Chapter 8

Quasi-geostrophic motion

Scale analysis for synoptic-scale motions

- Simplification of the basic equations can be obtained for synoptic scale motions.
- Consider the Boussinesq system $\Rightarrow$ ρ is assumed to be constant in as much as it affects the fluid inertia and continuity.
- Introduce nondimensional variables, (′), and typical scales (in capitals) as follows:

$(x, y) = L(x', y')$ $z = Hz'$ $t = (L/U)t'$

$(u,v) = U(u', v')$ $w = Ww'$ $p = Pp'$; $b = \Sigma b'$; and $f = f_0 f'$,

f_0 is a typical middle latitude value of f.

The horizontal component of the momentum equation takes the nondimensional form

$$\mathrm{Ro}\left[\frac{\partial}{\partial t'}+\mathbf{u}_h'\cdot\nabla_h'+\left[\frac{W}{U}\frac{L}{H}\right]w'\frac{\partial}{\partial z'}\right]\mathbf{u}_h'+f'\,\mathbf{k}\wedge\mathbf{u}_h'=-\frac{P}{\rho U L f_0}\nabla_h' p'$$

where ∇'_h **denotes the operator** ($\partial/\partial x'$, $\partial/\partial y'$, 0) **and** Ro **is the nondimensional parameter** $U/(f_0L)$, **the Rossby number.**

- **Definition of scales => all** $(')$**-quantities have magnitude** ~ O(1).
- **Typical values of the scales for middle latitude synoptic systems are:** $L = 10^6$ m, $H = 10^4$ m, $U = 10$ ms^{-1} , $P = 10^3$ Pa (10 mb), $b = g\delta T/T = 10*3/300 = 10$ ms^{-2}, $\rho = 1$ kg m^{-3} **and** $f_0 \sim 10^{-4}$ s.
- **Clearly, we can take** $P = ULf_0$.
- **Then, assuming that** (WL/UH) ~ O(1), **the key parameter is the Rossby number.**

$$\mathrm{Ro}\left[\frac{\partial}{\partial t'}+\mathbf{u}_h'\cdot\nabla_h'+\left[\frac{W}{U}\frac{L}{H}\right]w'\frac{\partial}{\partial z'}\right]\mathbf{u}_h'+f'\,\mathbf{k}\wedge\mathbf{u}_h'=-\frac{P}{\rho U L f_0}\nabla_h' p'$$

- **For synoptic scale motions at middle latitudes,** Ro ~ 0.1 **so that, to a first approximation, the** $D'\mathbf{u}'_h/Dt'$ **can be neglected and the equation reduces to one of geostrophic balance.**
- **In dimensional form it becomes**

$$f\mathbf{k}\wedge\mathbf{u}_h=-\frac{1}{\rho}\nabla_h p$$

We solve it by taking $\mathbf{k}\wedge$ **of both sides.**

$$\mathbf{u}_g=+\frac{1}{\rho f}\mathbf{k}\wedge\nabla_h p$$

This equation defines the geostrophic wind. Our scaling shows to be a good approximation to the total horizontal wind $\mathbf{u}_h$.

- As noted earlier, it is a diagnostic equation from which the wind can be inferred at a particular time when the pressure gradient is known.
- In other words, the limit of

$$Ro\left[\frac{\partial}{\partial t'} + \mathbf{u}_h' \cdot \nabla_h' + \left[\frac{W}{U}\frac{L}{H}\right] w' \frac{\partial}{\partial z'}\right]\mathbf{u}_h' + f'\,\mathbf{k}\wedge\mathbf{u}_h' = -\frac{P}{\rho U L f_0}\nabla_h' p'$$

as $Ro \to 0$ is degenerate in the sense that time derivatives drop out.

- We cannot use the geostrophic equation to predict the evolution of the wind field.
- If f is constant the geostrophic wind is horizontally nondivergent; i.e., $\nabla_h \cdot \mathbf{u_g} = 0$.

The difference between the horizontal wind and the geostrophic wind is called the ageostrophic wind:

$$\mathbf{u}_a = \mathbf{u}_h - \mathbf{u}_g$$

Now $$Ro\left[\frac{\partial}{\partial t'} + \mathbf{u}_h' \cdot \nabla_h' + \left[\frac{W}{U}\frac{L}{H}\right] w' \frac{\partial}{\partial z'}\right]\mathbf{u}_h' + f'\,\mathbf{k}\wedge\mathbf{u}_h' = -\frac{P}{\rho U L f_0}\nabla_h' p'$$

⇒ for $Ro \ll 1$, $\mathbf{u_h} \sim \mathbf{u_g}$ while $\mathbf{u_a}$ is of order Ro.

⇒ A suitable scale for $|\mathbf{u_a}|$ is URo.

Because $\nabla_h \cdot \mathbf{u_g} = 0$, the continuity equation reduces to the nondimensional form (assuming that f is constant).

The second term of $\quad Ro\,\nabla'_h \cdot \mathbf{u}'_a + \left[\frac{W}{H}\frac{L}{U}\right]\frac{\partial w'}{\partial z'} = 0$

is important if $\quad \frac{W}{H}\frac{L}{U} = Ro \approx 0.1$

a typical scale for w is $U(H/L)Ro = 10^{-2}\ ms^{-1}$.

the operator $\mathbf{u}_{h'} \cdot \nabla_{h'}$ in

$$Ro\left[\frac{\partial}{\partial t'} + \mathbf{u}'_h \cdot \nabla'_h + \cancel{\left[\frac{W}{U}\frac{L}{H}\right] w' \frac{\partial}{\partial z'}}\right]\mathbf{u}'_h + f'\,\mathbf{k} \wedge \mathbf{u}'_h = -\frac{P}{\rho U L f_0}\nabla'_h p'$$

is much larger than $\partial w'/\partial z'$.

To a first approximation, advection by the vertical velocity can be neglected, both in the momentum and thermodynamic equations.

Two important results

To a first approximation, advection by the vertical velocity can be neglected, both in the momentum and thermodynamic equations.

Also the dominant contribution to $\mathbf{u}_{h'} \cdot \nabla_{h'}$ is $\mathbf{u}'_g \cdot \nabla_{h'}$

in quasi-geostrophic motion, advection is by the geostrophic wind.

Vertical momentum equation

In nondimensional form, the vertical momentum equation is

$$Ro\left[\frac{W}{H}\frac{L}{U}\right]\frac{D'w'}{Dt'} = -\frac{\partial p'}{\partial z'} + \frac{\Sigma H}{ULf_0}b'$$

It is easy to check that $\Sigma H/(ULf_0) = 1$

$\Rightarrow$ $Ro(WH/UL) = Ro^2(H/L)^2 = 10^{-6}$.

$\Rightarrow$ Synoptic scale perturbations are in a very close state of hydrostatic balance.

The governing equations for quasi-geostrophic motion in dimensional form

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right]\mathbf{u}_g + f\mathbf{k} \wedge \mathbf{u}_a = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = b$$

$$\nabla_h \cdot \mathbf{u}_a + \frac{\partial w}{\partial z} = 0$$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right]b + N^2 w = 0$$

an approximated form of the thermodynamic equation

$$\mathbf{u}_g = +\frac{1}{\rho f}\mathbf{k} \wedge \nabla_h p$$

where at present f is assumed to be a constant.

A prediction equation for the flow at small Ro

First derive the vorticity equation:

Use: $$\mathbf{u}_g \cdot \nabla \mathbf{u}_g = \nabla(\tfrac{1}{2}\mathbf{u}_g{}^2) - \mathbf{u}_g \wedge (\nabla \wedge \mathbf{u}_g)$$

$$\Rightarrow \quad \frac{\partial \mathbf{u}_g}{\partial t} + \nabla(\tfrac{1}{2}\mathbf{u}_g{}^2) - \mathbf{u}_g \wedge (\nabla \wedge \mathbf{u}_g) + f\mathbf{k} \wedge \mathbf{u}_a = 0$$

Taking $\mathbf{k} \cdot \nabla \wedge$ gives $$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right](\zeta_g + f) = -f\nabla_h \cdot \mathbf{u}_a$$

where $\zeta_g = \mathbf{k} \wedge u_g$ is the vertical component of relative vorticity computed using the geostrophic wind.

If f is constant, $$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right] f = 0 \ .$$

Assume N^2 is constant

Take $\frac{\partial}{\partial z}$ **of** $$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right] b + N^2 w = 0$$

$$\Rightarrow \quad \left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right]\frac{\partial b}{\partial z} + \frac{\partial \mathbf{u}_g}{\partial z} \cdot \nabla_h b + N^2 \frac{\partial w}{\partial z} = 0$$

$\frac{\partial}{\partial z}$ **of** $\mathbf{u}_g = +\frac{1}{\rho f}\mathbf{k} \wedge \nabla_h p \quad \Rightarrow \quad \frac{\partial \mathbf{u}_g}{\partial \mathbf{z}} = +\frac{1}{\rho f}\mathbf{k} \wedge \nabla_h \frac{\partial p}{\partial z}$

Using $\frac{1}{\rho}\frac{\partial p}{\partial z} = b \quad \Rightarrow \quad \frac{\partial \mathbf{u}_g}{\partial \mathbf{z}} = +\frac{1}{f}\mathbf{k} \wedge \nabla_h b$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right]\frac{\partial b}{\partial z} + \cancel{\frac{\partial \mathbf{u}_g}{\partial z} \cdot \nabla_h b} + N^2 \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = -\nabla_h \cdot \mathbf{u}_a$$

$$\left[\frac{\partial}{\partial t}+\mathbf{u}_g\cdot\nabla_h\right]\frac{\partial b}{\partial z}=-N^2\nabla_h\cdot\mathbf{u}_a$$

$$\left[\frac{\partial}{\partial t}+\mathbf{u}_g\cdot\nabla_h\right](\zeta_g+f)=-f\nabla_h\cdot\mathbf{u}_a$$

$$\left[\frac{\partial}{\partial t}+\mathbf{u}_g\cdot\nabla_h\right]\left[\zeta_g+f+\frac{f}{N^2}\frac{\partial b}{\partial z}\right]=0$$

- **Assumes that f is a constant (then we can omit the single f in the middle bracket).**
- **If the meridional displacement of air parcels is not too large, we can allow for meridional variations in f within the small Rossby number approximation - see exercise 8.1.**

The quasi-geostrophic potential vorticity equation

- **Suppose that $f = f_0 + \beta y$, then**

$$\left[\frac{\partial}{\partial t}+\mathbf{u}_g\cdot\nabla_h\right]q=0$$

where $$q=\zeta_g+f+\frac{f_0}{N^2}\frac{\partial b}{\partial z}$$

- **This is an equation of fundamental importance in dynamical meteorology; it is the quasi-geostrophic potential vorticity equation**
- **It states that the quasi-geostrophic potential vorticity q is conserved along geostrophically computed trajectories.**
- **It is the prognostic equation which enables us to calculate the time evolution of the geostrophic wind and pressure fields.**

Expression of q in terms of pressure

$$\zeta_g = \mathbf{k}\cdot\nabla\wedge\mathbf{u}_g = \frac{1}{\rho f_0}\nabla_h{}^2 p \qquad b = \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$q = \zeta_g + f + \frac{f_0}{N^2}\frac{\partial b}{\partial z}$$

$$q = \frac{1}{\rho f_0}\nabla_h{}^2 p + f + \frac{f_0}{\rho N^2}\frac{\partial^2 p}{\partial z^2}$$

Solution procedure

Write $\left[\frac{\partial}{\partial t} + \mathbf{u}_g\cdot\nabla_h\right]q = 0$ **in the form**

$$\frac{\partial q}{\partial t} = -\mathbf{u}_g\cdot\nabla_h q$$

$$\mathbf{u}_g = \frac{1}{\rho f_0}\mathbf{k}\wedge\nabla_h p$$

- Suppose that we make an initial measurement of the pressure field $p(x,y,z,0)$ at time $t = 0$.
- Calculate $q(x,y,z,0)$ using

$$q = \frac{1}{\rho f_0}\nabla_h{}^2 p + f + \frac{f_0}{\rho N^2}\frac{\partial^2 p}{\partial z^2}$$

- Calculated $\mathbf{u}_\mathbf{g}(x,y,z,0)$ using

$$\mathbf{u}_g = \frac{1}{\rho f_0}\mathbf{k}\wedge\nabla_h p$$

- Predict the distribution of $q(x,y,z,\Delta t)$ using

$$\frac{\partial q}{\partial t} = -\mathbf{u}_g\cdot\nabla_h q$$

- Diagnose $p(x,y,z,\Delta t)$ by solving the elliptic partial differential equation for p:

$$\boxed{\nabla_h{}^2 p + \frac{f_0^2}{N^2}\frac{\partial^2 p}{\partial z^2} = \rho f_0(q - f)}$$

- Diagnose $\mathbf{u}_\mathbf{g}(x,y,z,\Delta t)$ using

$$\boxed{\mathbf{u}_g = \frac{1}{\rho f_0}\mathbf{k}\wedge\nabla_h p}$$

- Repeat the process ...

Boundary conditions

- In order to carry out the integrations, appropriate boundary conditions must be prescribed.
- For example, for flow over level terrain, $w = 0$ at $z = 0$.

Use $\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right] b + N^2 w = 0$ and $b = \frac{1}{\rho}\frac{\partial p}{\partial z}$

$$\Rightarrow \left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right] \frac{\partial p}{\partial z} = 0 \qquad \text{at } z = 0$$

More on the approximated thermodynamic equation

- When N is a constant, the nondimensional form of this equation is

$$\frac{D'b'}{Dt'} + \frac{1}{B} w' = 0$$

where $B = \frac{f_0^2 L^2}{N^2 H^2} = \frac{L^2}{L_R^2}$ and $L_R = \frac{NH}{f_0}$

the Burger number

the Rossby radius of deformation

- An important feature of quasi-geostrophic motion is the assumption that $L \sim L_R$, or equivalently that $B \sim 1$.

When B ~ 1, $\dfrac{D'b'}{Dt'} + \dfrac{1}{B} w' = 0$ ⟹

The rate-of-change of buoyancy (and temperature) experienced by fluid parcels is associated with vertical motion in the presence of a stable stratification.

- **Since in quasi-geostrophic theory the total derivative** D/Dt **is approximated by** $\partial/\partial t + \mathbf{u_g} \cdot \nabla_h$**, the rate-of-change of buoyancy is computed following the (horizontal) geostrophic velocity** $\mathbf{u_g}$.
- **The vertical advection of buoyancy** $w\partial/\partial z$ **is negligible.**
- **Thus quasi-geostrophic flows "see" only the stratification of the basic state characterized by** $N^2 = (g/\theta)d\theta_0/dz$ **---- this is independent of time; such flows cannot change the 'effective static stability' characterized locally by** $N^2 + \partial b/\partial z$ **.**

The quasi-geostrophic equation for a compressible atmosphere

- **The derivation of the potential vorticity equation for a compressible atmosphere is similar to that for a Boussinesq fluid.**

$s = c_p \ln\theta$

- **The equation for the conservation of entropy, or equivalently, for potential temperature** θ, **replaces the equation for the conservation of density:**

$$\frac{D\rho}{Dt} = 0 \quad \Longrightarrow \quad \frac{D\theta}{Dt} = 0$$

- **Other changes are**

$$b = -g\frac{\rho - \rho_o}{\rho^*} \quad \Longrightarrow \quad b = g\frac{\theta - \theta_o}{\theta^*} \qquad N^2 = \frac{g}{\theta_*}\frac{d\theta_o}{dz}$$

The theory applies to small departures from an adiabatic atmosphere in which $\theta_0(z)$ is approximately constant, equal to θ^*.

For a deep atmospheric layer, the continuity equation must include the vertical density variation $\rho_0(z)$:

$$\nabla_h \cdot \mathbf{u}_a + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0$$

The vorticity equation is

$$\frac{D}{Dt}(\zeta_g + f) = \frac{f_0}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w)$$

The potential vorticity equation is

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla_h\right]\left[\zeta_g + f + \frac{f_0^{\,2}}{\rho_0(z)}\frac{\partial}{\partial z}\left[\frac{\rho_0(z)}{N^2}\frac{\partial \psi}{\partial z}\right]\right] = 0$$

Quasi-geostrophic flow over a bell-shaped mountain

- For steady flow ($\partial/\partial t \equiv 0$) the quasi-geostrophic potential vorticity equation takes the form $\mathbf{u_g} \cdot \nabla_h q = 0$.
- Assume that f is a constant,
- $\mathbf{u_g} \cdot \nabla_h q = 0$ is satisfied (e.g.) by solutions of the form $q = f$.
- For these solutions, p satisfies

$$\nabla_h^{\,2} p + \frac{f_0^2}{N^2}\frac{\partial^2 p}{\partial z^2} = \rho f_0 (q - f) = 0 \quad \textbf{or} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2 \psi}{\partial z^2} = 0$$

ψ is the geostrophic streamfunction ($= p/\rho f$).

These solutions have zero perturbation potential vorticity

Omit the zero subscript on f**, and assume that** N **is a constant.**

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2}=0$$

Put $\bar{z}=(N/f)z$ $\Rightarrow$ **Laplace's equation**

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial \bar{z}^2}=0$$

Two particular solutions are:

$\psi = -Uy$ $\Rightarrow$ $u = -\psi_y = U$ **a uniform flow**

$\psi = -S/4\pi r$ $\Rightarrow$ **a source solution, strength** S **at** $z = -\bar{z}*$

$$r=\sqrt{x^2+y^2+(\bar{z}+\bar{z}_*)^2}$$

Streamfunction equation is linear

$\Rightarrow$ $\psi = -Uy - S/4\pi r$ **is a solution.**

In qg-flow $b=\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}=f\dfrac{\partial\psi}{\partial z}$ **because** $\psi = p/\rho f$

The vertical displacement of a fluid parcel, η **is related to** σ **by**

$$\eta=-\frac{b}{N^2}$$

Since b **is a constant on isentropic surfaces, the displacement of the isentropic surface from** z = constant **for the flow defined by** $\psi = -Uy - S/4\pi r$ **is given (in dimensional form) by**

$$\eta=-\frac{S}{4\pi f}\left[x^2+y^2+\frac{N^2}{f^2}(z+z_*)^2\right]^{-3/2}(z+z_*)$$

$$\eta = -\frac{S}{4\pi f}\left[x^2 + y^2 + \frac{N^2}{f^2}(z+z_*)^2\right]^{-3/2}(z+z_*)$$

The displacement of fluid parcels which, in the absence of motion would occupy the plane at $z = 0$ **is**

$$h(x,y) = -\frac{Sz_*}{4\pi f}\left[x^2 + y^2 + \frac{N^2}{f^2}z_*^2\right]^{-3/2}$$

$$= \frac{h_m}{\left[(R/R_*)^2 + 1\right]^{3/2}}$$

$h_m = -S/(4\pi N R_*^2)$

$R_* = N z_*/f$

$R = \sqrt{(x^2 + y^2)}$

$$h(x,y) = \frac{h_m}{\left[(R/R_*)^2 + 1\right]^{3/2}}$$

is an isentropic surface of the quasi-geostrophic flow defined by $\psi = -Uy - S/4\pi r$**.**

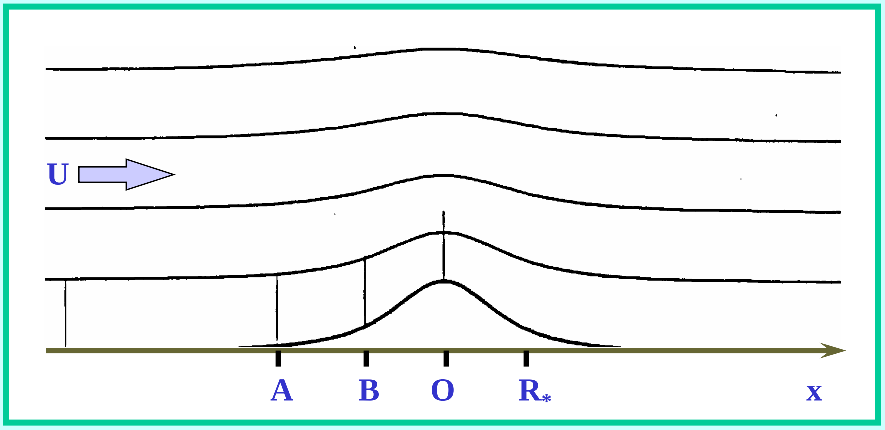
When $S = 4\pi N R_*^2 h_m$ **and** $z = f R_*/N$, $\psi = -Uy - S/4\pi r$ **represents the flow in the semi-infinite region** $z >= h$ **of a uniform current** U **past the bell-shaped mountain with circular contours given by** h(x,y).

The mountain height is h_m **and its characteristic width is** R_*.

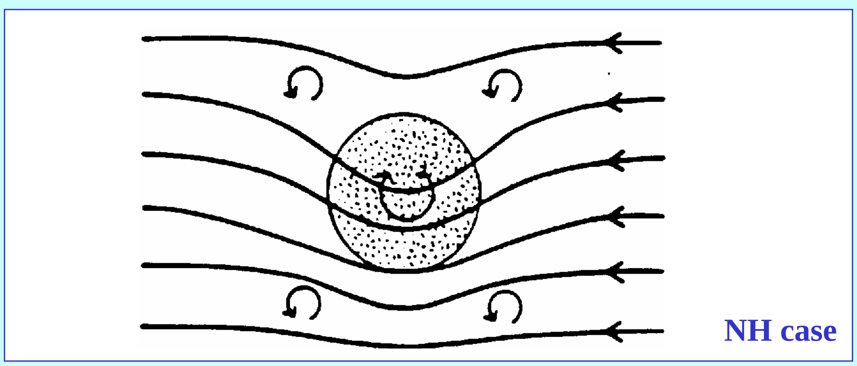
In terms of h_m **etc., the displacement of an isentropic surface in this flow is**

$$\eta(x,y,z) = \frac{h_m(z/z_* + 1)}{\left[(R/R_*)^2 + (z/z_* + 1)^2\right]^{3/2}}$$

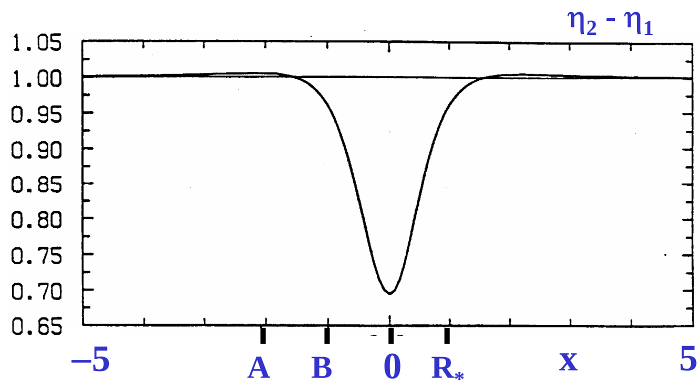
The vorticity changes in stratified quasi-geostrophic flow over an isolated mountain



The streamline pattern in quasi-geostrophic stratified flow over an isolated mountain



The incident flow is distorted by the mountain anticyclone, but the perturbation velocity and pressure field decay away from the mountain (after R. B. Smith, 1979).



Height of the lowest isentrope above the topography in as a function of x**.**

Unit scale equals the length of the four vertical lines in Fig. 8.1.

End of Chapter 8