Chapter 7

The Vorticity Equation in a Rotating Stratified Fluid

The vorticity equation for a rotating, stratified, viscous fluid

- The vorticity equation in one form or another and its interpretation provide a key to understanding a wide range of atmospheric and oceanic flows.
- The full Navier-Stokes' equation in a rotating frame is

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p_T - g\mathbf{k} + \nu\nabla^2\mathbf{u}$$

where p is the total pressure and $\mathbf{f} = f\mathbf{k}$.

- We allow for a spatial variation of **f** for applications to flow on a beta plane.

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p_T - g\mathbf{k} + \nu\nabla^2\mathbf{u}$$

Now $\mathbf{u} \cdot \nabla\mathbf{u} = \nabla(\tfrac{1}{2}\mathbf{u}^2) + \omega \wedge \mathbf{u}$

$$\Rightarrow \quad \frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\mathbf{u}^2\right) + \left(\omega + \mathbf{f}\right) \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p_T - g\mathbf{k} + \nu\nabla^2\mathbf{u}$$

take the curl $\Rightarrow$

$$\frac{D}{Dt}(\omega + \mathbf{f}) = (\omega + \mathbf{f}) \cdot \nabla\mathbf{u} - (\omega + \mathbf{f})\nabla \cdot \mathbf{u} + \frac{1}{\rho^2}\nabla\rho \wedge \nabla p_T + \nu\nabla^2\omega$$

or $$\frac{D\omega}{Dt} = -\mathbf{u} \cdot \nabla\mathbf{f} + \ldots.$$

Note that $\nabla \wedge [\omega + \mathbf{f}] \wedge \mathbf{u}] = \mathbf{u} \cdot \nabla (\omega + \mathbf{f}) + (\omega + \mathbf{f}) \nabla \cdot \mathbf{u} - (\omega + \mathbf{f}) \cdot \nabla\mathbf{u}$,
and $\nabla \cdot [\omega + \mathbf{f}] \equiv 0$.

Terminology

$\omega_a = \omega + \mathbf{f}$ **is called the absolute vorticity - it is the vorticity derived in an a inertial frame**

ω **is called the relative vorticity, and**

$\mathbf{f}$ **is called the planetary-, or background vorticity**

Recall that solid body rotation corresponds with a vorticity 2Ω.

Interpretation

$$\frac{D}{Dt}(\omega + \mathbf{f}) = (\omega + \mathbf{f}) \cdot \nabla \mathbf{u} - (\omega + \mathbf{f}) \nabla \cdot \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \wedge \nabla p_T + \nu \nabla^2 \omega$$

$\frac{D\omega}{Dt}$ is the rate-of-change of the relative vorticity

$-\mathbf{u} \cdot \nabla \mathbf{f}$: If **f** varies spatially (i.e., with latitude) there will be a change in ω as fluid parcels are advected to regions of different **f**.

Note that it is really $\omega + \mathbf{f}$ whose total rate-of-change is determined.

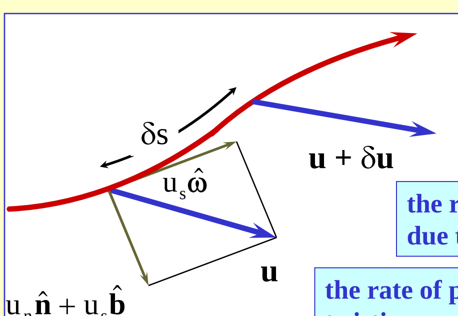
$$\frac{D}{Dt}(\omega + \mathbf{f}) = \underline{(\omega + \mathbf{f}) \cdot \nabla \mathbf{u}} - (\omega + \mathbf{f}) \nabla \cdot \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \wedge \nabla p_T + \nu \nabla^2 \omega$$

$(\omega + \mathbf{f}) \cdot \nabla \mathbf{u}$ consider first $\omega \cdot \nabla \mathbf{u}$, or better still, $(\omega/|\omega|) \cdot \nabla \mathbf{u}$.

unit vector along the vortex line

$$\Rightarrow \quad \hat{\omega} \cdot \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial s} = \underline{\frac{\partial}{\partial s}(u_s \hat{\omega})} + \underline{\frac{\partial}{\partial s}(u_n \hat{\mathbf{n}} + u_b \hat{\mathbf{b}})}$$

principal normal and binormal directions



the rate of relative vorticity production due to the stretching of relative vorticity

the rate of production due to the bending (tilting, twisting, reorientation, etc.) of relative vorticity

$$\frac{D}{Dt}(\boldsymbol{\omega} + \mathbf{f}) = (\boldsymbol{\omega} + \mathbf{f})\cdot\nabla\mathbf{u} - (\boldsymbol{\omega} + \mathbf{f})\nabla\cdot\mathbf{u} + \frac{1}{\rho^2}\nabla\rho\wedge\nabla p_T + \nu\nabla^2\boldsymbol{\omega}$$

$$\mathbf{f}\cdot\nabla\mathbf{u} = f\frac{\partial\mathbf{u}}{\partial z} = f\frac{\partial\mathbf{u}_h}{\partial z} + f\frac{\partial w}{\partial z}\mathbf{k}$$

the rate of vorticity production due to the bending of planetary vorticity

the rate of vorticity production due to the stretching of planetary vorticity

$$-(\boldsymbol{\omega} + \mathbf{f})\nabla\cdot\mathbf{u}$$

$= (1/\rho)(D\rho/Dt)(\boldsymbol{\omega} + \mathbf{f})$ **using the full continuity equation**

a relative increase in density $\Rightarrow$ a relative increase in absolute vorticity.

Note that this term involves the total divergence, not just the horizontal divergence, and it is exactly zero in the Boussinesq approximation.

$$\frac{D}{Dt}(\boldsymbol{\omega} + \mathbf{f}) = (\boldsymbol{\omega} + \mathbf{f})\cdot\nabla\mathbf{u} - (\boldsymbol{\omega} + \mathbf{f})\nabla\cdot\mathbf{u} + \frac{1}{\rho^2}\nabla\rho\wedge\nabla p_T + \nu\nabla^2\boldsymbol{\omega}$$

$$\frac{1}{\rho^2}\nabla\rho\wedge\nabla p_T$$

sometimes denoted by B, this is the baroclinicity vector and represents baroclinic effects.

B is identically zero when the isoteric (constant density) and isobaric surfaces coincide.

Denote $\phi = \ln\theta = s/c_p$,
s = **specific entropy** $= \tau^{-1}\ln p_T - \ln\rho$ + constants,
where $\tau^{-1} = 1 - \kappa$.

$$\mathbf{B} = \frac{1}{\rho}\nabla p_T\wedge\nabla\phi$$

$$\mathbf{B} = \frac{1}{\rho} \nabla p_T \wedge \nabla \phi$$

- **B** **represents an anticyclonic vorticity tendency in which the isentropic surface (constant** s, ϕ, θ **) tends to rotate to become parallel with the isobaric surface.**
- **Motion can arise through horizontal variations in temperature even though the fluid is not buoyant (in the sense that a vertical displacement results in restoring forces); e.g. frontal zones, sea breezes.**

$\nu \nabla^2 \boldsymbol{\omega}$ **represents the viscous diffusion of vorticity into a moving fluid element.**

The vorticity equation for synoptic scale atmospheric motions

The equations appropriate for such motions are

$$\frac{\partial \mathbf{u_h}}{\partial t} + \mathbf{u_h} \cdot \nabla \mathbf{u}_h + w \frac{\partial \mathbf{u}_h}{\partial z} + \mathbf{f} \wedge \mathbf{u}_h = -\frac{1}{\rho} \nabla_h p \qquad \textbf{(a)}$$

and

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \sigma$$

Let

$$\omega_h = \nabla \cdot \mathbf{u}_h = \left(-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Take the curl of (a)

$$\frac{\partial \boldsymbol{\omega}_h}{\partial t} + (\boldsymbol{\omega}_h + \mathbf{f})\nabla_h \cdot \mathbf{u}_h + \mathbf{u}_h \cdot \nabla(\boldsymbol{\omega}_h + \mathbf{f})$$

$$- (\boldsymbol{\omega}_h + \mathbf{f}) \cdot \nabla \mathbf{u}_h + w\frac{\partial \boldsymbol{\omega}_h}{\partial z} + \nabla w \wedge \frac{\partial \mathbf{u}_h}{\partial z} = +\frac{1}{\rho^2}\nabla\rho \wedge \nabla_h p$$

We use $\mathbf{u}_h \cdot \nabla \mathbf{u}_h = \nabla\left(\frac{1}{2}\mathbf{u}_h^2\right) + \boldsymbol{\omega}_h \wedge \mathbf{u}_h$ and $\nabla \wedge (\phi \mathbf{a}) = \nabla\phi \wedge \mathbf{a} + \phi \nabla \wedge \mathbf{a}$

The vertical component of this equation is

$$\frac{\partial \zeta}{\partial t} = -\mathbf{u}_h \cdot \nabla(\zeta + f) - w\frac{\partial \zeta}{\partial z} - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

where $\zeta = \mathbf{k} \cdot \boldsymbol{\omega}_h = \partial v / \partial x - \partial u / \partial y$

An alternative form is

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\quad " \quad\right) + \left(\quad " \quad\right)$$

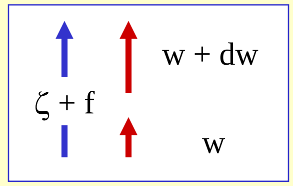
- **The rate of change of the vertical component of absolute vorticity (which we shall frequently call just the absolute vorticity) following a fluid parcel.**

The term $-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ **is the divergence term**

For a Boussinesq fluid: $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ **=>**

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = (\zeta + f)\frac{\partial w}{\partial z}$$

$$-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=(\zeta+f)\frac{\partial w}{\partial z}$$

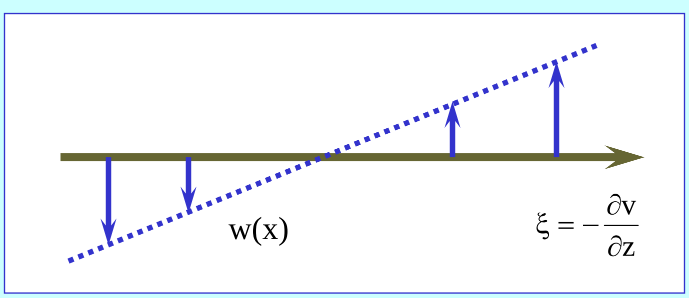


$(\zeta + f)\partial w/\partial z$ **corresponds with a rate of production of absolute vorticity by stretching.**

For an anelastic fluid (one in which density variations with height are important) the continuity equation is:
$\partial u/\partial x + \partial v/\partial y + (1/\rho_0)\partial(\rho_0 w)/\partial z = 0$ **=>**

$$\boxed{-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=(\zeta+f)\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}}$$

The term $\left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right)$ **in the vorticity equation is the tilting term; this represents the rate of generation of absolute vorticity by the tilting of horizontally oriented vorticity** $\boldsymbol{\omega}_{\mathbf{h}} = (-\partial v/\partial z, \partial u/\partial z, 0)$ **into the vertical by a non-uniform field of vertical motion** $(\partial w/\partial x, \partial w/\partial y, 0) \neq 0$**.**



- The last term in the vorticity equation is the solenoidal term.
- This, together with the previous term, is generally small in synoptic scale atmospheric motions as the following scale estimates show:

$$\left[\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right]\le\frac{W}{H}\frac{U}{L}=10^{-11}\,s^{-2},$$

$$\frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]\le\frac{\delta\rho}{\rho^2}\frac{\delta p}{L^2}=2\times10^{-11}\,s^{-2};$$

The sign $\le$ indicates that these may be overestimated due to cancellation.

End of Chapter 7