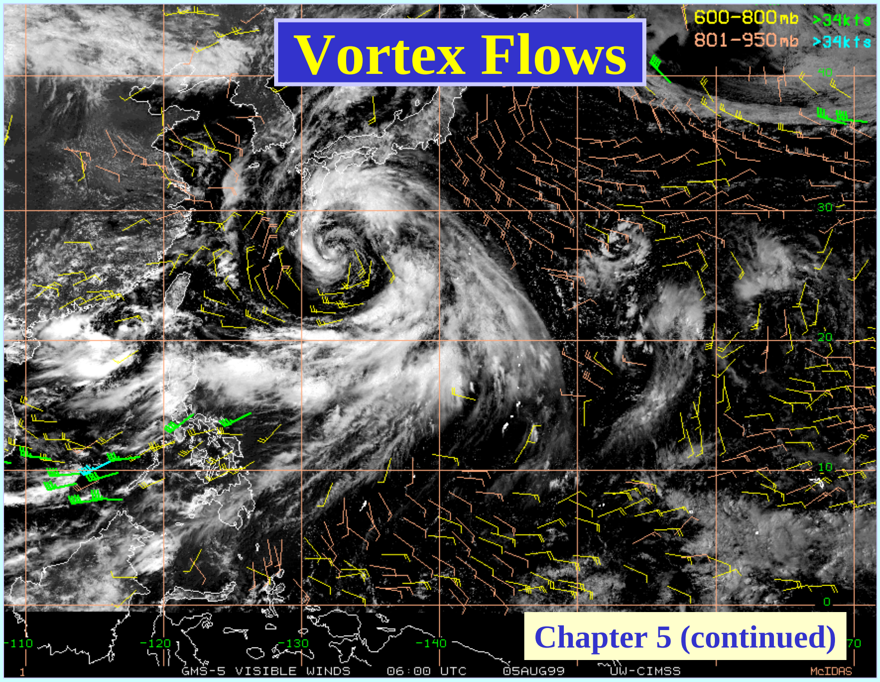
Vortex Flows



Chapter 5 (continued)

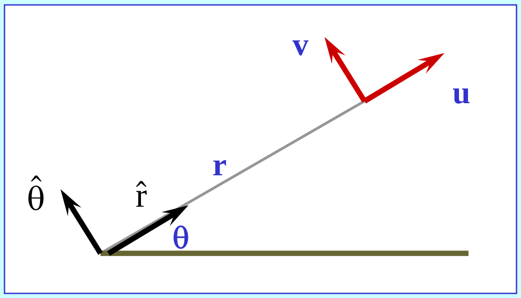
Vortex flows: the gradient wind equation

- Strict geostrophic motion requires that the isobars be straight, or, equivalently, that the flow be uni-directional.
- We investigate now balanced flows with curved isobars, including vortical flows in which the motion is axi-symmetric.
- It is convenient to express Euler's equation in cylindrical coordinates.
- We begin by deriving an expression for the total horizontal acceleration $D\mathbf{u}_h/Dt$ in cylindrical coordinates.

Let the horizontal velocity be expressed as

$$\mathbf{u}_h = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}}$$

unit vectors in the radial and tangential directions.



$$\frac{D\mathbf{u}_h}{Dt} = \frac{Du}{Dt}\hat{\mathbf{r}} + u\frac{D\hat{\mathbf{r}}}{Dt} + \frac{Dv}{Dt}\hat{\boldsymbol{\theta}} + v\frac{D\hat{\boldsymbol{\theta}}}{Dt}$$

Now

$$\frac{D\hat{\mathbf{r}}}{Dt} = \frac{\partial\hat{\mathbf{r}}}{\partial t} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{D\hat{\boldsymbol{\theta}}}{Dt} = \frac{\partial\hat{\boldsymbol{\theta}}}{\partial t} = -\dot{\theta}\hat{\mathbf{r}}$$

where

$$\dot{\theta} = d\theta / dt = v / r$$

Then

$$\frac{D\mathbf{u}_h}{Dt} = \frac{Du}{Dt}\hat{\mathbf{r}} + u\frac{D\hat{\mathbf{r}}}{Dt} + \frac{Dv}{Dt}\hat{\boldsymbol{\theta}} + v\frac{D\hat{\boldsymbol{\theta}}}{Dt}$$

$$\Rightarrow \quad \frac{D\mathbf{u}_h}{Dt} = \left[\frac{Du}{Dt} - \frac{v^2}{r}\right]\hat{\mathbf{r}} + \left[\frac{Dv}{Dt} + \frac{uv}{r}\right]\hat{\boldsymbol{\theta}}$$

The radial and tangential components of Euler's equation may be written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

The axial component is

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

The case of pure circular motion with $u = 0$ **and** $\partial/\partial\theta \equiv 0$**.**

$$\Rightarrow \quad \frac{v^2}{r} + fv = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

- **This is called the gradient wind equation.**
- **It is a generalization of the geostrophic equation which takes into account centrifugal as well as Coriolis forces.**
- **This is necessary when the curvature of the isobars is large, as in an extra-tropical depression or in a tropical cyclone.**

The gradient wind equation

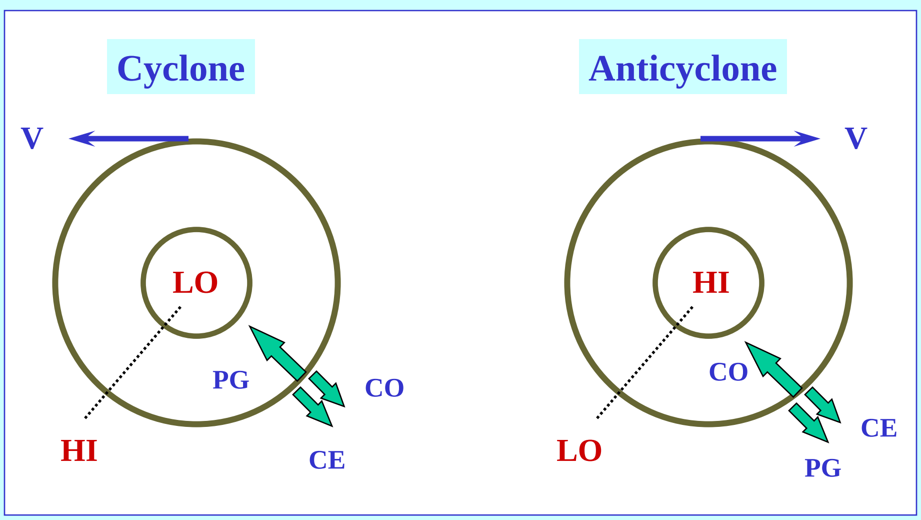
Write

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{v^2}{r} + fv$$

terms interpreted as forces

- **The equation expresses a balance of the centrifugal force (v^2/r) and Coriolis forces (fv) with the radial pressure gradient.**
- **This interpretation is appropriate in the coordinate system defined by $\hat{\mathbf{r}}$ and $\hat{\theta}$, which rotates with angular velocity v/r.**

Force balances in low and high pressure systems



The equation $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{v^2}{r} + fv$$

is a diagnostic equation for the tangential velocity v in terms of the pressure gradient:

$$v = -\frac{1}{2}fr + \left[\frac{1}{4}f^2r^2 + \frac{r}{\rho}\frac{\partial p}{\partial r}\right]^{\frac{1}{2}}$$

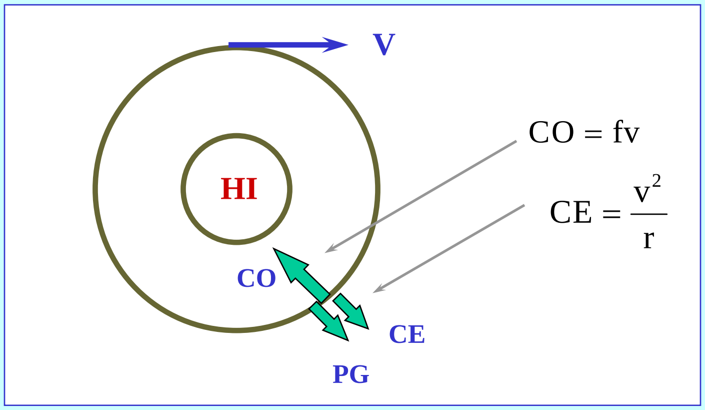
The positive sign is chosen in solving the quadratic equation so that geostrophic balance is recovered as $r \to \infty$ (for finite v, the centrifugal force tends to zero as $r \to \infty$).

$$v = -\frac{1}{2}fr + \left[\frac{1}{4}f^2r^2 + \frac{r}{\rho}\frac{\partial p}{\partial r}\right]^{\frac{1}{2}}$$

- In a low pressure system, $\partial p/\partial r > 0$ andthere is no theoretical limit to the tangential velocity v.
- In a high pressure system, $\partial p/\partial r < 0$ and the local value of the pressure gradient cannot be less than $-\rho r f^2/4$ in a balanced state.
- Therefore the tangential wind speed cannot locally exceed $rf/2$ in magnitude.
- This accords with observations in that wind speeds in anticyclones are generally light, whereas wind speeds in cyclones may be quite high.

Limited wind speed in anticyclones

In the anticyclone, the Coriolis force increases only in proportion to v: => this explains the upper limit on v predicted by the gradient wind equation.



The local Rossby number

- **In vortical type flows we can define a local Rossby number at radius r :**

$$Ro(r) = \frac{v}{rf}$$

- **This measures the relative importance of the centrifugal acceleration to the Coriolis acceleration in the gradient wind equation.**
- **For radii at which** $Ro(r) \ll 1$**, the centrifugal acceleration** $\ll$ **the Coriolis acceleration and the motion is approximately geostrophic.**

Cyclostrophic balance

- **If** Ro(r) >> 1**, the centrifugal acceleration >> the Coriolis acceleration and we refer to this as cyclostrophic balance.**
- **Cyclostrophic balance is closely approximated in strong vertical flows such as tornadoes, waterspouts and tropical cyclones in their inner core.**
- **We can always define a geostrophic wind** v_g **in terms of the pressure gradient, i.e.,** $v_g = (1/\rho f)\, \partial p/\partial r$. **Then**

$$\frac{v^2}{r} + fv = \frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{v_g}{v} = 1 + \frac{v}{rf}$$

$$\frac{v_g}{v} = 1 + \frac{v}{rf}$$

For cyclonic flow ($v\,\mathrm{sgn}(f) > 0$**),** $|v_g| > |v|$ ⇒

The geostrophic wind gives an over-estimate of the gradient wind v**.**

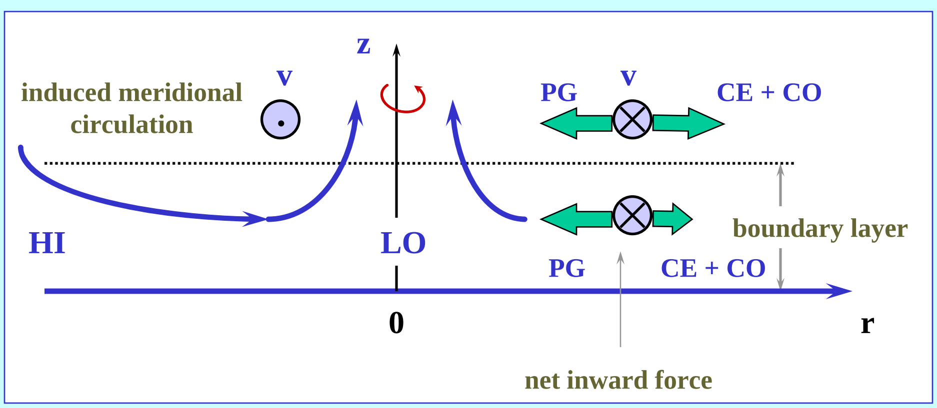
For anticyclonic flow ($v\,\mathrm{sgn}(f) < 0$**),** $|v_g| < |v|$ ⇒

The geostrophic wind gives an under-estimate of the gradient wind.

Vortex boundary layers

- Consider a **low pressure system** with **circular isobars** and in **gradient wind balance**, situated over a **rigid frictional boundary**.
- In the region near the boundary, friction reduces the tangential flow velocity and hence both the centrifugal and Coriolis forces.
- This leaves a state of imbalance in the boundary layer with a net radially inwards pressure gradient.
- This radial pressure gradient drives fluid across the isobars towards the vortex centre, leading to vertical motion at inner radii.

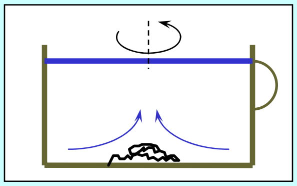
Frictionally-induced secondary circulation in a vortex



Schematic cross-section illustrating the effect of friction at the terminating boundary of a low pressure vortex.

Frictionally-induced meridional circulation

- Frictional effects in the terminating boundary of a vortex induce a meridional circulation (i.e., one in the r-z plane) in the vortex with upflow at inner radii.
- This meridional circulation is vividly illustrated by the motion of tea leaves in a stirred pot of tea.

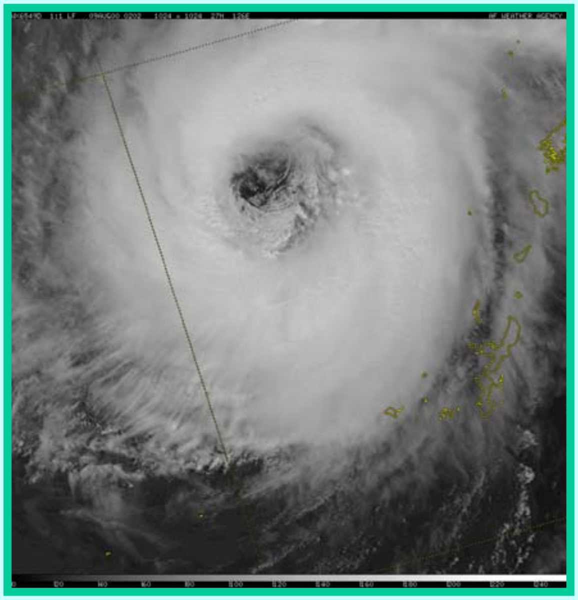


- A short time after stirring the tea, the tea leaves congregate on the bottom near the centre of the tea pot as a result of the inward motion induced in the frictional boundary layer.

Effects of the meridional circulation

- In a tropical cyclone, the frictionally-induced convergence near the sea surface transports moist air to feed the towering cumulonimbus clouds surrounding the central eye, thereby maintaining an essential part of the storm's heat engine.
- In high pressure systems, frictional effects result in a net outwards pressure gradient near the surface and this leads to boundary layer divergence with subsiding motion near the anticyclone centre.
- However, as shown in Chapter 10, subsidence occurs in developing anticyclones in the absence of friction.

A tropical cyclone



End of Chapter 5