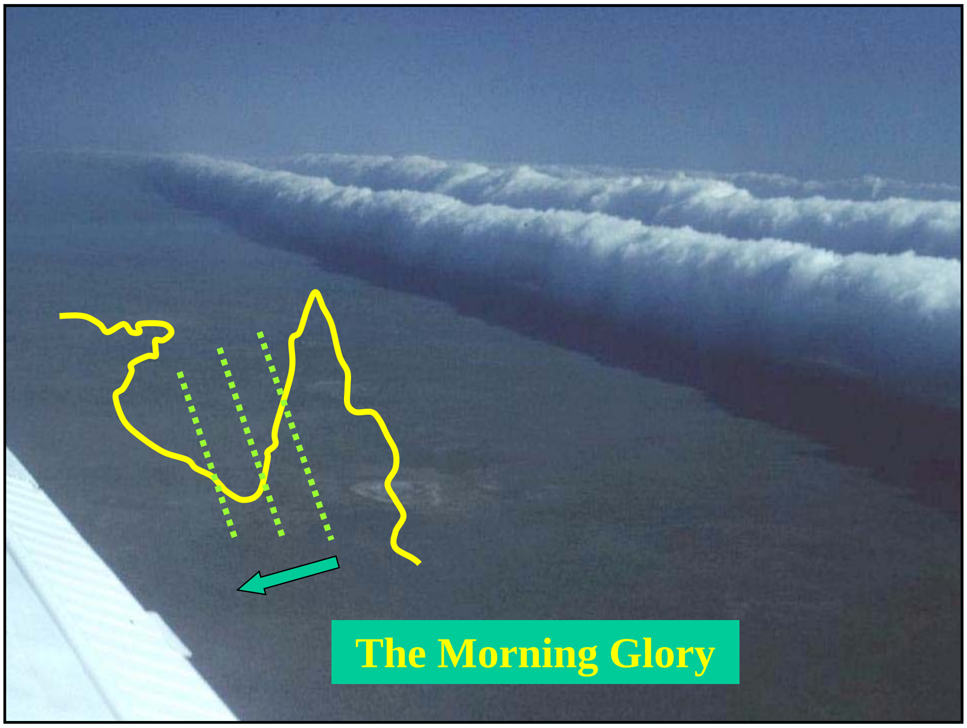
Force balance on a fluid parcel undergoing pure inertial oscillations (northern hemisphere).



There is no pressure gradient in the flow - the only forces are the centrifugal and Coriolis forces. In circular motion these must balance. This is possible only in anticyclonic motion.

Inertial effects

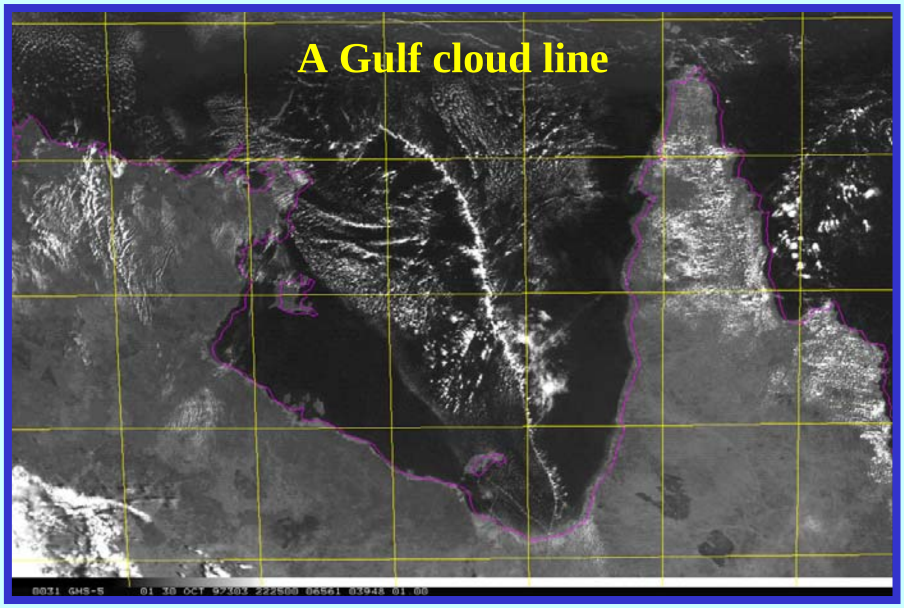
- **Pure inertial oscillations do not seem to be important in the earth's atmosphere, but**
- **Time spectra of ocean currents often exhibit significant amounts of energy at the inertial frequency (see Holton, p.60).**
- **Nevertheless, inertial effects are observed in the atmosphere.**
- **Examples are:**
  - **sea breeze circulations (=> next pictures)**
  - **the nocturnal low level jet**



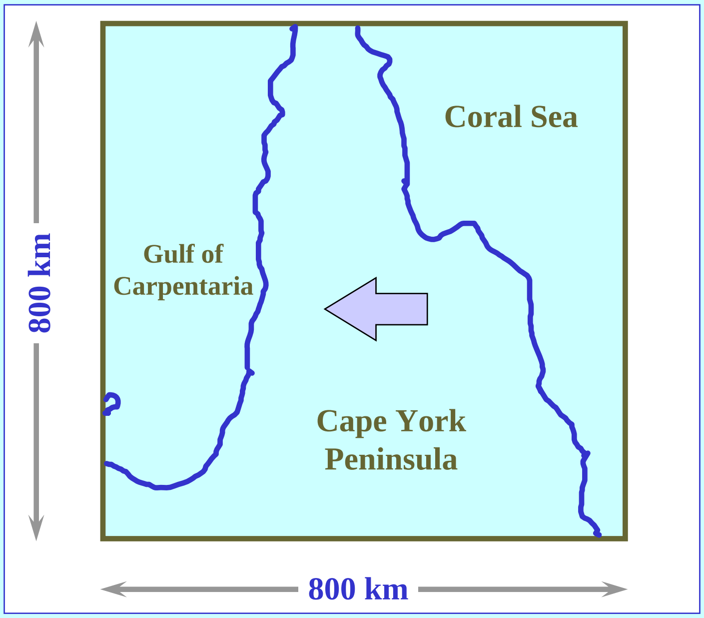
The Morning Glory

A Gulf cloud line

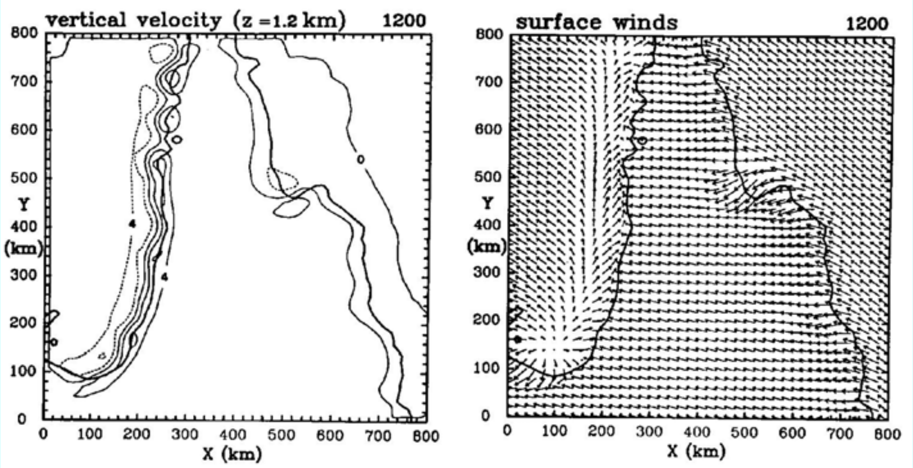
A Gulf cloud line



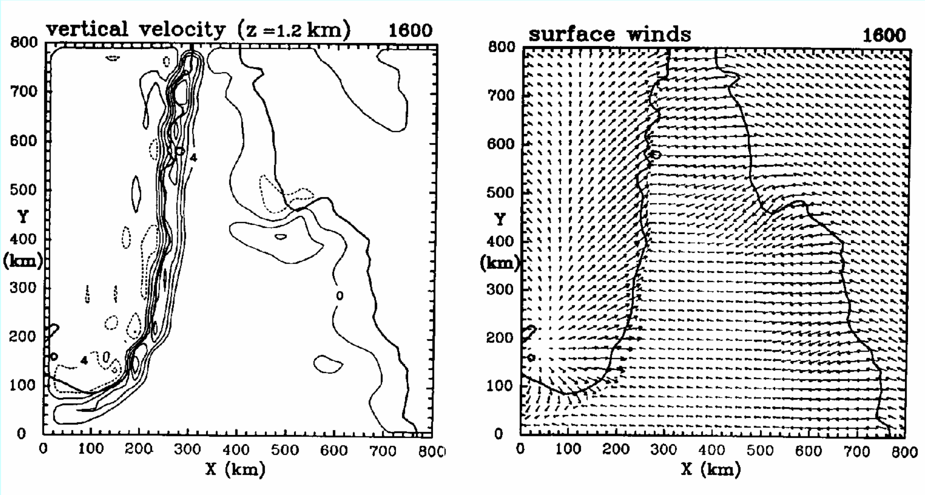
Sea breeze circulations over Cape York Peninsula



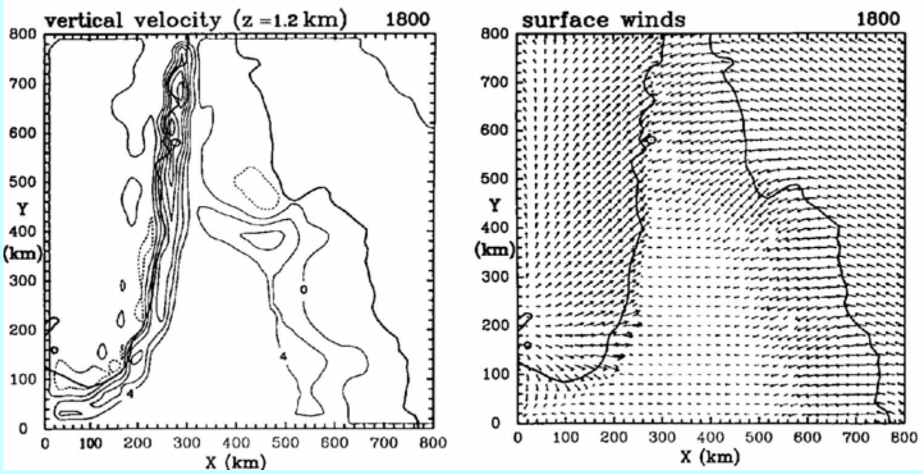
1200 h


vertical velocity (z =1.2 km)
1200
surface winds
1200
Y
(km)
X (km)
0
100
200
300
400
500
600
700
800
4
0

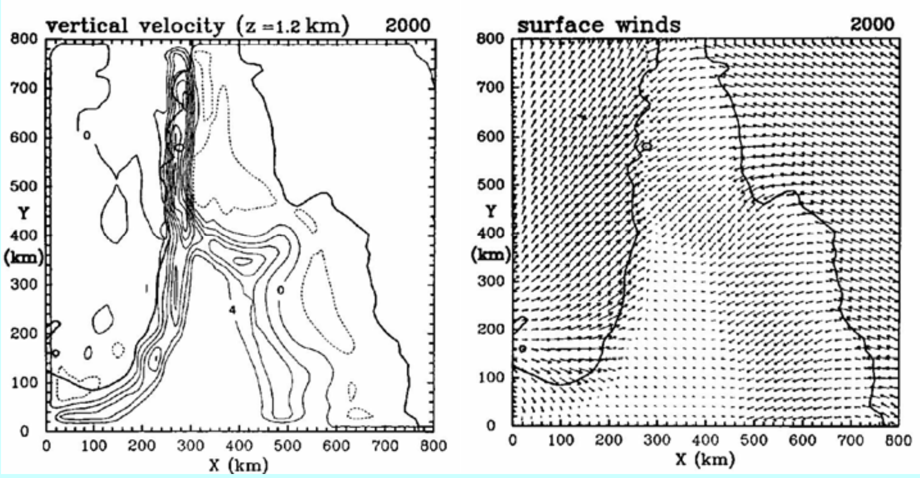
1600 h


vertical velocity (z =1.2 km)
1600
surface winds
1600
Y
(km)
X (km)
0
100
200
300
400
500
600
700
800
4
0

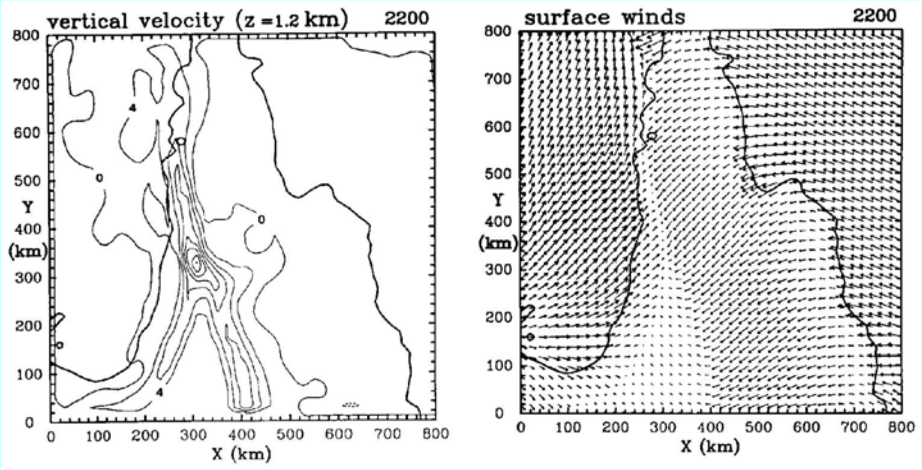
1800 h


vertical velocity (z = 1.2 km)
1800
surface winds
1800
Y (km)
X (km)
0
100
200
300
400
500
600
700
800
0
4

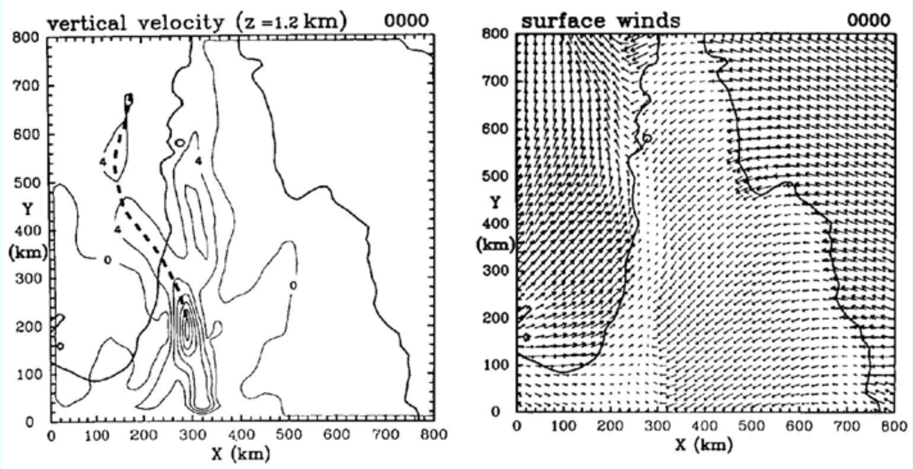
2000 h


vertical velocity (z = 1.2 km)
2000
surface winds
2000
Y (km)
X (km)
0
100
200
300
400
500
600
700
800
0
4

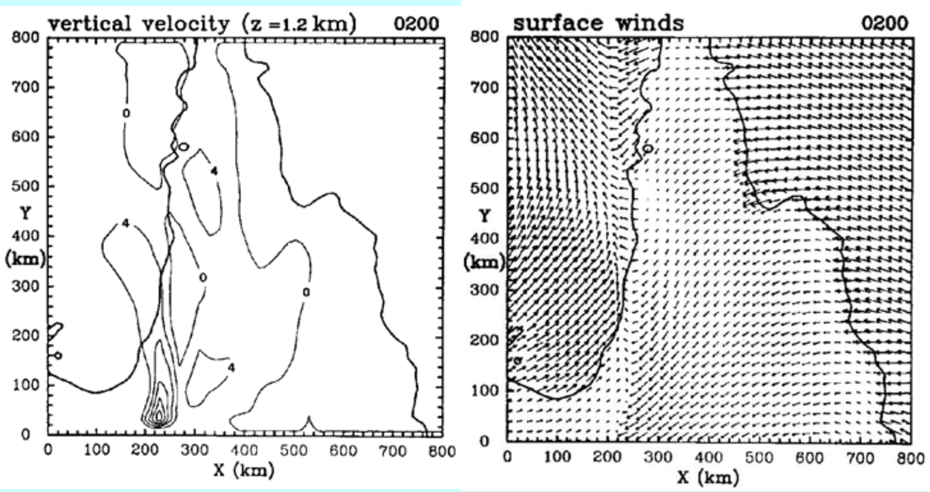
2200 h


vertical velocity (z =1.2 km)
2200
surface winds
2200
Y (km)
X (km)
0
100
200
300
400
500
600
700
800
4
0

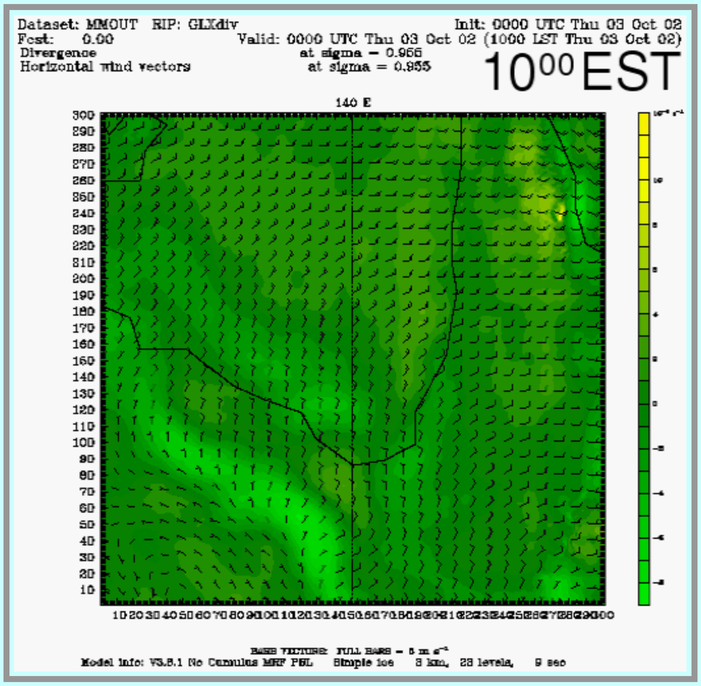
0000 h

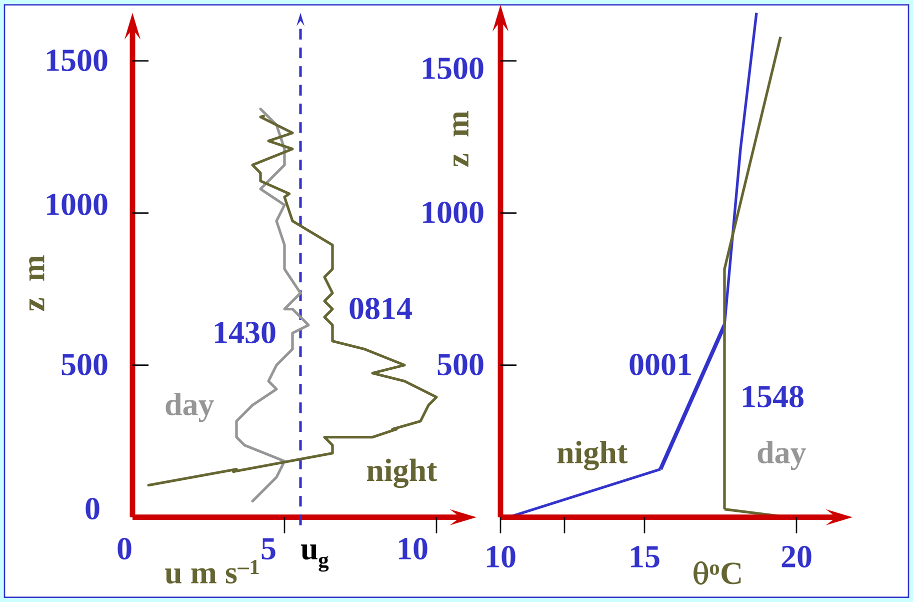

vertical velocity (z =1.2 km)
0000
surface winds
0000
Y (km)
X (km)
0
100
200
300
400
500
600
700
800
4
0

0200 h



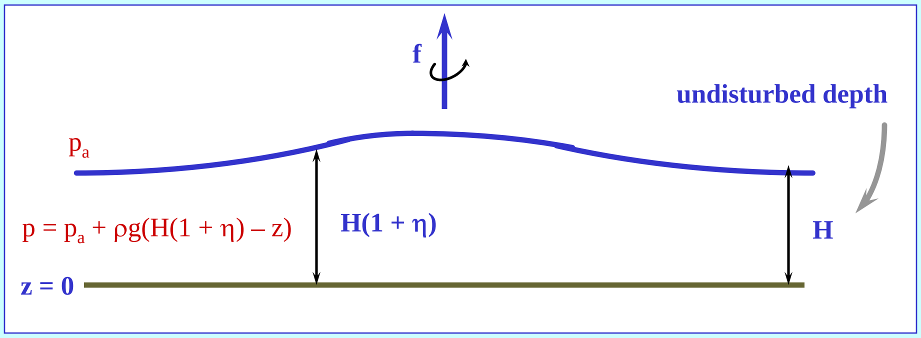
Movie by Gerald Thomsen





The profile of wind component in the direction of the geostrophic wind ($\mathbf{u}_g$) showing a **nocturnal jet**, compared with the profile the previous afternoon.

Inertia-gravity waves



Shallow water model configuration

- In **pure inertial wave motion**, horizontal pressure gradients are zero.
- Consider now **waves** in a layer of rotating fluid with a **free surface** where horizontal pressure gradients are associated with free surface displacements.

Consider hydrostatic motions - then

$$p(x,y,z,t) = p_a + \rho g[H\{1+\eta(x,y,t)\} - z]$$

⇒

$$\frac{1}{\rho}\nabla_h p = gH\nabla_h \eta$$

independent of z

⇒ **The fluid acceleration is independent of z.**

⇒ **If the velocities are initially independent of z, then they will remain so.**

Linearized equations - no basic flow

$$\frac{\partial u}{\partial t} - fv = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -gH\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \eta}{\partial t}$$

Consider wave motions which are independent of y.

A solution exists of the form

$$u = \hat{u}\cos(kx - \omega t),$$

$$v = \hat{v}\sin(kx - \omega t),$$

$$\eta = \hat{\eta}\cos(kx - \omega t),$$

if ⇒

$\hat{u}$, $\hat{v}$ and $\hat{\eta}$ constants

$$\omega\hat{u} - f\hat{v} - gHk\hat{\eta} = 0\,,$$
$$f\hat{u} - \omega\hat{v} = 0\,,$$
$$-k\hat{u} + \omega\hat{\eta} = 0\,,$$

These algebraic equations for $\hat{u}$, $\hat{v}$ and $\hat{\eta}$ **have solutions only if**

$$\begin{vmatrix} \omega & -f & -gHk \\ f & -\omega & 0 \\ -k & 0 & \omega \end{vmatrix} = -\omega^3 + \omega\,(f^2 + gHk^2) = 0$$

➡ $\boxed{\omega = 0 \;\text{ or }\; \omega^2 = f^2 + gHk^2}$

The solution with $\omega = 0$ **corresponds with the steady solution (**$\partial/\partial t = 0$**) of the equations and represents a steady current in strict geostrophic balance in which**

$$v = \frac{gH}{f}\frac{\partial\eta}{\partial x}$$

The other two solutions correspond with so-called inertia-gravity waves, with the dispersion relation

$$\omega^2 = f^2 + gHk^2$$

The phase speed of these is

$$c_p = \omega\,/\,k = \pm\sqrt{[gH + f^2\,/\,k^2]}$$

➡ **The waves are dispersive**

$$\omega^2 = f^2 + gHk^2 \qquad c_p = \frac{\omega}{k} = \pm\sqrt{gH + \frac{f^2}{k^2}}$$

- The existence of a positive and negative root for ω (or c_p) shows that these waves can propagate in either x-direction.
- In the limit as $k \to 0$, $\omega \to f$ and the x-dependence of the motion drops out.
- Then the motion corresponds with the pure inertial wave discussed earlier.
- In the absence of rotation (i.e., $f = 0$) the theory reduces to that for small amplitude surface gravity waves on shallow water.
- Such waves are non-dispersive: $c_p = \pm\sqrt{gH}$

Note that the hydrostatic assumption restricts the validity of the whole analysis to long waves on shallow water; strictly for waves with wavelength $\lambda >> H$.

$$\omega^2 = f^2 + gHk^2$$

Rotation is important if $f^2 / k^2 \sim gH$ =>

$$\lambda = 2\pi(gH)^{1/2} / f = 2\pi L_R$$

the Rossby radius of deformation
for a homogeneous fluid

In the deep ocean, $H \sim 4\ \text{km} = 4 \times 10^3\ \text{m}$

At 45° **latitude,** $f = 10^{-4}\,s^{-1}$ ➡

$$\lambda = 2\pi \times (10 \times 4 \times 10^3)^{1/2} / 10^{-4} = 1.3 \times 10^7\,\text{m} \sim 10^4\ \text{km}$$

and $\sqrt{gH} \div 200\ \text{ms}^{-1}$ **.**

➡

Rotation effects are important only on the planetary scale.

Inclusion of the beta effect

- **Replace the** v-**momentum equation by the vorticity equation =>**

$$\frac{\partial u}{\partial t} - f_0 v = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \zeta}{\partial t} + \beta v = f_0\frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Assume that f **can be approximated by its value** f_0 **at a particular latitude, except when differentiated with respect to** y **in the vorticity equation. This is justified provided that meridional particle displacements are small.**

- Again assume that $\partial/\partial y \equiv 0$ and consider travelling-wave solutions of the form:

$$u = \hat{u}\cos(kx - \omega t),$$
$$v = \hat{v}\sin(kx - \omega t),$$
$$\eta = \hat{\eta}\cos(kx - \omega t),$$

➡

$$\omega\hat{u} - f_0\hat{v} - gHk\hat{\eta} = 0\ ,$$
$$(k\omega + \beta)\hat{v} - f_0\omega\hat{\eta} = 0\ ,$$
$$-k\hat{u} + \omega\hat{\eta} = 0\ ,$$

These are consistent only if the determinant is zero ➡

Now

expanding by the second row

$$\begin{vmatrix} \omega & -f_0 & -gHk \\ 0 & +(k\omega + \beta) & -\omega f_0 \\ -k & 0 & \omega \end{vmatrix} = \underline{(\omega^2 - gHk^2)(\omega k + \beta) - f_0^2\,\omega k = 0}$$

A cubic equation for ω with three real roots.

When $\omega >> \beta/k$, the two non-zero roots are given approximately by the formula

$$\omega^2 = gHk^2 + f_0^2$$

This is precisely the dispersion relation for inertia-gravity waves.

When $\omega^2 << gHk^2$, there is one root given approximately by

$$\omega = -\beta k/(k^2 + f_0^2/gH)$$

- For wavelengths small compared with 2π times the Rossby length $2\pi\sqrt{gH}\ /\ f_0$

$$\omega = -\frac{\beta k}{k^2 + \frac{f_0^2}{gH}}$$

reduces to the dispersion relation for nondivergent Rossby waves.

For longer wavelengths,

$$k \leq 0(f_0\sqrt{gH})$$

the effects of divergence due to variations in the free surface elevation become important, or even dominant.

For extremely long waves

$$\omega \sim \frac{\beta k g H}{f_0^2}$$

These are nondispersive.

- The importance of divergence effects on the ultra-long waves explains why the calculated phase speeds for planetary waves of global wave-numbers-1 and -2 were unrealistically large.
- The latter were obtained from the phase speed formula for nondivergent waves.
- In the atmosphere, divergence effects are not associated with the displacement of a free surface!

- To understand the effect of divergence on planetary waves, we introduce the meridional displacement ξ of a fluid parcel.
- This is related to the meridional velocity component by $v = D\xi /Dt$, or to a first approximation by

$$\frac{\partial \xi}{\partial t} = v$$

- Substituting for v in $\partial_t \zeta + \beta v = f_0 \partial_t \eta$ and integrating with respect to time, assuming that all perturbation quantities vanish at $t = 0$, gives

$$\zeta = -\beta \xi + f_0 \eta$$

This formula is equivalent (within a linear analysis) to the conservation of potential vorticity (see exercise 11.2).

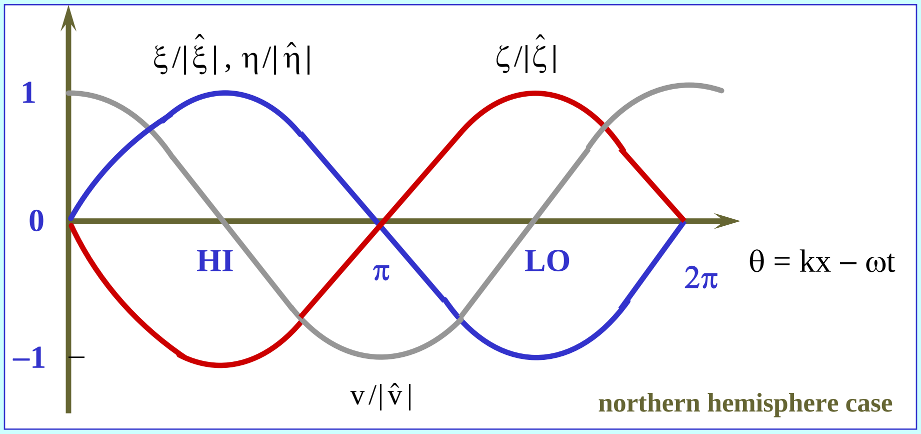
In the absence of divergence,

$$f + \zeta = f_0 + \beta \xi + \zeta = f_0$$

holds for a fluid parcel which has ξ and ζ initially zero so that $\zeta = -\beta \xi$.

The term $f_0 \eta$ represents the increase in relative vorticity due to the stretching of planetary vorticity associated with horizontal convergence (positive η).

Phase diagram for Rossby waves



The relative phases of the quantities v, ξ, η and ζ over one wavelength

The relative vorticity tendency due to stretching opposes that due to meridional motion and thereby reduces the restoring tendency of the induced velocity field.

Filtering

Equations:

$$\cancel{\frac{\partial u}{\partial t}} - f_0 v = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \zeta}{\partial t} + \beta v = f_0\frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v = \frac{gH}{f_0}\frac{\partial \eta}{\partial x}$$

If $\partial/\partial y \equiv 0$ as before, the vorticity equation reduces to an equation for η, and since $\zeta = \partial v/\partial x$ =>

$$\frac{\partial}{\partial t}\left(\frac{gH}{f_0}\frac{\partial^2 \eta}{\partial x^2} - f_0\eta\right) + \beta\frac{gH}{f_0}\frac{\partial \eta}{\partial x} = 0$$

This has the solution $\eta = \hat\eta\cos(kx - \omega t)$ **, where**

Dispersion relation for a divergent planetary wave

$$\omega = -\beta k / (k^2 + f_0^2 / gH)$$

- There is no other solution for ω as there was before.
- In other words, making the geostrophic approximation when calculating v has filtered out in the inertia-gravity wave modes from the equation set, leaving only the low frequency planetary wave mode.
- This is not too surprising since the inertia-gravity waves, by their very essence, are not geostrophically-balanced motions.

Filtered equations

- The idea of filtering sets of equations is an important one in geophysical applications.
- The quasi-geostrophic equations are often referred to as 'filtered equations' since, as in the above analysis, the consequence of computing the horizontal velocity geostrophically from the pressure or stream-function suppresses the high frequency inertia-gravity waves which would otherwise be supported by the Boussinesq equations.
- Furthermore, the Boussinesq equations themselves form a filtered system in the sense that the approximations which lead to them filter out compressible, or acoustic waves.