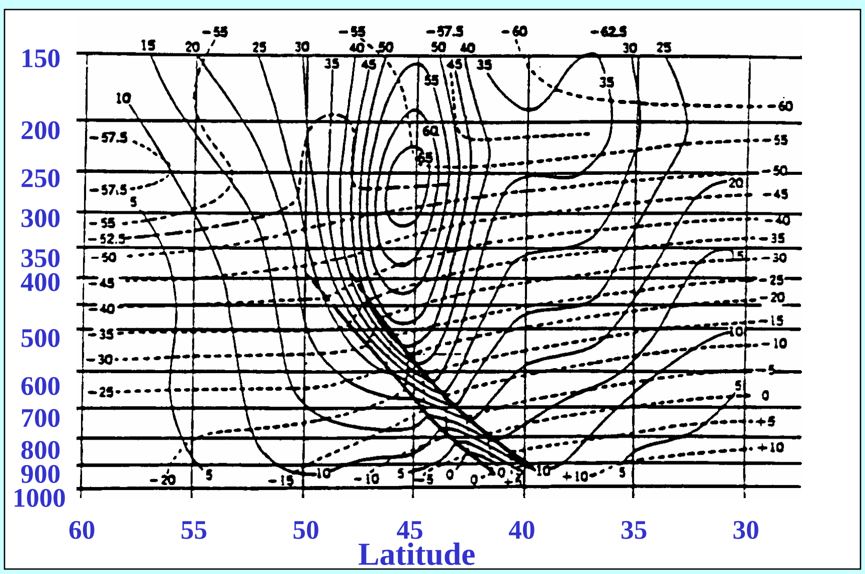
Fronts, Ekman Boundary Layers and Vortex Flows

Chapter 5

Fronts

- A front refers to the sloping interfacial region of air separating two air masses, each of more or less uniform properties.
- An example is the polar front, a zone of relatively large horizontal temperature gradient in the mid-latitudes that separates air masses of more uniform temperatures that lie poleward and equatorward of the zone.
- Other examples are the cold and warm fronts associated with extra-tropical cyclones.

Cold Front over Munich



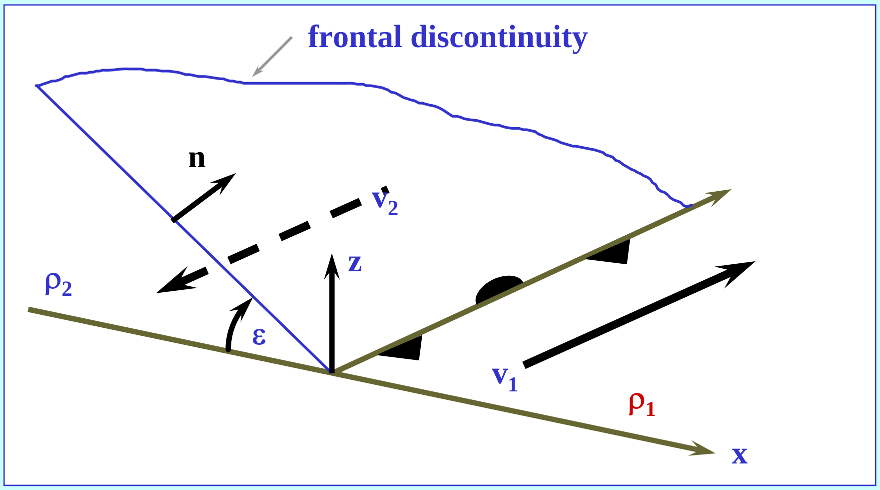
Composite meridional cross-section at 80°W of mean temperature and the zonal component of geostrophic wind computed from 12 individual cross-sections in December, 1946.

Intense atmospheric fronts

- **Often, quite sharp temperature differences occur across a frontal surface - a few degrees over a few kilometres.**
- **Melbourne's famous summertime "cool change", Sydney's "southerly buster" and New Zealand's "southerly change" are examples *par excellence.***
- **The first two are fronts which cross southeastern Australia and make the sharp transition region between a very warm air mass originating from deep over the continent and much cooler air from the Southern Ocean.**

Southerly Buster over Sydney

Margules' model



The simplest model representing a frontal "discontinuity". The front is idealized as a **sharp, plane, temperature discontinuity** separating **two inviscid, homogeneous, geostrophic flows.**

Assumptions of Margules' model

- **(i) the Boussinesq approximation; in particular that the temperature difference between the air masses is small in the sense that** $(T_1 - T_2)/T^* << 1$

  $T^* = (T_1 + T_2)/2$ **is the mean temperature of the two air masses and** T_2 **the temperature of the cold air;**

- **(ii) that the flow is everywhere parallel with the front and that there are no along-front variations in it; i.e.,** $\partial/\partial y \equiv 0$**; and**

- **(iii) that diffusion effects are absent so that the frontal discontinuity remains sharp.**

Equations of Motion

geostrophic equations

$$-fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x}$$

$$fu = 0$$

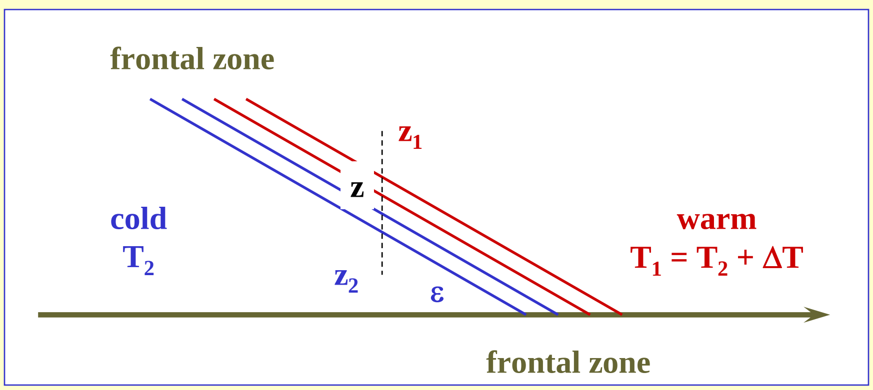
hydrostatic equation

$$0 = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} + g\left[\frac{T - T_2}{T_*}\right]$$

continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

We consider Margules' solution to be the limiting case of the situation where the temperature gradients are finite, but very small, except across the frontal zone where they are very large.



On any isotherm, $\delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial z}\delta z = 0$

The local slope of an isotherm in the frontal zone is $\varepsilon(x,z)$, given by

$$\tan\,\varepsilon = -\frac{\delta z}{\delta x} = \frac{\frac{\partial T}{\partial x}}{\frac{\partial T}{\partial z}}$$

Note that $\delta x > 0$ implies $\delta z < 0$ if, as shown, $0 < \varepsilon < \pi/2$.

Eliminate p from

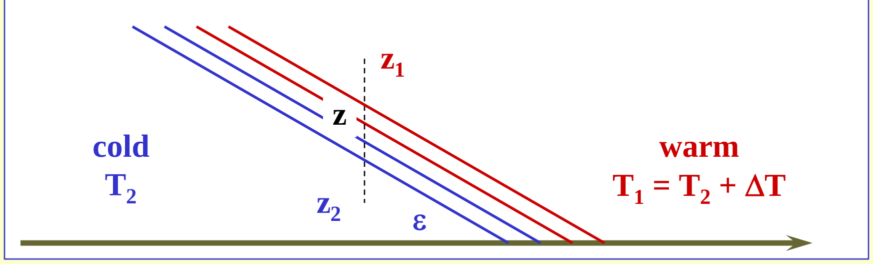
$$-fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x} \quad \text{and} \quad 0 = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} + g\frac{T - T_2}{T_*}$$

$$\Rightarrow \quad f\frac{\partial v}{\partial z} = \frac{1}{\rho_*}\frac{\partial^2 p}{\partial x \partial z} = \frac{g}{T_*}\frac{\partial T}{\partial x} = \frac{g}{T_*}\tan\,\varepsilon\,\frac{\partial T}{\partial z}$$

⇒ the thermal wind equation ⇒ the vertical shear across the front to the horizontal temperature contrast across it.

Integrate $f\frac{\partial v}{\partial z} = \frac{1}{\rho_*}\frac{\partial^2 p}{\partial x \partial z} = \frac{g}{T_*}\frac{\partial T}{\partial x} = \frac{g}{T_*}\tan\,\varepsilon\,\frac{\partial T}{\partial z}$

vertically across the front from z_2 to z



$$v(x,z) = v(x,z_2) + \frac{g}{fT_*}\int_{z_2}^{z}\tan\,\varepsilon\,\frac{\partial T}{\partial z}dz$$

$$\Rightarrow \quad v_1 = v_2 + \frac{g}{fT_*}(T_1 - T_2)\tan\,\varepsilon^*$$

ε^* is the angle of some intermediate isotherm between z_2 and z_1

$$v_1 = v_2 + \frac{g}{fT_*}(T_1 - T_2)\tan\varepsilon^*$$

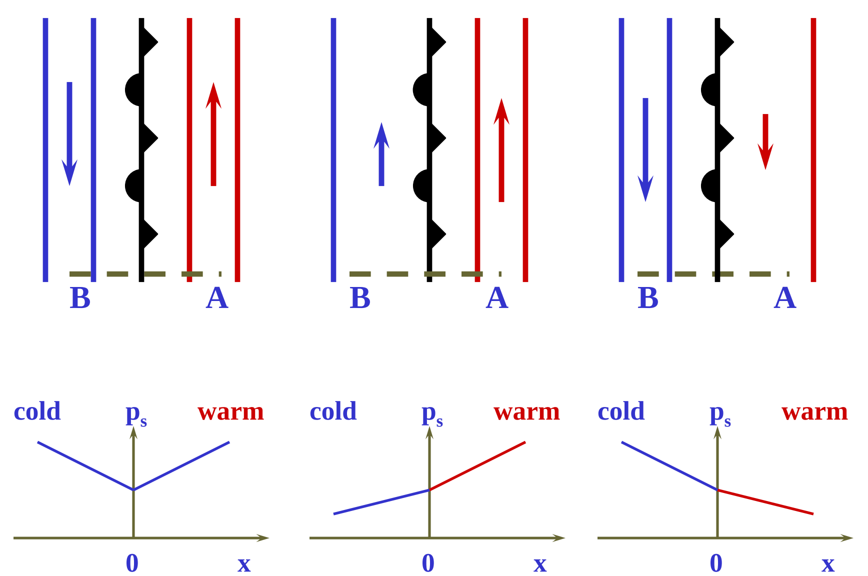
$$\Rightarrow \quad \delta v = \frac{g\delta T}{fT_*}\tan\varepsilon \qquad \textbf{Margules' formula}$$

Margules' formula relates the change in geostrophic wind speed across the front to the temperature difference across the front and to the frontal slope.

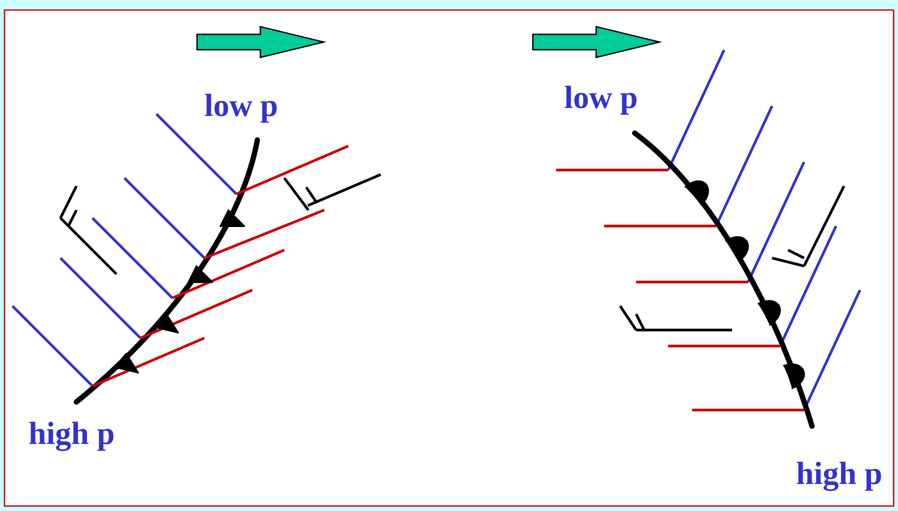
Note, with $0 < \varepsilon < \pi/2$

- **(i)** $\delta T = T_1 - T_2 > 0$, **otherwise the flow is gravitationally unstable, and,**
- **(ii)** $\delta v < 0$ (> 0) **if** $f < 0$ (> 0) **i.e., there is always a cyclonic change in** v **across the frontal surface.**
- **(iii) it is not necessary that** $v_1 < 0$ (> 0) **and** $v_2 > 0$ (< 0) **separately; only the change in** v **is important.**

Three possible configurations



- Margules' solution i.e. v_1 and v_2 related as shown and u and w everywhere zero, is an **exact** solution of the Euler equations of motion in a rotating frame.
- Margules' formula is a diagnostic one for a stationary, or quasi-stationary front; it tells us nothing about the formation (**frontogenesis**) or decay (**frontolysis**) of fronts.
- It is of little practical use in forecasting, since active fronts, which are responsible for a good deal of the '**significant weather**' in middle latitudes, are always associated with rising vertical motion and are, therefore, normally accompanied by precipitation.
- There are difficulties even in constructing an extension of Margules' model to a front that translates with a uniform geostrophic flow (**Sutcliffe, 1938; Smith, 1989**).
- Fronts occur also in the ocean.



Schematic representation of a **translating cold front** and a **translating warm front** as they might be drawn on a **mean sea level** synoptic chart for the northern hemisphere. Note the sharp cyclonic change in wind direction reflected in the discontinuous slope of the isobars.

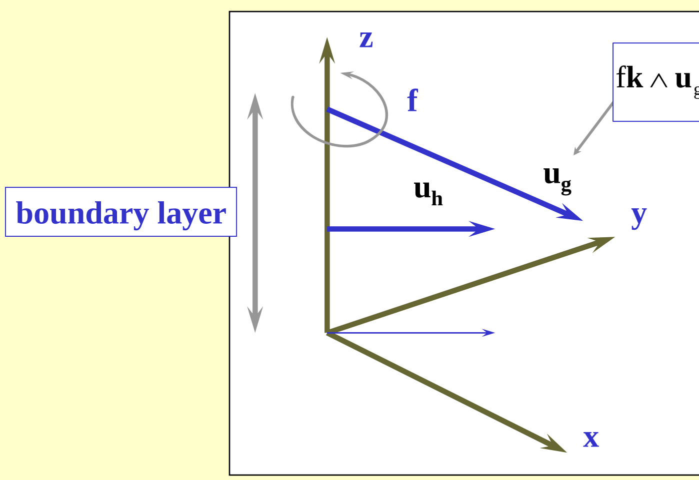
Viscous boundary layers: Ekman's solution

- Viscous boundary layers play an important role in the dynamics of rotating fluids because of their ability to induce motion normal to a boundary that is perpendicular to the axis of rotation.
- Consider laminar viscous flow adjacent to a rigid boundary at $z = 0$, the axis of rotation being as usual in the z direction.
- Assume that far from the boundary viscous effects can be neglected and the flow is geostrophic with velocity $\mathbf{u_g}$ parallel to the x-y plane.

NH flow configuration

The outer inviscid flow satisfies

$$f\mathbf{k} \wedge \mathbf{u}_g = -\frac{1}{\rho}\nabla p$$



In the boundary layer

$$f\mathbf{k} \wedge \mathbf{u_h} = -\frac{1}{\rho}\nabla_h p + \nu\frac{\partial^2 \mathbf{u_h}}{\partial z^2}$$

The vertical momentum equation with the boundary-layer approximation is

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

This is consistent with the geostrophic approximation!

→ The geostrophic pressure gradient is transmitted through to the boundary.

→ $$f\mathbf{k} \wedge (\mathbf{u_h} - \mathbf{u}_g) = \nu \frac{\partial^2 \mathbf{u_h}}{\partial z^2}$$

In components

$$f(v_g - v) = \nu \frac{\partial^2 u}{\partial z^2}$$

$$-f(u_g - u) = \nu \frac{\partial^2 v}{\partial z^2}$$

Solution of the boundary-layer equations

Put $\mathbf{U} = u + iv$ $\quad i = \sqrt{-1}$

Add i times $f(v_g - v) = \nu \frac{\partial^2 u}{\partial z^2}$ to $-f(u_g - u) = \nu \frac{\partial^2 v}{\partial z^2}$

→ a single complex equation for **U**:

$$\frac{\partial^2 U}{\partial z^2} - \alpha^2 U = -\alpha^2 U_g$$

where $\alpha^2 = \frac{f}{\nu} i$ or $\alpha = \pm \alpha_*$, $\quad \alpha_* = (f/\nu)^{1/2}(1+i)/\sqrt{2}$

Boundary conditions: no-slip ($\mathbf{u} = \mathbf{0}$) at $z = 0$, and
$u \to u_g$ as $z \to \infty$.

$$\frac{\partial^2 U}{\partial z^2} - \alpha^2 U = -\alpha^2 U_g \quad \Rightarrow \quad U = U_g(1 - e^{-\alpha_* z})$$

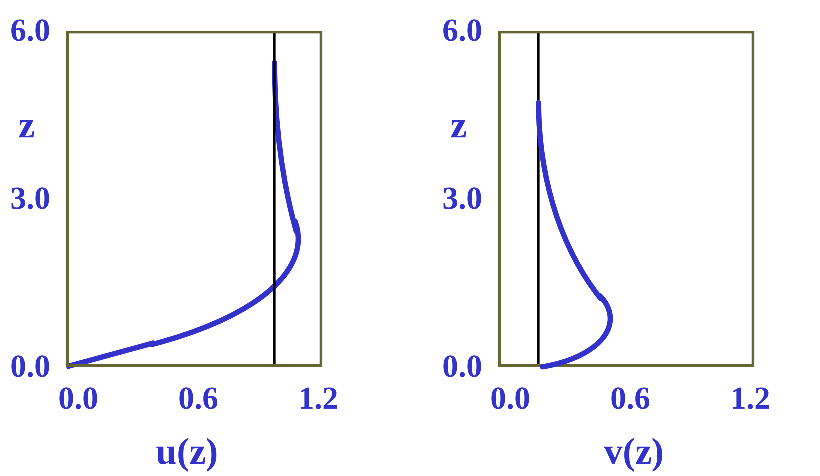
Choose axes so that $v_g = 0$ (i.e., $\mathbf{u_g}$ is in the x-direction) and set

$$\delta = (2\nu / f)^{\frac{1}{2}}$$ $\Rightarrow$ **the boundary layer depth scale.**

$$u = u_g(1 - e^{-z/\delta}\cos(z/\delta))$$
$$v = u_g e^{-z/\delta}\sin(z/\delta)$$

These velocity profiles are shown in the next slide, together with the hodograph of $\mathbf{u}(z)$, and the surface stress vector τ.

Ekman velocity profiles



The surface stress

The surface stress is defined as

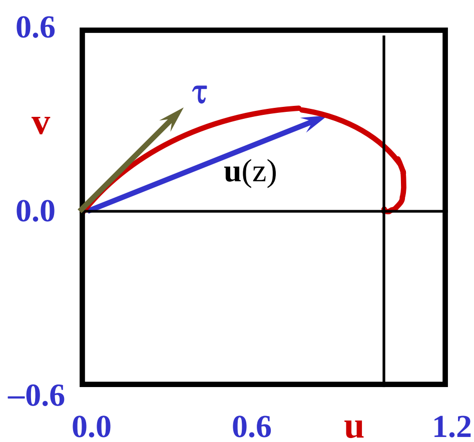
$$\boldsymbol{\tau} = \mu(\partial \mathbf{u} / \partial z)_{z=0}$$

In complex notation

$$\tau = \mu \frac{\partial U}{\partial z}\bigg]_{z=0} = \mu \alpha_* U_g = \frac{\sqrt{2}}{\delta} \mu\, e^{i\pi/4} U_g$$

The surface stress acts at 45 deg. to the left (NH), right (SH), of the geostrophic velocity $\mathbf{u}_g$.

Ekman spiral hodograph



Notes

- If $\mathbf{u}_g$ is spatially and temporally constant, the Ekman solution given is an exact solution of the full Navier-Stokes' equation.
- At 45 deg. latitude, $f \sim 10^{-4}$ s^{-1} and for air and water at room temperature, ν takes the respective values: 1.5×10^{-5} m^2s^{-1} and 1.0×10^{-6} m^2s^{-1}.
- Thus, calculated values of δ at latitude 45 deg (Munich is 48 deg) are for air 0.55 m; for water 0.14 m.
- For larger rotation rates, e.g., a laboratory tank rotating at 1 radian/sec. (approx. 10 revs. per min.), $\delta_{air} = 0.0033$ m and $\delta_{water} = 0.0008$ m.

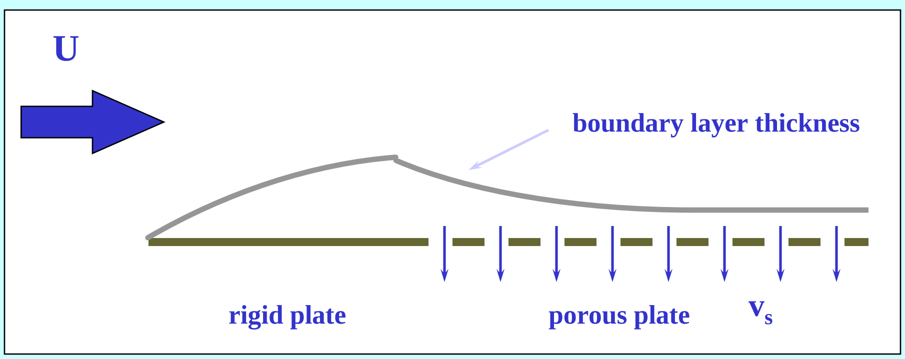
More notes

- Calculations apply to laminar flow only.
- The δ's do not relate to the atmosphere or oceans where the flows are generally turbulent and the effective viscosities are much greater.
- Observations show that frictional effects in the atmospheric boundary layer extend through a depth of about 1 km.
- Assume (crude!) that turbulent momentum transport can be characterized by a constant "eddy" viscosity K_M, analogous to laminar viscosity:
- $\Rightarrow$ an effective value of K_M at 45° lat. of order 10 m^2s^{-1} (compare with ν for air which is ~ 10^{-5} m^2s^{-1}).

The Ekman boundary layer has constant thickness

- One particularly interesting feature of the Ekman boundary layer is its **constant thickness**, measured by δ.
- In most aerodynamic flows at high Reynolds' numbers, the boundary layers **thicken downstream** as fluid retarded by friction accumulates near the boundary.
- One type of boundary layer that does have a uniform thickness is the **asymptotic suction boundary layer** over a **porous flat plate**.

The asymptotic suction boundary layer



The establishment of the asymptotic suction boundary layer over a porous plate.

The scale normal to the plate is greatly exaggerated.

The asymptotic suction boundary layer

- Over the rigid leading section of the plate, the boundary layer thickens in proportion to the square root of the downstream distance.
- From the point at which suction commences, the boundary layer evolves to a uniform thickness in which state the rate at which fluid is retarded is just balanced by the rate at which it is removed.
- It can be shown that the boundary layer thickness in the asymptotic state is proportional to the fluid viscosity and inversely proportional to the suction velocity.

Aircraft wing design

The constant thickness Ekman layer

- In the case of the Ekman layer, the disruption of geostrophic balance by friction leaves a net pressure gradient force towards low pressure.
- Thus, fluid which is retarded in the downstream direction is "re-energized" and flows across the isobars towards low pressure.
- This induced cross-isobaric mass flux in the Ekman layer has important consequences:
- If $\mathbf{u}_g$ varies spatially, there exists a mass flux convergence leading to a vertical velocity component at the outer edge of the boundary layer.
- This induced velocity can have a profound effect on the interior flow outside the boundary layer.

End of Chapter 5 Part I