Chapter 9

Synoptic-scale instability and cyclogenesis – Part II

A two-layer model

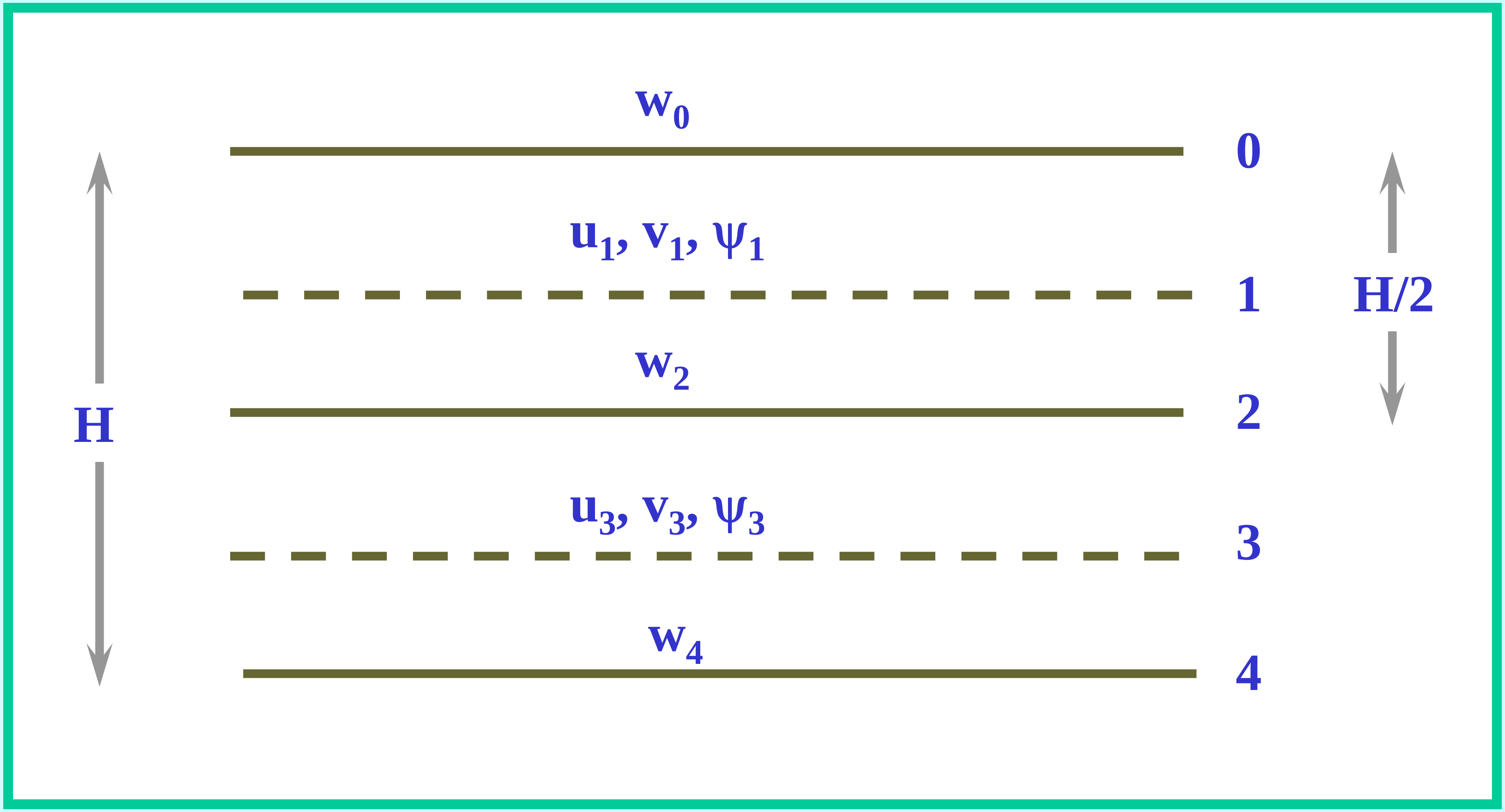
- **Some of the algebraic details in the Eady solution are complicated - especially:**

  - the calculation of w **and** b**, and**

  - the inclusion of a beta effect ($\partial f/\partial y \neq 0$**) renders the eigenvalue problem analytically intractable.**

- **An even simpler model which does not suffer these limitations may be formulated at the sacrifice of vertical resolution.**

- **The procedure is to divide the atmosphere into two layers:**



w_0
0
u_1, v_1, ψ_1
1
H/2
w_2
2
H
u_3, v_3, ψ_3
3
w_4
4

- **Vertical derivatives in the quasi-geostrophic equations are then replaced by central-difference approximations.**
- **In each layer,**

$$u_n = -\frac{\partial \psi_n}{\partial y}, \; v_n = \frac{\partial \psi_n}{\partial x}, \; \left[\frac{\partial}{\partial t} + u_n \frac{\partial}{\partial x} + v_n \frac{\partial}{\partial y}\right](\nabla^2 \psi_n + f) = f_0 \left[\frac{\partial w}{\partial z}\right]_n$$

We express $[\partial w/\partial z]_n$ **as central differences =>**

$$\left[\frac{\partial w}{\partial z}\right]_1 = \frac{w_0 - w_2}{\frac{1}{2}H}, \quad \left[\frac{\partial w}{\partial z}\right]_3 = \frac{w_2 - w_4}{\frac{1}{2}H}$$

We impose the boundary conditions $w_0 = 0$**,** $w_4 = 0$**.**

$$\left[\frac{\partial}{\partial t} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y}\right](\nabla^2\psi_1 + f) = -\frac{2f_0}{H}w_2$$

and $$\left[\frac{\partial}{\partial t} + u_3\frac{\partial}{\partial x} + v_3\frac{\partial}{\partial y}\right](\nabla^2\psi_3 + f) = +\frac{2f_0}{H}w_2$$

w_2 **satisfies** $$\left[\frac{\partial}{\partial t} + u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\right]f\frac{\partial\psi_2}{\partial z} + N^2 w_2 = 0$$

Since u_2 **and** v_2 **are not carried, we compute them by averaging** u_1 **and** u_3**,**

$$\partial\psi_2 / \partial z = (\psi_1 - \psi_3) / \tfrac{1}{2}H$$

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}(u_1+u_3)\frac{\partial}{\partial x} + \frac{1}{2}(v_1+v_3)\frac{\partial}{\partial y}\right](\psi_1-\psi_3) + \frac{HN^2}{2f_0}w_2 = 0$$

$$\left[\frac{\partial}{\partial t}+\frac{1}{2}(u_1+u_3)\frac{\partial}{\partial x}+\frac{1}{2}(v_1+v_3)\frac{\partial}{\partial y}\right](\psi_1-\psi_3)+\frac{HN^2}{2f_0}w_2=0$$

The coefficient of w_2 **may be written as**

$$\frac{2f_0}{H}\cdot\frac{N^2H^2}{4f_0^2}=\frac{2f_0}{H}\frac{L^2{}_R}{4}=\gamma\mu^{-2}$$

where $\gamma = 2f_0/H$ **and** $\mu = 2/L_R$

L_R **is the Rossby length**

Full set of nonlinear equations

$$\left[\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y}\right](\nabla^2 \psi_1 + f) = -\frac{2f_0}{H} w_2$$

$$\left[\frac{\partial}{\partial t} + u_3 \frac{\partial}{\partial x} + v_3 \frac{\partial}{\partial y}\right](\nabla^2 \psi_3 + f) = +\frac{2f_0}{H} w_2$$

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}(u_1 + u_3)\frac{\partial}{\partial x} + \frac{1}{2}(v_1 + v_3)\frac{\partial}{\partial y}\right](\psi_1 - \psi_3) = -\gamma\mu^{-2} w_2$$

Perturbation method

Let the streamfunction of the basic zonal flow in each layer

$$\overline{\psi}_n = -yU_n \quad (n = 1, 3)$$

and consider small perturbations to this

$$\psi_n = \overline{\psi}_n + \psi'_n \qquad \textbf{where} \qquad \left|\psi'_n\right| << \left|\overline{\psi}_n\right|$$

The linearized equations

$$\left[\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right]\frac{\partial^2\psi'_1}{\partial x^2}+\beta\frac{\partial\psi'_1}{\partial x}=-\gamma w_2$$

$$\left[\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right]\frac{\partial^2\psi'_3}{\partial x^2}+\beta\frac{\partial\psi'_3}{\partial x}=\gamma w_2$$

$$\left[\frac{\partial}{\partial t}+\frac{1}{2}(U_1+U_3)\frac{\partial}{\partial x}\right](\psi'_1-\psi'_3)-\frac{(U_1-U_3)}{2}\frac{\partial}{\partial x}(\psi'_1+\psi'_3)=-\gamma\mu^{-2}w_2$$

assuming a perturbation for which $\partial/\partial y \equiv 0$**.**

Solution method

- **The equations form a linear system with constant coefficients and therefore have solutions of the form**

$$\begin{bmatrix} \psi'_1 \\ \psi'_3 \\ w_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_3 \\ \tilde{w} \end{bmatrix} e^{ik(x-ct)},$$

constants

- **Substitution gives a set of linear homogeneous algebraic equations:**

$$ik\left[(c - U_1)k^2 + \beta\right]\phi_1 + \gamma\tilde{w} = 0$$

$$ik\left[(c - U_3)k^2 + \beta\right]\phi_3 - \gamma\tilde{w} = 0$$

$$-ik\left[c - U_3\right]\phi_1 + ik\left[c - U_1\right]\phi_3 + \gamma\mu^{-2}\tilde{w} = 0$$

These have a non-trivial solution for ϕ_1, ϕ_3 and $\tilde{w}$ if and only if the determinant of coefficients is zero.

A quadratic equation for the eigenvalues c**:**

Eigenvalue equation

$$c^2\left[k^4 + 2k^2\mu^2\right] + 2c\left[\beta(k^2 + \mu^2) - \tfrac{1}{2}(U_1 + U_3)(k^4 + 2k^2\mu^2)\right]$$

$$+ \; U_1U_3(k^4 + 2k^2\mu^2) + \beta^2 - (U_1 + U_3)\beta(k^2 + \mu^2) + k^2\mu^2(U_1 - U_3)^2 = 0 \; .$$

Write

$$U_m = \tfrac{1}{2}(U_1 + U_3), \qquad U_T = \tfrac{1}{2}(U_1 - U_3)$$

$$c = U_m - \frac{\beta(k^2 + \mu^2)}{k^2(k^2 + 2\mu^2)} + \delta^{1/2}$$

$$\delta = \frac{\beta^2\mu^4}{k^4(k^2 + 2\mu^2)^2} - U^2{}_T \frac{(2\mu^2 - k^2)}{(k^2 + 2\mu^2)}$$

$$ik\left[(c-U_1)k^2+\beta\right]\phi_1+\gamma\tilde{w}=0$$

$$ik\left[(c-U_3)k^2+\beta\right]\phi_3-\gamma\tilde{w}=0$$

$$\frac{\phi_1}{\phi_3}=-\frac{(c-U_3)k^2+\beta}{(c-U_1)k^2+\beta}=-\frac{U_Tk^2+\dfrac{\beta(k^2+\mu^2)}{k^2+2\mu^2}+k^2\delta^{1/2}}{U_Tk^2-\dfrac{\beta(k^2+\mu^2)}{k^2+2\mu^2}-k^2\delta^{1/2}}$$

Put

$$q=\frac{\beta\mu^2}{k^2(k^2+2\mu^2)U_T} \qquad \delta^{1/2}=U_Tp \qquad p^2=q^2-\left[\frac{2\mu^2-k^2}{k^2+2\mu^2}\right]$$

$$\phi_1=\left[\frac{1+q+p}{1-q-p}\right]\phi_3$$

Some special cases

1. No vertical shear, $U_T = 0$**, i.e.,** $U_1 = U_3$**.**

$$\delta^{1/2} = \pm \frac{\beta\mu^2}{k^2(k^2 + 2\mu^2)}$$

Then

$$c = U_m - \frac{ß}{k^2}$$

or

$$c = U_m - \frac{ß}{k^2 + 2\mu^2}$$

No vertical shear, $U_T = 0$, **i.e.,** $U_1 = U_3$.

$$c = U_m - \frac{ß}{k^2} \Rightarrow \phi_1 = \phi_3 , \quad \tilde{w} = 0$$

$\Rightarrow$ ψ'_1 **and** ψ'_3 **are exactly in phase =>**
the ridges and troughs are in phase.

$\tilde{w} = 0 \Rightarrow$ **there is no interchange of fluid between the two layers.**

This solution corresponds with a barotropic Rossby wave as the dispersion relation suggests.

No vertical shear, $U_T = 0$, **i.e.,** $U_1 = U_3$.

$$c = U_m - \frac{ß}{k^2 + 2\mu^2} \Rightarrow \phi_1 = -\phi_3$$

- **The waves in the upper and lower layers are exactly out of phase, i.e.,** $\psi'_1 = \psi'_3 e^{i\pi}$.
- **Thus at meridians (x-values) where the perturbation velocities are poleward in the upper layer, they are equatorward in the lower layer and vice versa.**
- **This mode is called a baroclinic, or internal, Rossby wave.**
- **The presence of the free mode of this type is a weakness of the two-layer model; see Holton, p. 220.**
- **The mode does not correspond with any free oscillation of the atmosphere, but such wave modes do exist in the oceans.**

2. No beta effect, $\beta = 0$ **, finite shear** $U_T \neq 0$**.**

$$\delta^{1/2} = U_T \left[\frac{k^2 - 2\mu^2}{k^2 + 2\mu^2} \right]^{1/2}$$

imaginary if $k^2 < 2\mu^2$

Let $\delta^{1/2} = ic_i$ ⇒ $e^{ik(x-ct)} = e^{ik(x-U_m t)}\, e^{kc_i t}$

⇒ **The wave grows or decays exponentially with time, according to the sign of** c_i**, and propagates zonally with phase speed** U_m**.**

No beta effect, $\beta = 0$, **finite shear** $U_T \neq 0$.

In $$\phi_1 = \left[\frac{1+q+p}{1-q-p}\right]\phi_3 \quad , \qquad q = 0$$

When $k^2 < 2\mu^2$, $p^2 = -(2\mu^2 - k^2)/(2\mu^2 + k^2) = -p^2_0$, **say.**

$$\Rightarrow \quad \frac{\phi_1}{\phi_3} = \frac{1+ip_0}{1-ip_0} = \frac{(1+ip_0)^2}{1+p_0{}^2} = e^{2i\theta}$$

where $\theta = \tan^{-1} p_0$.

Note that $|p_0| < 1$ **and if** $p_0 > 0$, $0 < \theta < \frac{\pi}{4}$.

No beta effect, $\beta = 0$ **, finite shear** $U_T \neq 0$**.**

$$\psi_1' = \phi_3 e^{kc_i t} e^{ik(x - U_m t) + 2i\theta},$$

$$\psi_3' = \phi_3 e^{kc_i t} e^{ik(x - U_m t)},$$

- **If** $c_i > 0$**, the upper wave is** 2θ **radians in advance of the lower wave => again the trough and ridge positions are displaced westwards with height, as in the growing Eady wave.**
- **The threshold for instability occurs when** $k^2 = 2\mu^2$**, or** $k = 2.82/L_R$, **waves of large wavenumber (shorter wavelength) being stable.**
- **This should be compared with the Eady stability criterion which requires that** $s^2 < 1.2$ **or** $k < 2.4/L_R$**.**

Forecasting rule

- It is evident that the growth rate of a disturbance is related to the degree of westward displacement of the trough with height.
- This accords with synoptic experience and provides forecasters with a rule for judging whether or not a lower pressure centre will intensify during a forecast period.
- This rule is based on a comparison of the positions of the upper-level trough (which may even have one or two closed isopleths) and the surface low.
- The rule will be investigated further in the next chapter.

3. The general case, $\beta \neq 0$, $U_T \neq 0$.

Algebraically more complicated.

$$c = U_m - \frac{\beta(k^2 + \mu^2)}{k^2(k^2 + 2\mu^2)} + \delta^{1/2}$$

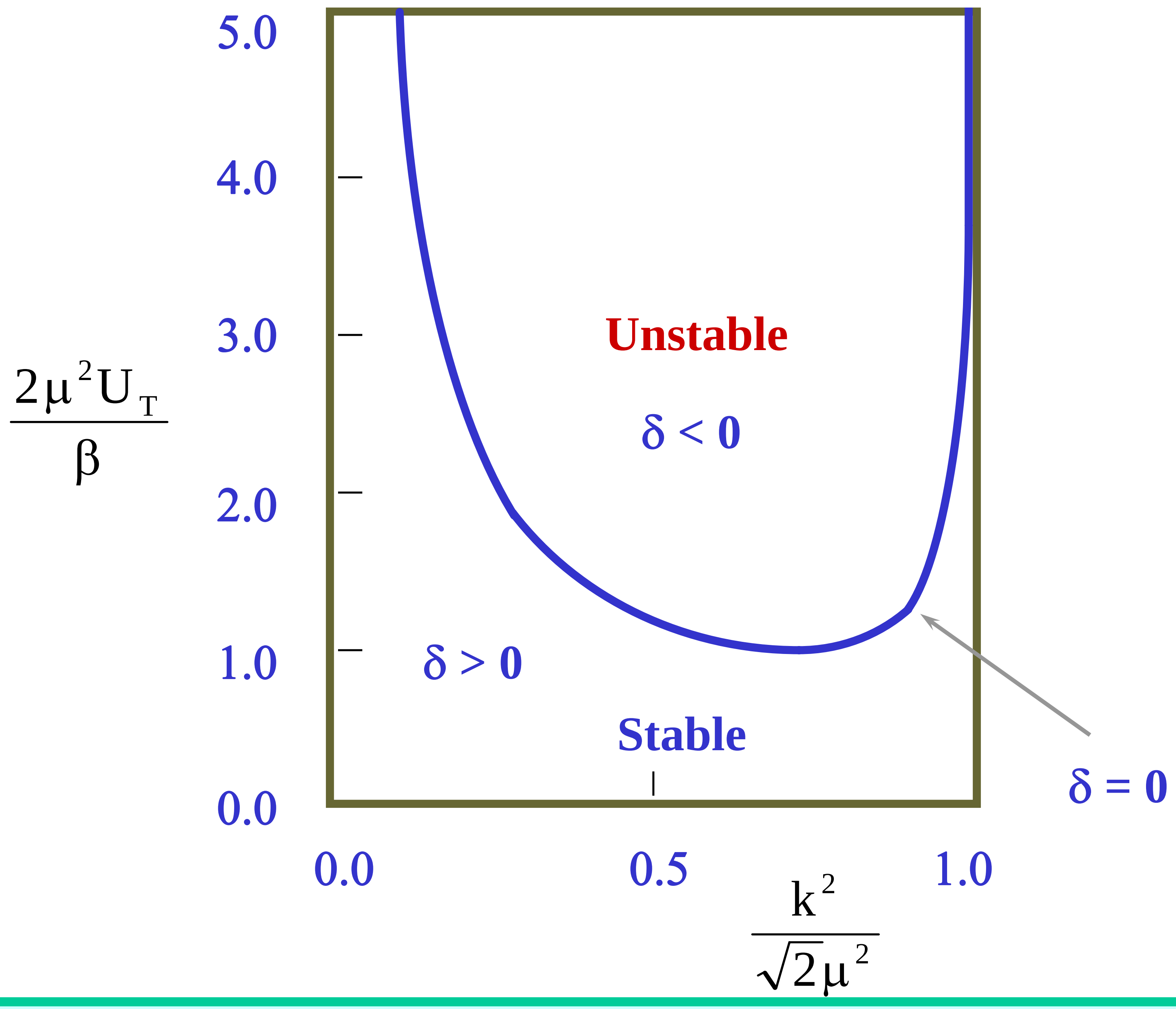
As before

$$\delta = \frac{\beta^2\mu^4}{k^4(k^2 + 2\mu^2)^2} - U_T^2 \frac{(2\mu^2 - k^2)}{(k^2 + 2\mu^2)}$$

Neutral stability <=> $\delta = 0$

$$\frac{\beta^2\mu^4}{k^4(k^2 + 2\mu^2)^2} = U_T{}^2 \frac{(2\mu^2 - k^2)}{(2\mu^2 + k^2)},$$

$$\frac{k^4}{2\mu^4} = 1 \pm \left[1 - \frac{\beta^2}{4\mu^4 U_T{}^2}\right]^{1/2} .$$



5.0
4.0
3.0
2.0
1.0
0.0
$\frac{2\mu^2 U_T}{\beta}$
Unstable
δ < 0
δ > 0
Stable
δ = 0
0.0
0.5
1.0
$\frac{k^2}{\sqrt{2}\mu^2}$

The energetics of baroclinic waves

- **The relative simplicity of the two-layer model makes it especially suitable for studying the energy conversions associated with baroclinic waves.**
- **The following discussion closely parallels that of Holton, §9.3.2.**

The energy equations

Multiply by $-\psi'_1$

$$\left[\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right]\frac{\partial^2\psi'_1}{\partial x^2}+\beta\frac{\partial\psi'_1}{\partial x}=-\gamma w_2$$

Multiply by $-\psi'_3$

$$\left[\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right]\frac{\partial^2\psi'_3}{\partial x^2}+\beta\frac{\partial\psi'_3}{\partial x}=\gamma w_2$$

Multiply by $\psi'_1 - \psi'_3$

$$\left[\frac{\partial}{\partial t}+\frac{1}{2}(U_1+U_3)\frac{\partial}{\partial x}\right](\psi'_1-\psi'_3)-\frac{(U_1-U_3)}{2}\frac{\partial}{\partial x}(\psi'_1+\psi'_3)=-\gamma\mu^{-2}w_2$$

omit primes, and take zonal averages denoted by

$$<(\quad)>=\frac{1}{\lambda}\int_0^{\lambda}(\;)dx$$

λ **is the perturbation wavelength.**

$$\Rightarrow \quad \frac{d}{dt} < \frac{1}{2} v_1{}^2 > = \gamma < w_2 \psi_1 >$$

$$\frac{d}{dt} < \frac{1}{2} v_3{}^2 > = -\gamma < w_2 \psi_3 >$$

$$\frac{d}{dt} < \frac{1}{2} (\psi_1 - \psi_3)^2 > = U_T < (\psi_1 - \psi_3) \frac{\partial}{\partial x} (\psi_1 + \psi_3) >$$

$$- \gamma \mu^{-2} < w_2 (\psi_1 - \psi_3) > .$$

Note that $$- < \psi_1 \frac{\partial}{\partial t} \left[\frac{\partial^2 \psi_1}{\partial x^2} \right] > = - < \psi_1 \frac{\partial^2}{\partial x^2} \left[\frac{\partial \psi_1}{\partial t} \right] > = - < \frac{\partial}{\partial x} \left[\psi_1 \frac{\partial}{\partial x} \left[\frac{\partial \psi_1}{\partial t} \right] \right] > +$$

$$< \frac{\partial \psi_1}{\partial x} \frac{\partial}{\partial t} \left[\frac{\partial \psi_1}{\partial x} \right] > = 0 + \frac{1}{2} < \frac{\partial}{\partial t} (v_1{}^2) > .$$

Likewise $$- U_1 < \psi_1 \frac{\partial^2}{\partial x^2} \left[\frac{\partial \psi_1}{\partial x} \right] > = \frac{1}{2} U_1 < \frac{\partial}{\partial x} v_1{}^2 > = 0 .$$

Define $K' = \frac{1}{2} H < v_1^2 + v_3^2 >$

the perturbation kinetic energy averaged over a wavelength per unit meridional direction.

$$\frac{d}{dt} < \frac{1}{2} v_1{}^2 >= \gamma < w_2 \psi_1 > \quad + \quad \frac{d}{dt} < \frac{1}{2} v_3{}^2 >= -\gamma < w_2 \psi_3 >$$

$$\frac{dK'}{dt} = \frac{1}{2} H\gamma < w_2 (\psi_1 - \psi_3) >= \frac{1}{2} H < w_2 b_2 >$$

an equation for the rate-of-change of the average perturbation kinetic energy.

Recall that

$$\sigma_2 = f_0(\partial\psi / \partial z)_2 = f\left(\psi_1 - \psi_3\right) / \tfrac{1}{2}H = \gamma\left(\psi_1 - \psi_3\right)$$

In the continuous model, available potential energy is defined as

$$\int_v \frac{1}{2}\frac{b^2}{N^2}\, dV$$

We approximate the contribution to this from the perturbation by defining

$$P' = \tfrac{1}{2}H \times \frac{1}{2}\frac{< b_2{}^2 >}{N^2} = \frac{1}{4}H\mu^2 < (\psi_1 - \psi_3)^2 >$$

to be the average perturbation available potential energy per unit meridional direction.

You may wonder **why the operation** $\int dV$ **is replaced here by** $\frac{1}{2}H$ **rather than** H**.**

- **It turns out to be necessary to do this for energy consistency.**
- **Since** b **is defined only at one level (i.e., level 2), the system knows only about the available potential energy between levels 3 and 1.**
- **With this definition for** P'**, the model is formally equivalent to the two-layer model assuming immiscible fluids with a free fluid interface as studied by Pedlosky (1979; see §7.16).**
- **Holton does not point out this subtlety in defining** P'**.**

Then

$$\frac{d}{dt} < \frac{1}{2}(\psi_1 - \psi_3)^2 > = U_T < (\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 + \psi_3) >$$

$$- \gamma\mu^{-2} < w_2(\psi_1 - \psi_3) > .$$

gives

$$\frac{dP'}{dt} = \frac{2f_0}{N^2} U_T < b_2 v_2 > - \frac{H}{2} < w_2 b_2 >$$

Interpretation

$$\frac{dP'}{dt} = \frac{2f_0}{N^2} U_T < b_2 v_2 > - \frac{H}{2} < w_2 b_2 >$$

- **This term correlates upward motion with positive buoyancy and downward motion with negative buoyancy.**
- **It represents a conversion of perturbation available potential energy into perturbation kinetic energy.**
- **It is the only source of** K' **and is the sink term of** P'**.**

$$\frac{dK'}{dt} = \frac{1}{2} H\gamma < w_2(\psi_1 - \psi_3) >= \frac{1}{2} H < w_2 b_2 >$$

$$\frac{dP'}{dt} = \frac{2f_0}{N^2} U_T < b_2 v_2 > - \frac{H}{2} < w_2 b_2 >$$

- **This term correlates poleward motion with positive buoyancy between levels 1 and 3, and equatorward motion with negative buoyancy.**
- **It is proportional to the vertical shear of the basic flow U_T, or, equivalently, to the basic meridional temperature gradient.**
- **It represents the conversion of mean available potential energy of the basic flow into perturbation available potential energy and is a source term in the above equation.**
- **Clearly, this term must exceed the second term in the equation if the disturbance is to grow.**

Energy conversions in a block diagram

- **Introduce the notation** C(A, B) **to denote a rate-of-conversion of energy form A to energy form B.**
- **Then** C(A, B) = –C(B, A)**, and**
- **In particular**

$$C(\overline{P}, P') = \frac{2f_o}{N^2} U_T < b_2 v_2 >$$

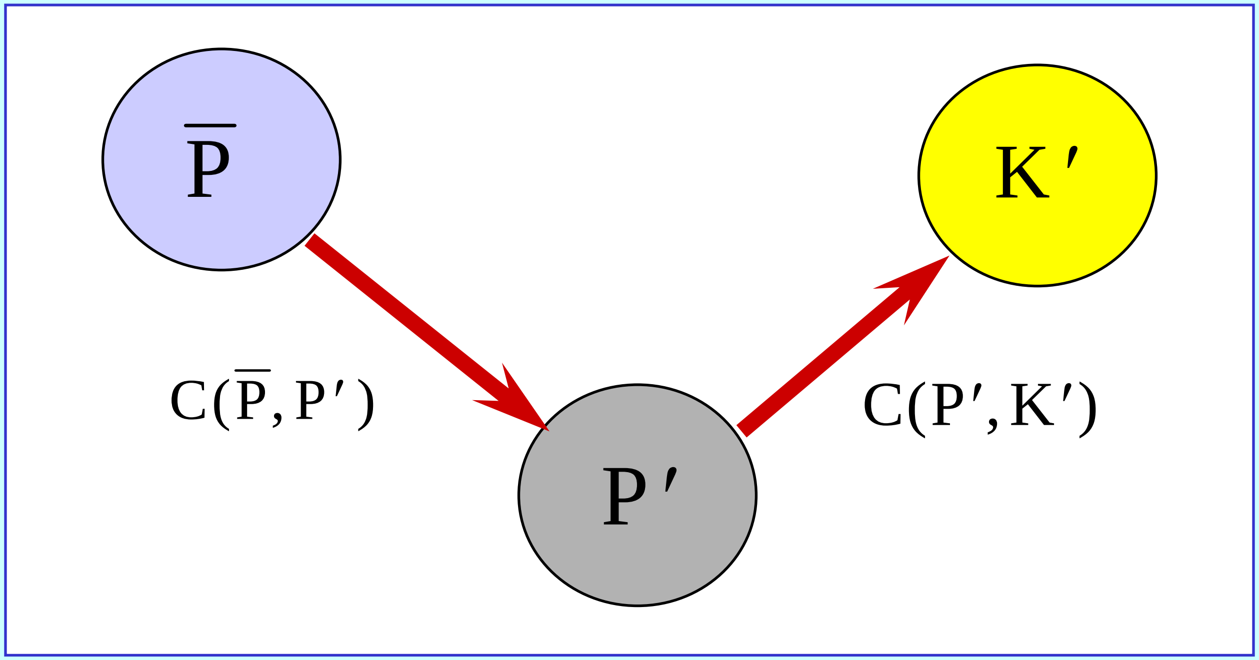
$$C(P', K') = \frac{1}{2} H < w_2 b_2 >$$

Then

$$\frac{dK'}{dt} = \frac{1}{2} H\gamma < w_2(\psi_1 - \psi_3) > = \frac{1}{2} H < w_2 b_2 >$$

$$\frac{dP'}{dt} = \frac{2f_0}{N^2} U_T < b_2 v_2 > - \frac{H}{2} < w_2 b_2 >$$

$$\frac{dK'}{dt} = C(P', K') \quad \textbf{and} \quad \frac{dP'}{dt} = C(\overline{P}, P') - C(P', K')$$



Adding the above equations gives

$$\frac{d}{dt}(K' + P') = C(\overline{P}, P')$$

The rate-of-change of total perturbation energy K' + P'**, is just** $C(\overline{P}, P')$ **.**

Large amplitude waves

- Since the available potential energy of the basic flow per unit volume is finite, exponential growth of a perturbation cannot continue indefinitely.
- The foregoing theories assume that the perturbation remains sufficiently small so that changes in the mean flow due to the presence of the wave can be ignored.
- When the wave amplitude grows to a significant amplitude, its interaction with the mean flow cannot be ignored and the depletion of the mean flow available potential energy is reflected in a reduced growth rate.

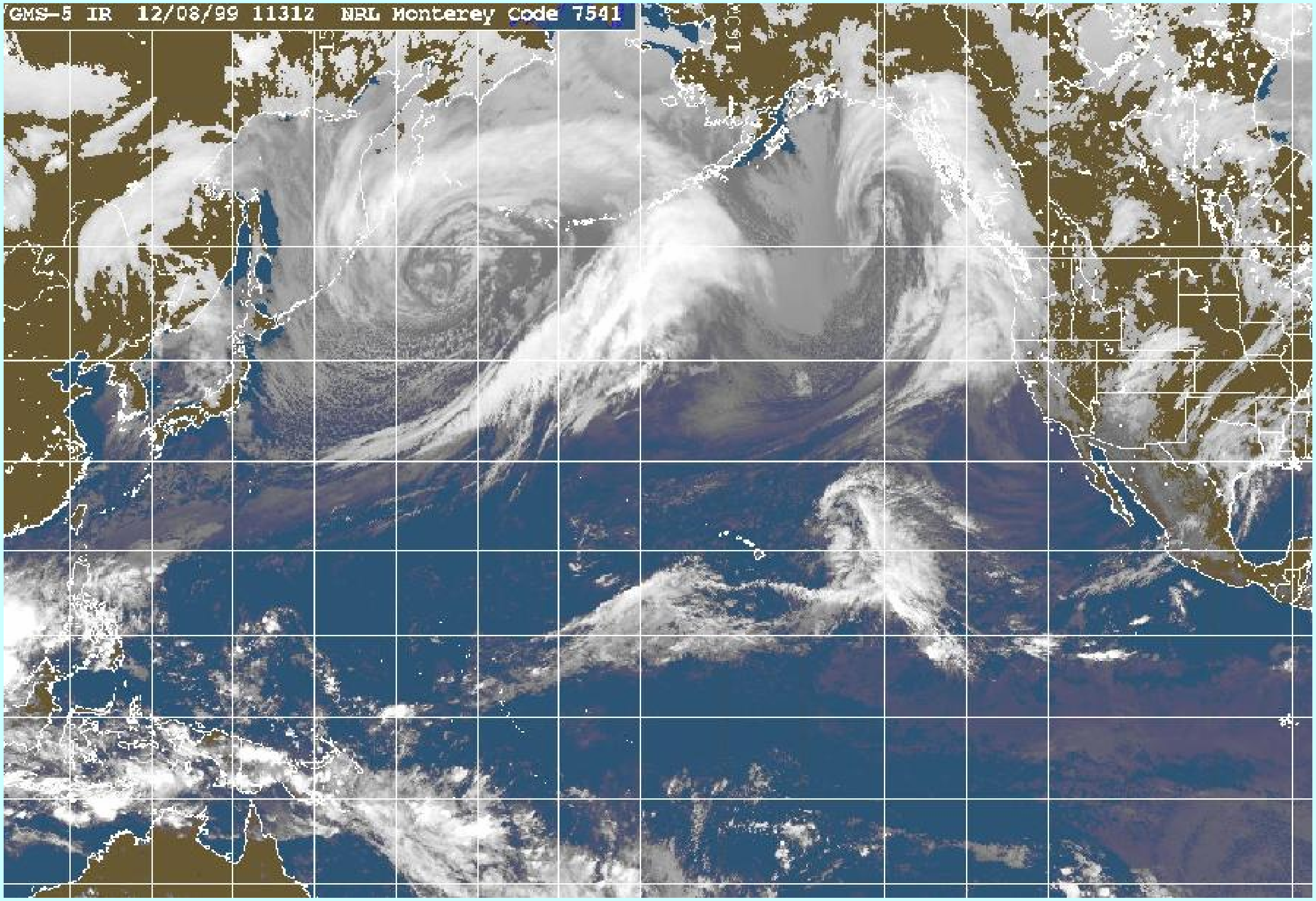
Nonlinear Theory

- **To study such finite amplitude effects necessarily requires a nonlinear analysis in which mean flow changes are determined as part of the solution.**

- **Such analyses are algebraically complicated and beyond the scope of these lecture.**

The role of baroclinic waves in the atmosphere's general circulation

- Differential solar heating between the equatorial and polar regions helps to maintain the available potential energy associated with the middle latitude westerly winds.
- The baroclinic instability of the westerlies leads to the growth of extra-tropical cyclones.
- These cyclones transport heat polewards and upwards, reducing the mean meridional temperature gradient. (i.e., depleting available potential energy) and increasing the vertical stability.



- Extra-tropical cyclones act together with planetary waves to reduce the meridional temperature contrasts which would occur if the earth's atmosphere were in radiative equilibrium.
- Therefore, both types of waves are important components of the atmosphere's "air-conditioning" system.

The End