Chapter 4 (continued)

More on Geostrophic Flows

The effects of stratification

- So far we have assumed a **homogeneous, incompressible fluid:**

  no buoyancy forces

  continuity equation $\nabla \cdot \mathbf{u} = 0$

- Consider now the additional effects of having an **inhomogeneous fluid**; i.e., variable ρ.
- Unless the density is a function of height only, buoyancy forces must be included in the analysis.
- The momentum equation becomes, assuming geostrophy,

$$2\Omega \wedge \mathbf{u} = -\frac{1}{\rho_*}\nabla p + b\mathbf{k}$$

ρ_* to be defined

The Boussinesq approximation

For an incompressible fluid we can still use the simple form of the continuity equation

$$\nabla \cdot \mathbf{u} = 0$$

under certain circumstances.

- **We make the Boussinesq approximation which assumes that density variations are important only:**

  - inasmuch as they give rise to buoyancy forces and

  - that variations in density as they affect the fluid inertia or continuity can be ignored.

  ρ_* **may be regarded as an average density over the whole flow domain, or the density at some particular height.**

The neglect of density variations with height requires strictly that $D/H_s \ll 1$.

H_S **the density height scale**

D **is the flow depth**

ρ_0 (z) **is the average density at height** z

The assumption is that $\delta\rho_0/\rho_0 \ll 1$

$\delta\rho_0$ **is the maximum difference in** ρ_0 (z)

The full continuity equation for an inhomogeneous compressible fluid is

$$\cancel{\frac{1}{\rho}\frac{D\rho}{Dt}} + \nabla \cdot \mathbf{u} = 0$$

in the Boussinesq approximation

Validity of the Boussinesq approximation

- **The Boussinesq approximation is an excellent one in the oceans where relative density differences nowhere exceed more than one or two percent.**
- **It is not very accurate in the atmosphere, except for motions in shallow layers.**
- **The reason is that air is compressible under its own weight to a degree that the density at the height of tropopause, say** 10 km**, is only about one quarter the density at sea level.**
- **For motions which occupy the whole depth of the troposphere,** $D \sim H_s$**.**

The anelastic approximation

At any height in the atmosphere departures of ρ **from** $\rho(z)$ **are small and an accurate form of the full continuity equation appropriate to deep atmospheric motions is**

$$\frac{1}{\rho_0}\mathbf{u}\cdot\nabla\rho_0 + \nabla\cdot\mathbf{u} = 0 \quad \textbf{or} \quad \nabla\cdot(\rho_0\mathbf{u}) = 0$$

- **The inclusion of** $\rho_0(z)$ **- the so-called anelastic approximation - complicates the mathematics without leading to new insights.**
- **We shall use the Boussinesq approximation in our study of atmospheric motions.**
- **The assumption is quite adequate for acquiring an understanding of the dynamics of these motions.**

The thermal wind equation

- To explore the effects of stratification we take again the curl of the momentum equation to obtain the **thermal wind equation**

$$2(\mathbf{\Omega}\cdot\nabla)\mathbf{u} = \mathbf{k}\wedge\nabla b$$

Now the Taylor-Proudman theorem no longer holds.

- In component form with z vertical and in the direction of $\mathbf{\Omega}$ as before, the thermal wind equation becomes

$$2\Omega\left[\frac{\partial u}{\partial z},\frac{\partial v}{\partial z},\frac{\partial w}{\partial z}\right] = \left[-\frac{\partial b}{\partial y},\frac{\partial b}{\partial x},0\right]$$

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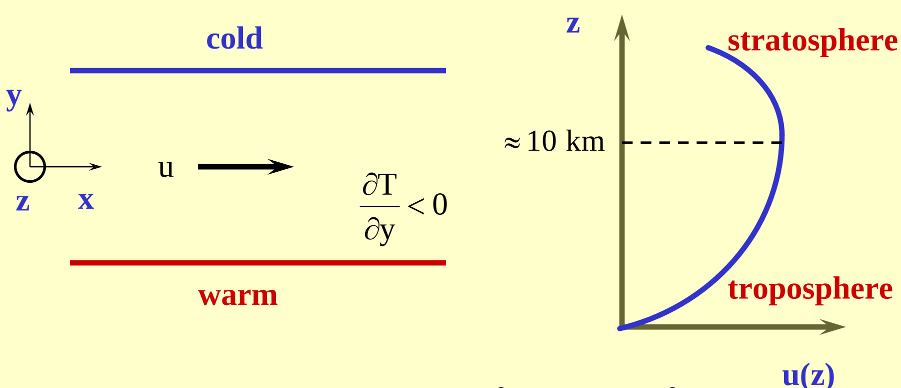
- Again, $w = 0$ at $z = 0 \Rightarrow w = 0$ in the entire flow.
- Later we shall show that for finite, but small Ro, w is not exactly zero, but is formally of order Ro.
- Here we are considering the limit $Ro \to 0$.

- **With the Boussinesq approximation, the buoyancy force can be approximated, either in terms of density or temperature as follows:**

$$b = \begin{cases} -g\dfrac{(\rho-\rho_0)}{\rho} \approx -g\dfrac{(\rho-\rho_0)}{\rho_*}\,, \\ g\dfrac{(T-T_0)}{T_0} \approx \; g\dfrac{(T-T_0)}{T_*}\,, \end{cases} \qquad T_0 = T_0\,(z)$$

T_* **is a constant temperature analogous to** ρ_*

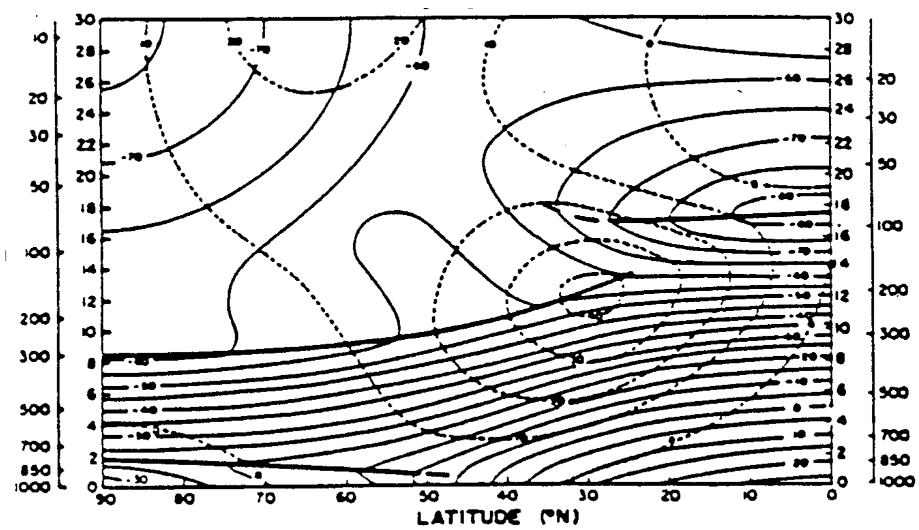
A simple zonal flow in thermal wind balance



Thermal wind equation ➡ $$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega T_o}\frac{\partial T}{\partial y}$$

(Northern hemisphere)

January



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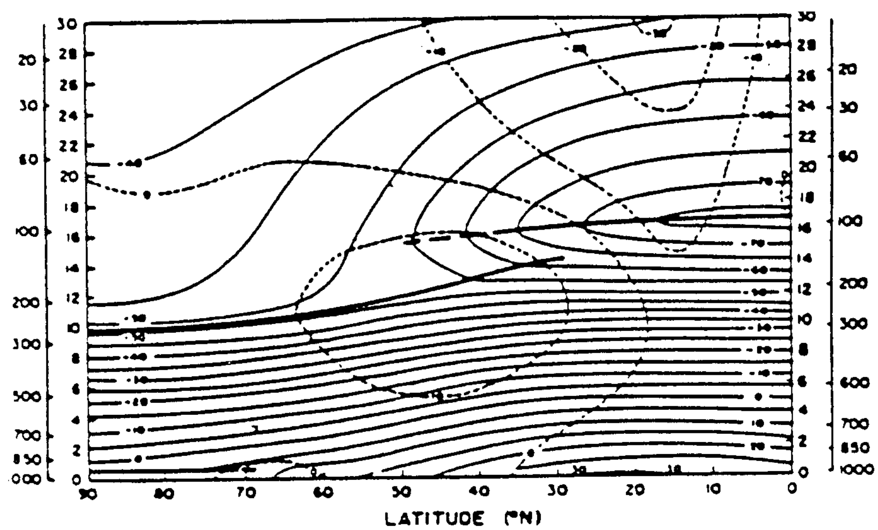


$$2(\mathbf{\Omega}\cdot\nabla)\mathbf{u} = \mathbf{k}\wedge\nabla b$$

$$\Rightarrow \quad 2\Omega\frac{\partial\mathbf{u}}{\partial\mathbf{z}} = \mathbf{k}\wedge\nabla b$$

Compare with $\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k}\wedge\nabla p$

The vertical wind gradient is parallel with the isotherms at any height and has low temperature on the left in the northern hemisphere and on the right in the southern hemisphere.

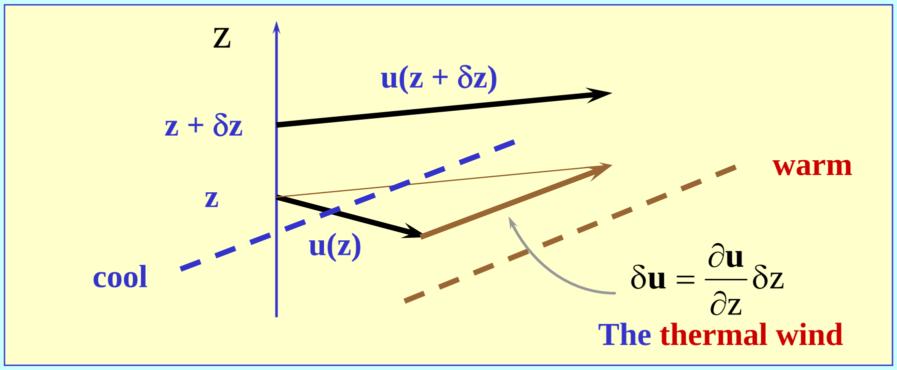
The vertical wind gradient is proportional to the magnitude of the temperature gradient.

General case

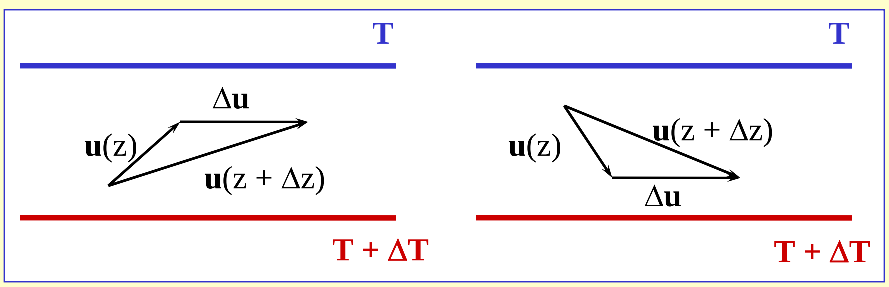
- **So far we assumed that the geostrophic wind and thermal wind are in the same direction.**
- **This happens if the isotherms have the same direction at all heights.**
- **In general this is not the case and we consider now the situation in which the geostrophic wind blows at an angle to the isotherms.**

- Suppose that the geostrophic wind at height **z** blows towards high temperature.
- The geostrophic wind at height z + δz, (δz small), can be written

$$\mathbf{u}(z+\delta z) = \mathbf{u}(z) + \frac{\partial \mathbf{u}}{\partial z}\delta z + 0(\delta z^2)$$



Warm air advection | **Cold air advection**



Turning of the geostrophic wind with height as a result of thermal wind effects (northern hemisphere case).

Veering and backing

- **When the wind direction turns clockwise, or anticyclonic, with height in the northern hemisphere we say that the wind veers with height.**
- **If the wind turns cyclonically with height we say it backs with height.**
- **In the southern hemisphere 'cyclonic' and 'anticyclonic' have reversed senses, but what is confusing is that the terms "veering" and "backing" still mean turning to the right or left respectively.**
- **Thus cyclonic means in the direction of the earth's rotation in the particular hemisphere (counterclockwise in the northern hemisphere, clockwise in the southern hemisphere).**

Thermal advection

- **In general, any air mass will have horizontal temperature gradients within it and the isotherms will be oriented differently at different heights.**
- **Therefore, unless the wind blows in a direction parallel with the isotherms, there will be local temperature changes at any point simply due to advection.**
- **If the temperature of fluid parcels is conserved during horizontal displacement, we may express this mathematically by the equation** $DT/Dt = 0$.
- **Then the local rate of change of temperature at any point,** $\partial T/\partial t$**, is given by**

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T$$

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T$$

called the **thermal advection**

- If warmer air flows towards a point $\mathbf{u} \cdot \nabla T < 0$, and $\partial T / \partial t > 0$.
- We call this **warm air advection**.
- It follows that there is a connection between thermal advection and the turning of the geostrophic wind vector with height.
- In the **northern** (southern) hemisphere, the wind **veers** (backs) with height in conditions of **warm air advection**.
- It **backs** (veers) with height when there is **cold air advection**.

Some notes concerning the thermal wind equation

- It is a diagnostic equation and as such is useful, in **checking analyses** of the observed wind and temperature fields **for consistency**.
- Secondly, the z component of the thermal wind equation is $0 = -\rho_*^{-1} \partial p / \partial z + b$, which shows that the density-, or buoyancy field is in hydrostatic equilibrium.
- Finally, the thermal wind **constraint** is important also in ocean current systems wherever there are horizontal density contrasts.

The thermodynamic equation

- When vertical motions are present, the equation

$$DT/Dt = 0$$

may be inaccurate since ascent or subsidence is associated also with a thermal tendency.

- However, when diabatic processes such as radiative heating and cooling can be neglected, and provided that condensation or evaporation does not occur, the potential temperature θ, of an air parcel is conserved, even when the parcel ascends or subsides.
- This is expressed mathematically by the formula $\frac{D\theta}{Dt} = 0$.
- This formula encapsulates the first law of thermodynamics.

The thermodynamic equation for a Boussinesq liquid

- For a Boussinesq fluid, i. e. one for which the Boussinesq approximation is satisfied, density is conserved following a fluid parcel, i.e.,

$$\frac{D\rho}{Dt} = 0$$

This is consistent with $\nabla \cdot \mathbf{u} = 0$ and $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$

In terms of the buoyancy force σ, $\frac{D\rho}{Dt} = 0$ may be written in the form

$$\frac{Db}{Dt} + N^2 w = 0$$

where $N^2 = -(g/\rho_*)\,(d\rho_0/dz)$ is the square of the Brunt-Väisälä frequency or the buoyancy frequency of the motion.

In a **shallow** atmosphere, the thermodynamic equation

$$\frac{D\theta}{Dt} = 0$$

reduces to the same form as $\frac{Db}{Dt} + N^2 w = 0$

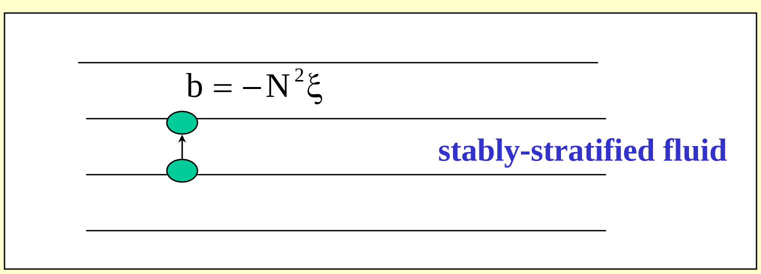
with σ given by $g(\theta - \theta_0)/\theta_*$ and with N^2 replaced by

$$N^2 = (g / \theta_*)\,(d\theta_0 / dz)$$

$\theta_0(z)$ is the **basic state** potential temperature distribution.

Interpretation of $\frac{Db}{Dt} + N^2 w = 0$

This equation represents the change in buoyancy force experienced by a fluid parcel as it moves around and ascends or descends.



If N^2 = constant $\qquad \frac{Db}{Dt} = -N^2\frac{D\xi}{Dt} = -N^2 w$

The use of pressure coordinates

- Some authors, including Holton, use a coordinate system in which pressure is used instead of the vertical coordinate z.
- This has certain advantages:

  (i) pressure is a quantity measured directly in the global meteorological data network and upper air data is normally presented on isobaric surfaces: i.e. on surfaces p = constant rather than z = constant;

  (ii) the continuity equation has a much simpler form in pressure coordinates.
- A major disadvantage of pressure coordinates is that the surface boundary condition analogous to, say, w = 0 at z = 0 over flat ground, is much harder to apply.

Comparison of the equations in height and pressure coordinates

- The simplifications of the pressure coordinate systems disappear in the case of nonhydrostatic motion.
- The following comparison is for the hydrostatic system of equations only.

horizontal momentum equations

$$\frac{D_h \mathbf{u}_h}{Dt} + w\frac{\partial \mathbf{u}_h}{\partial z} + f\mathbf{k} \wedge \mathbf{u}_h = -\frac{1}{\rho}\nabla_h p$$

$$\frac{D_p \mathbf{u}_h}{Dt} + \omega\frac{\partial \mathbf{u}_h}{\partial p} + f\mathbf{k} \wedge \mathbf{u}_h = -\nabla_p \phi$$

ϕ **is essentially** gz

z **the height of an isobaric surface**

ϕ **is called the geopotential.**

ω **plays the role of** w **in** p**-coordinates**

$$\omega = \frac{Dp_T}{Dt} = \frac{\partial p_T}{\partial t} + \mathbf{u}_h \cdot \nabla p_T + w\frac{\partial p_T}{\partial z} = \frac{D_h p_T}{Dt} - \rho g w\ .$$

vertical momentum equations

$$\frac{1}{\rho}\frac{\partial p_T}{\partial z} = -g \quad \text{or} \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = b$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p}$$

continuity equation

$$\nabla_h \cdot (\rho_0(z)\mathbf{u}_h) + \frac{\partial}{\partial z}(\rho_0(z)w) = 0$$

$$\nabla_p \cdot \mathbf{u}_h + \frac{\partial \omega}{\partial p} = 0$$

Some notes

- In the pressure coordinate system, p is the total pressure p_T.
- In most situations, $|D_h p_T/Dt| << |\rho g w|$ so that $\mathrm{sgn}(\omega) = -\,\mathrm{sgn}(w)$.
- Thus ω negative (positive) signifies ascending (descending) motion.
- In the pressure coordinate system, the subscripts p on the operators D_p/Dt and ∇_p signify that derivatives are computed with p held fixed.
- Because the isobaric surfaces are very close to horizontal, there is no practical difference between $D_h\,\mathbf{u_h}/Dt$ and $D_p\mathbf{u_h}/Dt$.
- Certainly such a difference could not be measured.

$$\cancel{\frac{D_p \mathbf{u}_h}{Dt} + \omega \frac{\partial \mathbf{u}_h}{\partial p}} + f\mathbf{k} \wedge \mathbf{u}_h = -\nabla_p \phi$$

- In pressure coordinates, the geostrophic equation is

$f\mathbf{k} \wedge \mathbf{u}_h = -\nabla\phi$ with solution $\mathbf{u}_h = -f\mathbf{k} \wedge \nabla\phi$

- The thermal wind equation is frequently used in the form

$$\mathbf{u}' = \mathbf{u}_{500mb} - \mathbf{u}_{1000mb} = \frac{1}{f}\mathbf{k} \wedge \nabla(\phi_{500} - \phi_{1000}) = \frac{1}{f}\mathbf{k} \wedge \nabla h'$$

geopotential thickness between 500 mb and 1000 mb

$h' = R\overline{T}\,\ln 2$ is proportional to the mean temperature between the two pressure surfaces.

Thickness charts

- **Thickness charts** are charts showing contours of equal thickness.
- They are used by weather forecasters, *inter alia*, to locate regions of cold and warm air in the lower troposphere.
- In Australia, the thickness isopleths are given in metres; that is, contours of h'/g are plotted.
- They are similar to 500 mb charts, because often

$$u_{500\ \text{mb}} >> u_{1000\ \text{mb}}$$ in magnitude.

Thickness advection

The thermal tendency equation is

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T$$

Because of the relationship between h' and $\overline{T}$ in pressure coordinates we may obtain a thickness tendency equation

$$\frac{\partial h'}{\partial t} = -\overline{\mathbf{u}}_h \cdot \nabla h'$$

where $\overline{\mathbf{u}}_h$ is a measure, in some sense, of the mean wind between 1000 mb and 500 mb.

Write $\overline{\mathbf{u}}_h = \mathbf{u}_s + \lambda\mathbf{u}' + \mathbf{u}_a$ **where** $\mathbf{u}_s$ **is the surface geostrophic wind,** $\mathbf{u}_a$ **is a measure of the horizontal ageostrophic wind and** λ **is constant.**

In many circumstances, $|\mathbf{u}_a|$ **is small compared with** $|\mathbf{u}_s|$ **and** $|\mathbf{u}'|$ **,** →

$\overline{\mathbf{u}}_h$ **is approximately the surface wind plus a weighted measure of the thermal wind, or wind difference between** 1000 mb **and** 500 mb**.**

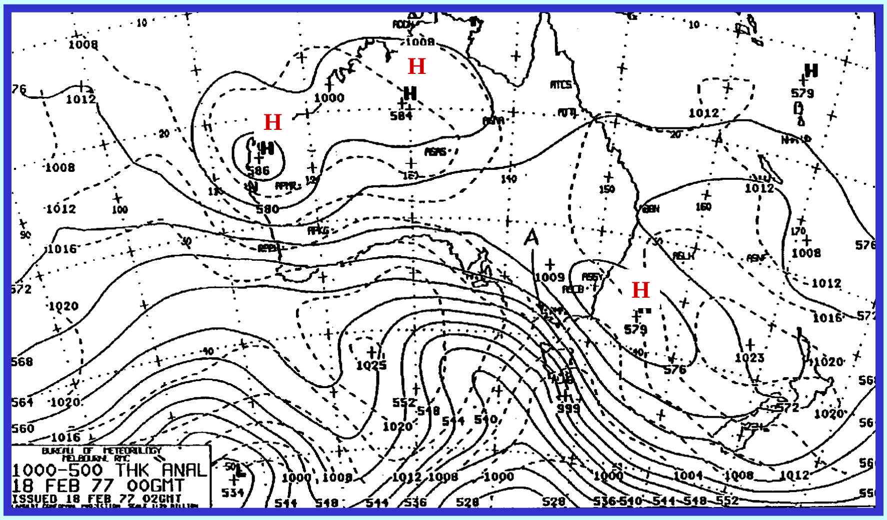
Since $\mathbf{u}' \cdot \nabla h' = f^{-1}\mathbf{k} \wedge \nabla h' \cdot \nabla h' \equiv 0$

$$\frac{\partial h'}{\partial t} \approx -\mathbf{u_s} \cdot \nabla h'$$

$$\frac{\partial h'}{\partial t} \approx -\mathbf{u_s} \cdot \nabla h' \quad \rightarrow$$

- **Under circumstances where the temperature field is merely advected (this is not always the case), the thickness tendency is due entirely to advection by the surface wind field.**
- **Accordingly the surface isobars are usually displayed on thickness charts so that** $\mathbf{u}_s$ **can be deduced readily in relation to** $\nabla h'$**.**
- **A typical thickness chart is shown in the next figure.**

A typical thickness chart



End of Chapter 4