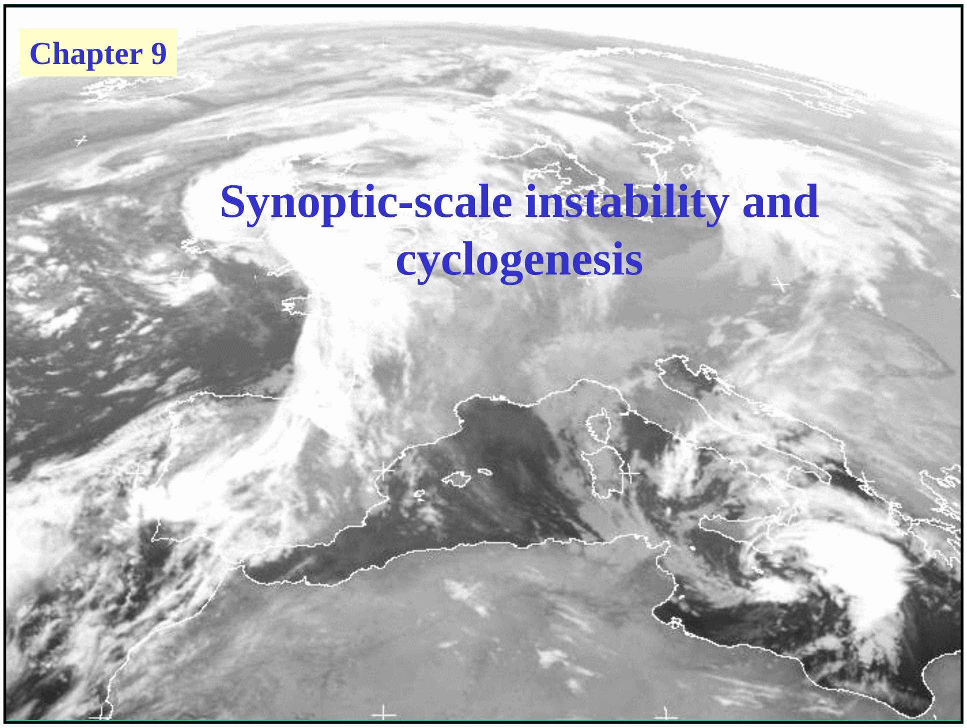
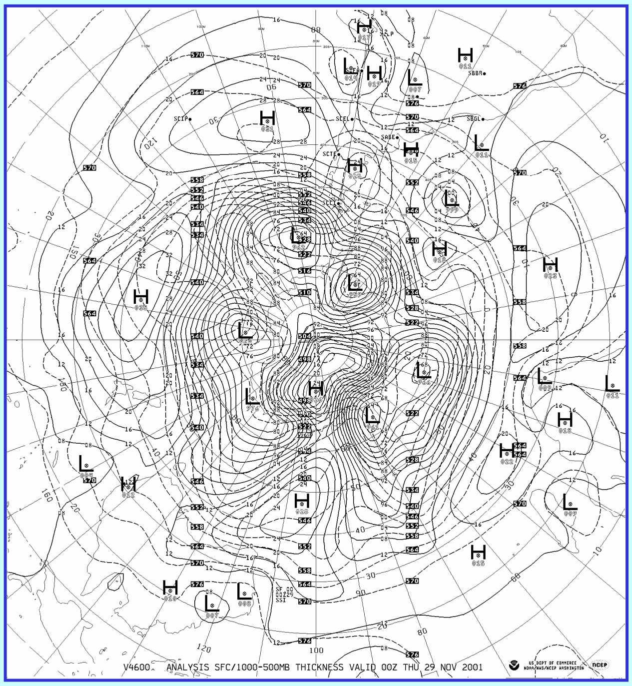
Chapter 9



Synoptic-scale instability and cyclogenesis

Extra-tropical cyclones

- Look at mean sea level isobaric charts => one notices synoptic-scale vortices or low pressure centres, also called extra-tropical cyclones, depressions, or simply ’lows’.

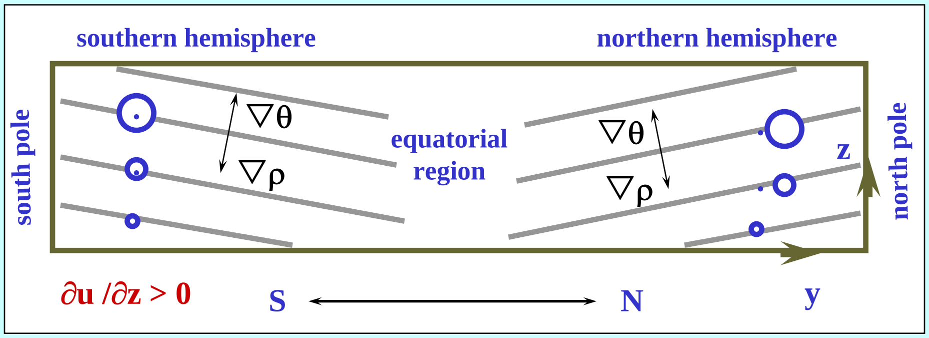


Extra-tropical cyclones

- These vortices play an important role in the dynamics of the atmosphere's general circulation and contribute together with their associated fronts to much of our 'bad weather'.
- The occurrence of extra-tropical cyclones is a manifestation of the inherent instability of the zonal 'westerly' winds of middle latitudes.
- We begin by considering the energy source for the instability.
- Then consider a simple model for cyclogenesis (i.e. cyclone growth).

The middle latitude 'westerlies'

- On average, the tropospheric winds in middle latitudes are westerly and increase in strength with height.
- They are also in approximate thermal wind balance with the poleward temperature gradient associated with differential solar heating.



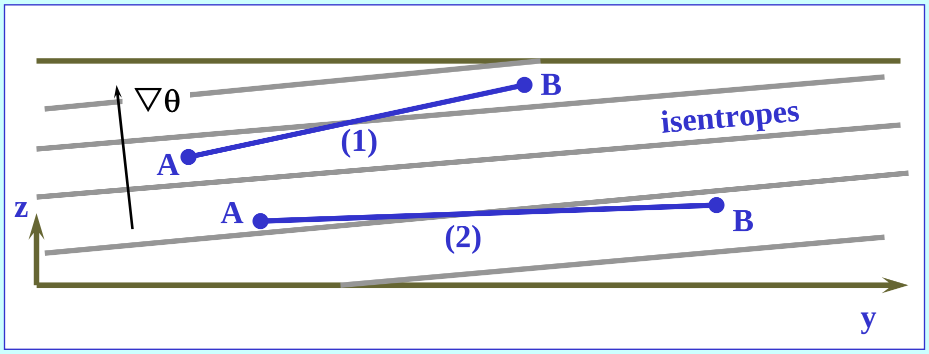
Available potential energy

- The atmosphere has an enormous potential energy measured in the usual way:

$$\int_{\text{atmosphere}} \rho g z \, dV$$

- Only a small fraction of this is available for conversion to kinetic energy.
- The precise amount available is the actual potential energy minus the potential energy obtained after an adiabatic rearrangement of the density field so that the isentropes (surfaces of constant θ) are horizontal and in stable hydrostatic equilibrium.

Consider the adiabatic interchange of two air parcels A and B in the meridional plane...



Let us write:

The horizontal average of $\rho - \rho_*$

$$\rho = \rho_* + \rho_0(z) + \overline{\rho}(y,z) + \rho'(x,y,z,t)$$

Either the volume average of ρ over the whole flow domain, or the surface density

The zonal average of $\rho - \rho_* - \rho_0(z)$

$\rho - [\rho_* + \rho_0(z) + \overline{\rho}(y,z)]$

A zonal average is an average in the x-, or eastward-direction

$$\left(\overline{}\right) = \frac{1}{X}\int_0^X (\ )dx$$

e.g. the length of a latitude circle

Note that $\bar{\rho}(y,z) + \rho'(x,y,z,t)$ **represents the deviation of the density field from hydrostatic equilibrium**

Also, by the definition of the averaging operator $\overline{\rho'} \equiv 0$

In practice, for a Boussinesq fluid

$$\rho_* >> \max\{|\rho_0(z)|, |\bar{\rho}(y,z)|, |\rho'(x,y,z,t)|\}$$

Buoyancy force $b = -g\dfrac{\rho - \rho_* - \rho_0(z)}{\rho_*}$

$$= -\frac{g\bar{\rho}}{\rho_*} - \frac{g\rho'}{\rho_*} = \bar{b} + b' \text{ , say}$$

An air parcel displaced a vertical distance ξ **from equilibrium experiences a restoring force** $b = -N^2\xi$ **per unit mass.**

The work done in producing such a displacement is

$$\int_0^\xi b\,dz = \tfrac{1}{2}N^2\xi^2 \qquad \underline{= \tfrac{1}{2}b^2 / N^2}$$

$b = -N^2\xi$ **assuming that N is a constant**

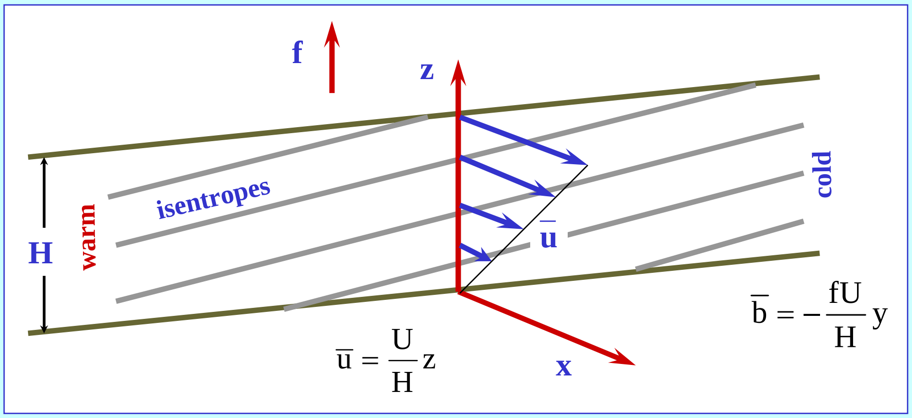
The change in potential energy due to an adiabatic rearrangement of the density field from equilibrium is

$$\int_{\text{atmosphere}} \frac{b^2}{2N^2}\,dV = \int_{\text{atmosphere}} \frac{1}{2N^2}(\overline{b^2} + \overline{b'^2})\,dV$$

A measure of the available potential energy - APE.

When no disturbance (b´ = 0), the APE of a zonal flow is related to the deviation of the local density from the horizontal average at that level.

Baroclinic instability: the Eady problem



Zonal flow configuration in the Eady problem (northern hemisphere).

The Eady model

- **Assumptions:**
  - Boussinesq liquid.
  - N is a constant.
  - f is a constant
- **Basic-state streamfunction =>** $\bar{\Psi} = -\frac{U}{H} yz$
- **Basic-state potential vorticity =>**

$$\bar{q} = \nabla^2{}_h \bar{\Psi} + f + \frac{f^2}{N^2} \frac{\partial^2 \bar{\Psi}}{\partial z^2} = f = \text{constant}$$

The zonal flow satisfies the potential vorticity equation exactly.

Perturbation solution

We consider small perturbations to the zonal flow: put

$$\psi = \overline{\psi} + \psi' \qquad q = \overline{q} + q'$$

$$\Rightarrow \quad \left[\frac{\partial}{\partial t} + (\overline{u} + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right](\overline{q} + q') = 0$$

linearize =>

$$\left[\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right]\left[\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2 \psi'}{\partial z^2}\right] = 0$$

Boundary conditions

z = H

z = 0

w = 0 **at the ground** (z = 0) **and at the model tropopause,** z = H**.**

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right] b + N^2 w = 0 \quad \Rightarrow$$

$$\frac{\partial \overline{b}}{\partial y} = -\frac{fU}{H}$$

$$\left[\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right](\overline{b} + b') = 0$$

$$b' = f\frac{\partial \psi'}{\partial z}$$

$$\Rightarrow \quad \left[\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right]\frac{\partial \psi'}{\partial z} - \frac{U}{H}\frac{\partial \psi'}{\partial x} = 0 \quad \text{at } z = 0, H$$

For maximum simplicity consider 2-D disturbances with $\partial/\partial y \equiv 0$.

Assume that an arbitrary disturbance can be expressed as a sum of Fourier modes.

Consider a single Fourier component with

$$\psi(x,z,t) = \hat{\psi}(z)e^{ik(x-ct)}$$

k **and** c **are constants and 'the real part' is implied.**

The objective is to determine c **as a function of wavenumber** k**, and the corresponding eigenfunction** $\hat{\psi}(z)$ **.**

Substitution =>

$$\frac{d^2\hat{\psi}}{dz^2} - \frac{N^2k^2}{f^2}\hat{\psi} = 0$$

$$\frac{d^2\hat{\psi}}{dz^2} - \frac{N^2k^2}{f^2}\hat{\psi} = 0$$

Put $z' = z/H$ **=>**

$$\frac{d^2\hat{\psi}}{dz'^2} - 4s^2\hat{\psi} = 0$$

$$4s^2 = \frac{N^2H^2}{f^2}k^2 = L_R{}^2k^2$$

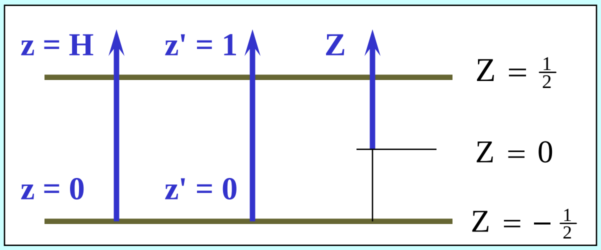
$L_R = NH/f$ **is called the Rossby radius of deformation.**

Mathematical niceties

A gain in symmetry is obtained if we put

$$z' = Z + \tfrac{1}{2} \qquad c = \tfrac{1}{2}U + UC$$

nondimensional phase speed of the wave



$$\frac{d^2\hat{\psi}}{dZ^2} - 4s^2\hat{\psi} = 0$$

$$(C - Z)\hat{\psi}_Z + \hat{\psi} = 0 \qquad \text{at} \qquad Z = \tfrac{1}{2}, -\tfrac{1}{2}$$

Solution is:

$$\hat{\psi}(Z) = A \sinh 2sZ + B \cosh 2sZ$$

Boundary conditions give

$$[2s(C + \tfrac{1}{2})\cosh s \sinh s]A + [\cosh s - 2(C + \tfrac{1}{2})\sinh s]B = 0$$

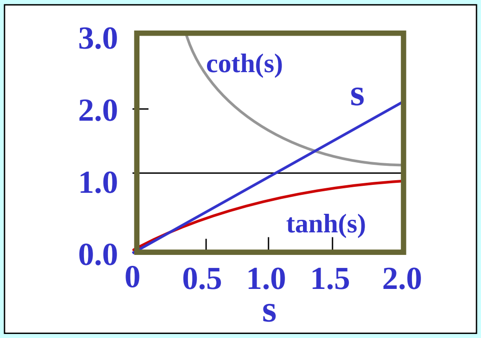
$$[2s(C - \tfrac{1}{2})\cosh s + \sinh s]A + [\cosh s - 2(C - \tfrac{1}{2})\sinh s]B = 0$$

A pair of homogeneous algebraic equations for A **and** B.

Solution exists only if the determinant of the coefficients is zero

$$4s^2C^2 = 1 + s^2 - 2s \coth 2s$$

$$c = \tfrac{1}{2}U \pm (U/2s)\left[(s - \coth s)(s - \tanh s)\right]^{1/2}$$



The expression $(s - \coth s)(s - \tanh s)$ **inside brackets is negative if** $s < s_0$ **and positive if** $s > s_0$**, where** $s_0 = \coth s_0$**.**

- **For** $s < s_0$**,** c **has the form** $c = \tfrac{1}{2}U \pm i c_i(s)$

  where $i = \sqrt{-1}$

Then $\psi' = \hat{\psi}(z) e^{\pm k c_i t} e^{ik(x - \frac{1}{2}Ut)}$

wave-type disturbances exist and for $s < s_0$ **they propagate with phase speed**

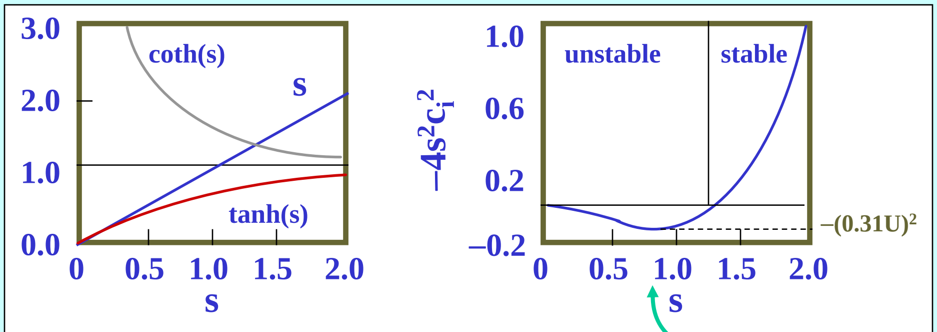
$$\mathrm{Re}(c) = \tfrac{1}{2}U,$$

$=$ **the zonal wind speed at** $z = \tfrac{1}{2}H$

This height, at which $c = \overline{u}(z)$ **is called the steering level for the disturbance.**

Such disturbances also grow or decay exponentially with time, the growth rate (or decay rate) being kc_i**, or** $2sc_i(s)/L_R$**.**

$$c = \tfrac{1}{2}U \pm (U/2s)\left[(s - \coth s)(s - \tanh s)\right]^{1/2}$$



The growth rate (or decay rate) is kc_i**, or** $2sc_i\,(s)/L_R$**.**

For the unstable wave with $kc_i > 0$**, the maximum growth rate occurs when** $s = s_m = 0.8$.

The unstable wave mode

Our interest is primarily in the amplifying mode.

For $s > s_0$**, both solutions are neutrally stable; i.e.,** $\mathrm{Im}(c) = 0$**.**

For the unstable wave with $kc_i > 0$**, the maximum growth rate occurs when** $s = s_m = 0.8$**, and**

$$(kc_i)_{max} = 2s_m c_i(s_m)/L_R = 0.31U/L_R$$

The half-wavelength of the fastest growing wave is

$$\tfrac{1}{2}\lambda_{max} = \pi / k_m = \pi L_R / 2s_m$$

this being the distance between the ridge (maximum p' **or** ψ'**) and trough (minimum** p' **or** ψ'**).**

Typical scales

Typical atmospheric values are f ~ 10^{-4} s^{-1} **(**45 **deg. latitude),** N ~ 10^{-2} s^{-1}**,** H ~ 10^4 m **(**10 km**) and** U ~ 40 ms^{-1}**, giving ,**

$$L_R = \frac{NH}{f} \sim 10^6 \mathrm{m} \;\; (10^3 \,\mathrm{km}) \,, \qquad (kc_i)^{-1}_{max} \sim 0.8 \times 10^5 \mathrm{s}$$

(about 1 day)

and $\frac{1}{2}\lambda_{max} \sim 2 \times 10^6 \mathrm{m} \;\; (2000 \;\; \mathrm{km})$

These values for $(kc_i)^{-1}_{max}$ **and** $\frac{1}{2}\lambda_{max}$ **are broadly typical of the observed e-folding times and horizontal length scales of extra-tropical cyclones in the atmosphere.**

Adding $[2s(C+\tfrac{1}{2})\cosh s \sinh s]A + [\cosh s - 2(C+\tfrac{1}{2})\sinh s]B = 0$

and $[2s(C-\tfrac{1}{2})\cosh s + \sinh s]A + [\cosh s - 2(C-\tfrac{1}{2})\sinh s]B = 0$

$\Rightarrow$ $B = 2sCA \,/\, (s \tanh s - 1) = iDc_i(s)A$ **, say.**

Then $\boxed{\psi' = \hat{\psi}(z) e^{\pm kc_i t} e^{ik(x - \frac{1}{2}Ut)}}$

The vertical structure of the disturbance is given by

$$\hat{\psi}(Z) = A[\sinh 2sZ + iDc_i(s) \;\cosh\; 2sZ] = \underline{\tilde{A}(Z)e^{i\gamma(Z)}}$$

where $\tilde{A}(Z) = A[\sinh^2 2sZ + D^2 c_i(s)^2 \;\; \cosh^2 \;\; 2sZ]^{1/2}$

and $\gamma(Z) = \arg[\sinh 2sZ + iDc_i(s) \;\cosh\; 2sZ]$

The perturbation streamfunction takes the form

$$\psi'(x,y,Z,t) = \tilde{A}(Z)e^{kc_i t}\cos[k(x - \tfrac{1}{2}Ut) + \gamma(Z)]$$

Hence the streamfunction (or pressure-) perturbation and other quantities have a phase variation with height.

Note: all the flow quantities are determined in terms of ψ'; e.g.,

$$v' = \psi'_x \propto \sin\left[k(x - \tfrac{1}{2}Ut) - \gamma(Z)\right], \quad u' = -\psi'_y = 0,$$

$$w = -\frac{fH}{N^2}\left[\left(\frac{\partial}{\partial t} + UZ\frac{\partial}{\partial x}\right)\frac{\partial\psi'}{\partial Z} - U\frac{\partial\psi'}{\partial x}\right], \text{ and } \quad b = f\psi'_Z = \ldots\ .$$

To evaluate the expressions for w and σ involves considerable algebra.

Structure of an unstable Eady wave

- The detailed structure of an unstable two-dimensional Eady wave is shown in the next figure including:
- Geostrophic quantities

  - the streamfunction (pressure) perturbation

  - the meridional velocity isotachs $v(x, z)$;

  - the buoyancy perturbation $b(x, z)$, proportional to the potential temperature deviation $\theta'(x,z) = (\theta - \theta_0(z))$;

  - the vertical component of relative vorticity $\zeta(x, z)$;

  And
- Ageostrophic quantities:

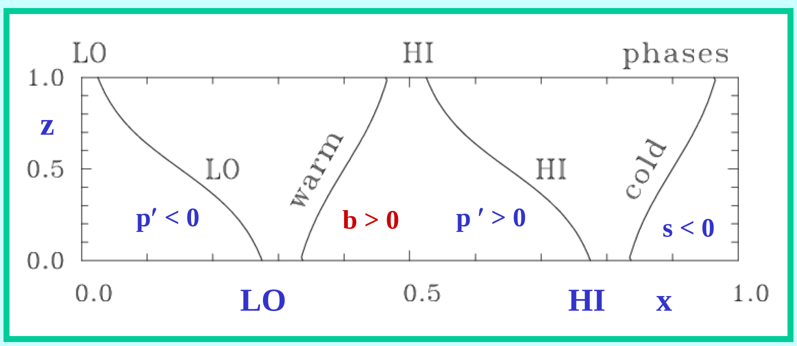
Structure of an unstable Eady wave

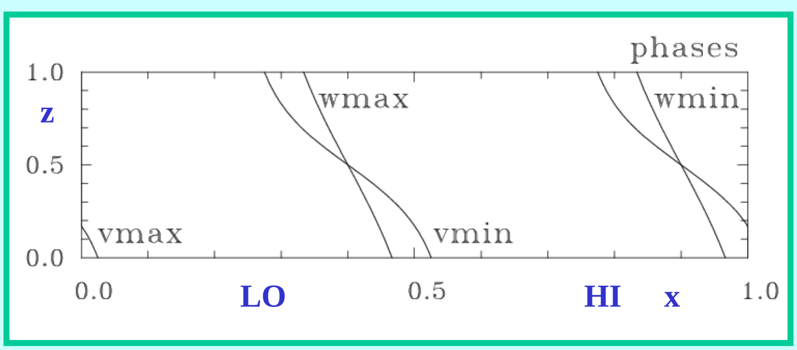
- **Ageostrophic quantities:**

  - the vertical velocity $w(x, z)$**;**

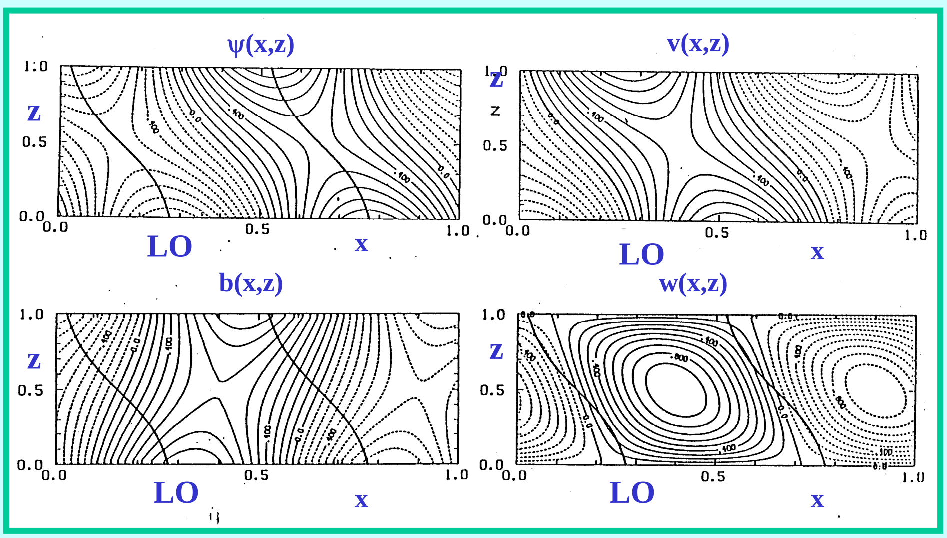
  - the streamfunction of the ageostrophic motion in a vertical plane, denoted here by $\Phi(x, z)$ **defined by** $u_a = \Phi_z$ **,** $w = -\Phi_x$ **satisfies the two-dimensional continuity equation,** $\partial u_a/\partial x + \partial w/\partial z = 0$**;**

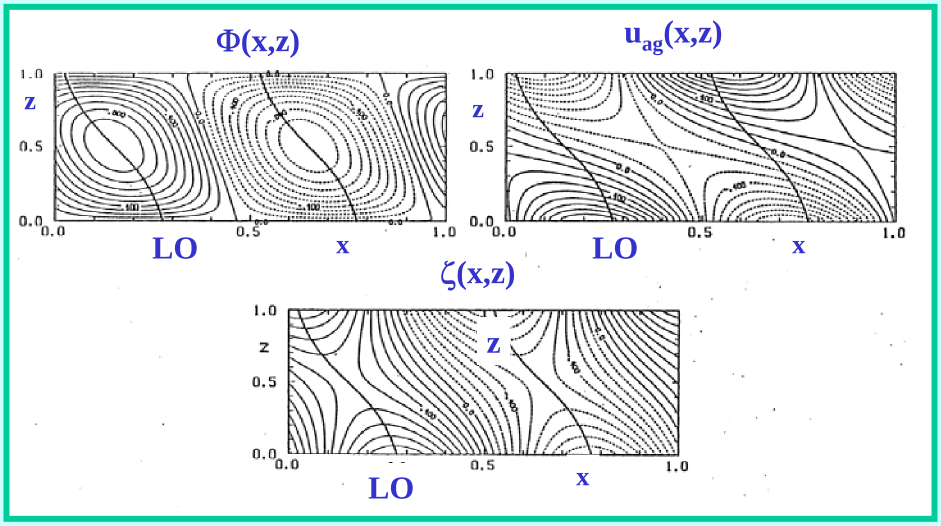
  - the ageostrophic wind $u_a(x, z)$**.**





- The minimum pressure perturbation (the pressure trough axis) and the maximum pressure perturbation (the ridge axis) tilt westwards with height.
- This is a characteristic feature of developing cyclones and anticyclones in the atmosphere.
- The warmest air ($b > 0$) is rising ($w > 0$) and the coldest air ($b < 0$) is subsiding ($w < 0$), an indication that available potential energy is being reduced.
- It is clear also that the warm air moves polewards ($v' > 0$ in NH) and the cold air moves equatorwards ($v' < 0$ in NH) so that the wave effects a poleward heat transport.
- Note that cyclonic ζ corresponds with negative values in the southern hemisphere.





Three-dimensional waves

- **So far we have assumed that the wave structure is independent of the meridional direction** y.
- **A slightly more realistic calculation vis-á-vis extra-tropical cyclones is to relax this assumption and to investigate wave disturbances confined to a zonal channel with rigid (frictionless) walls at** $y = 0$ **and** $y = Y$**, say.**
- **Then,** $\partial/\partial y \neq 0$ **and** $u' \neq 0$**, but** $v' = 0$ **at** $y = 0$ **and** Y**.**
- **In this case, the solution procedure is essentially the same as before, but we now take**

$$\psi'(x, y, z, t) = \hat{\psi}(z)e^{ik(x-ct)} \sin(m\pi y / Y)$$

m **is an integer to satisfy the condition** $v' = \psi'_x = 0$ **at** $y = 0, Y$**.**

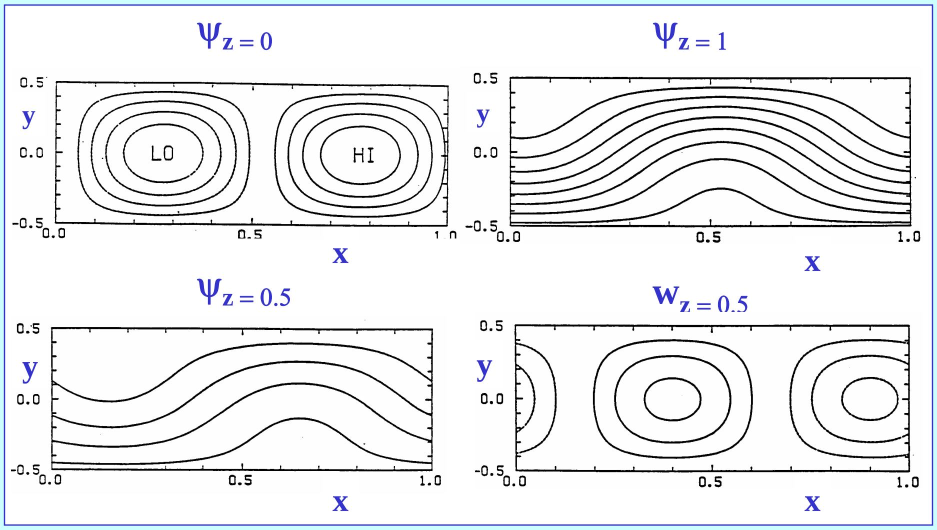
The only change to the foregoing analysis is to replace $4s^2 = L_R^2$ **with**

$$4s^2 = L_R^2(k^2 + m^2\pi^2 / Y^2)$$

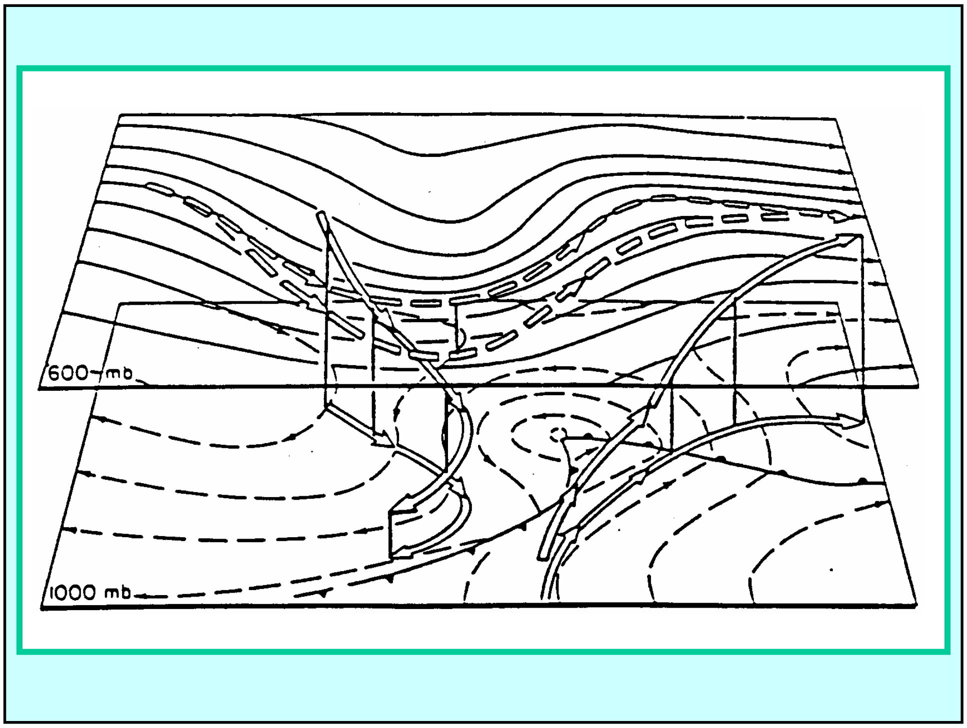
The next figure shows the pressure patterns corresponding to the total streamfunction

$$\psi = \overline{\psi} + \psi'$$

at the surface, in the middle troposphere and in the upper troposphere, for the wave with $m = 1$**.**



Isobar patterns: a) at the surface ($z' = 0$) **in the middle troposphere** ($z' = 0.5$) **and c) in the upper troposphere** ($z' = 1.0$) **in the Eady solution for a growing baroclinic wave with** $m = 1$**. Shown in d), is the isotach pattern of vertical velocity in the middle troposphere.**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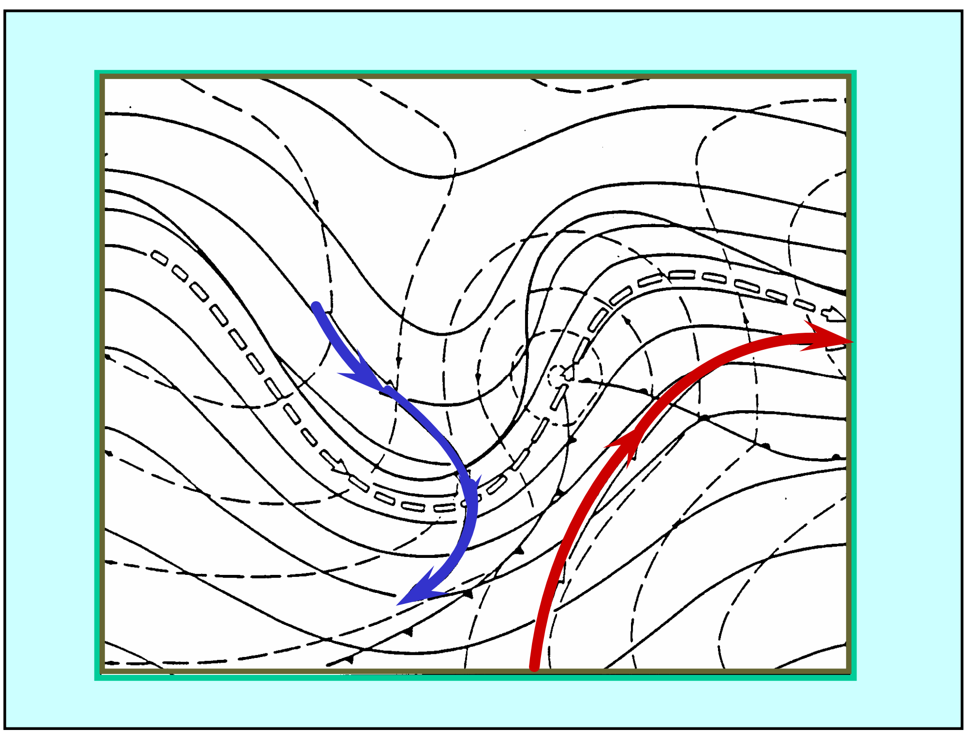

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