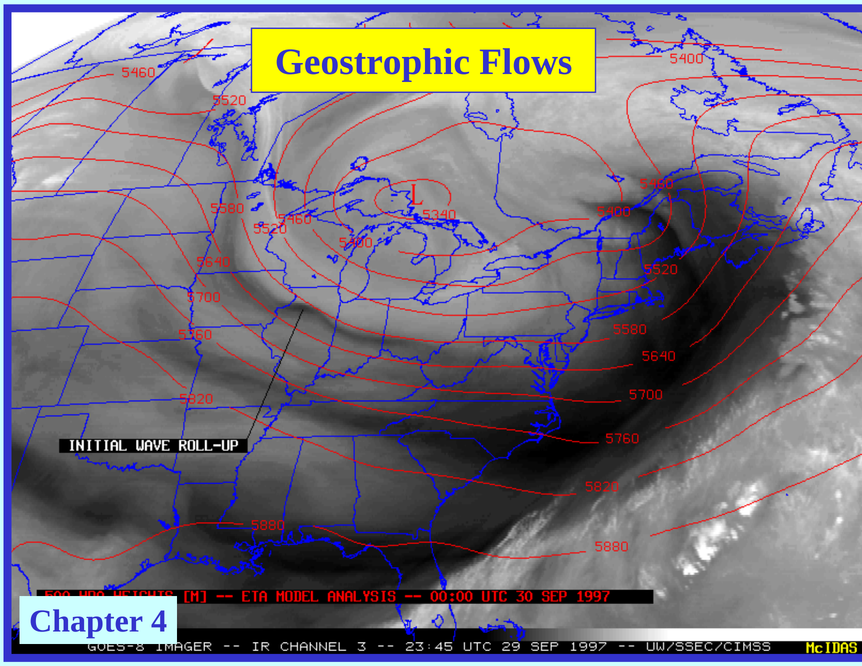
Geostrophic Flows



Chapter 4

The geostrophic approximation

For frictionless motion (**D = 0**) the momentum equation is

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

perturbation pressure

Let $Ro \rightarrow 0$

$$2\boldsymbol{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

This is called the geostrophic equation

We expect this equation to hold approximately in synoptic scale motions in the atmosphere and oceans, except possibly near the equator.

$$2\mathbf{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

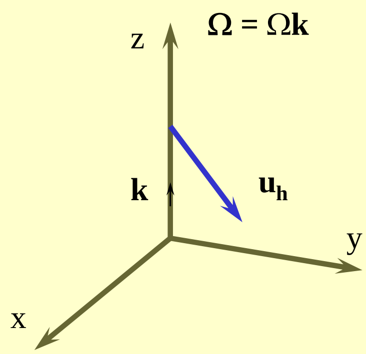
Take the scalar product with Ω

$$0 = -\frac{1}{\rho}\mathbf{\Omega} \cdot \nabla p$$

In geostrophic motion, the perturbation pressure gradient is perpendicular to Ω**.**

Choose rectangular coordinates:

$\mathbf{k} = (0,0,1)$



velocity components $\mathbf{u} = (u,v,w)$, $\mathbf{u} = \mathbf{u_h} + w\mathbf{k}$

$\mathbf{u_h} = (u,v,0)$ **is the horizontal flow velocity**

Take $\mathbf{k} \wedge$

$$2\Omega \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

$(\mathbf{k}\cdot\mathbf{u})\mathbf{k} = (0,\ 0\ w)$

$$2\Omega\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{u}) = 2\Omega\big[\underbrace{(\mathbf{k}\cdot\mathbf{u})\mathbf{k} - \mathbf{u}}_{-\mathbf{u}_h}\big] = -\frac{1}{\rho}\mathbf{k}\wedge\nabla p$$

$\nabla_h p = (\partial p/\partial x,\ \partial p/\partial y,\ 0)$

$$\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k}\wedge\nabla_h p$$

and $$0 = \frac{\partial p}{\partial z}$$

This is the solution for geostrophic flow.

The geostrophic wind

$$\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k}\wedge\nabla_h p$$

- The geostrophic wind blows parallel to the lines (or more strictly surfaces) of constant pressure - the isobars, with low pressure to the left.
- Well known to the layman who tries to interpret the newspaper "weather map", which is a chart showing isobaric lines at mean sea level.
- In the southern hemisphere, low pressure is to the right.

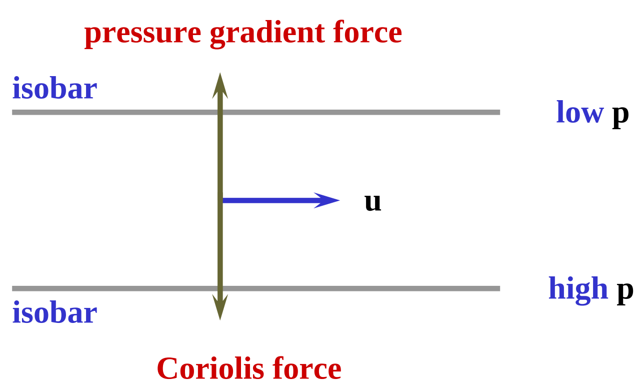
Choice of coordinates

- For simplicity, let us orientate the coordinates so that x points in the direction of the geostrophic wind.
- Then v = 0, implying that $\partial p/\partial x = 0$.

$$u = -\frac{1}{2\Omega\rho}\frac{\partial p}{\partial y}$$

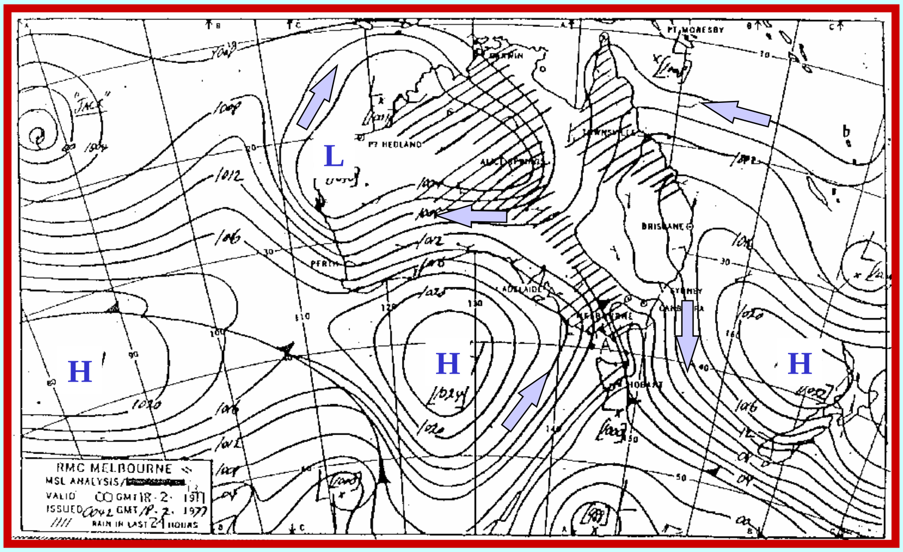
- Note that for fixed Ω , the winds are stronger when the isobars are closer together and, for a given isobar separation, they are stronger for smaller $|\Omega|$.

Geostrophic flow



(Northern hemisphere case: > 0)

A mean sea level isobaric chart over Australia



Streamlines and Isobars

Note that in geostrophic flow $\nabla_h \cdot \mathbf{u_h} = 0$

⇒ there exists streamfunction ψ such that

$$\mathbf{u}_h = (-\psi_y, \psi_x, 0) = \mathbf{k} \wedge \nabla_h \psi$$

Compare with $\mathbf{u}_h = \dfrac{1}{2\Omega\rho} \mathbf{k} \wedge \nabla_h p$

⇒ $\psi = p / 2\Omega\rho$ ⇒

The streamlines are coincident with the isobars ⇒ just another way of saying the flow is parallel with the isobars.

Note also that the solution $\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k} \wedge \nabla_h p$

and $0 = \frac{\partial p}{\partial z}$

tells us nothing about the vertical velocity w.

- For an incompressible fluid, $\nabla \cdot \mathbf{u} = 0$.
- Also, for geostrophic flow, $\nabla_h \cdot \mathbf{u}_\mathbf{h} = 0$.
- then $\partial w/\partial z = 0$ implying that w is independent of z.

If $w = 0$ at some particular z, say $z = 0$, which might be the ground, then $w \equiv 0$.

The geostrophic equation is degenerate!

- The geostrophic equation is degenerate, i.e. time derivatives have been eliminated in the approximation.
- We cannot use the equation to predict how the flow will evolve.
- Such equations are called diagnostic equations.
- In the case of the geostrophic equation, for example, a knowledge of the isobar spacing at a given time allows us to calculate, or 'diagnose', the geostrophic wind.
- We cannot use the equation to forecast how the wind velocity will change with time.

The Taylor-Proudman Theorem

The curl of the geostrophic equation gives

$$2(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = \mathbf{0}$$

In our rectangular coordinate frame,

$$\frac{\partial \mathbf{u}}{\partial z} = 0$$

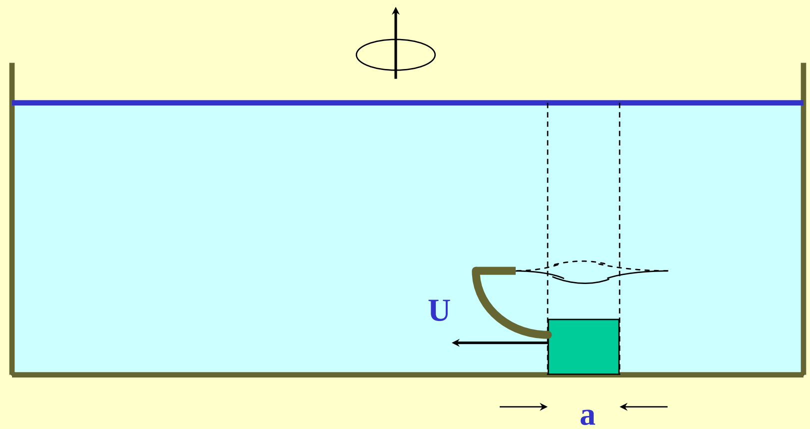
⇨ $\mathbf{u} = \mathbf{u}(x, y, t)$ **only; it is independent of** z**.**

This is the Taylor-Proudman theorem which asserts that geostrophic flows are strictly two-dimensional.

Note that $2(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = \mathbf{0}$ **is the vorticity equation for geostrophic flow of a homogeneous fluid.**

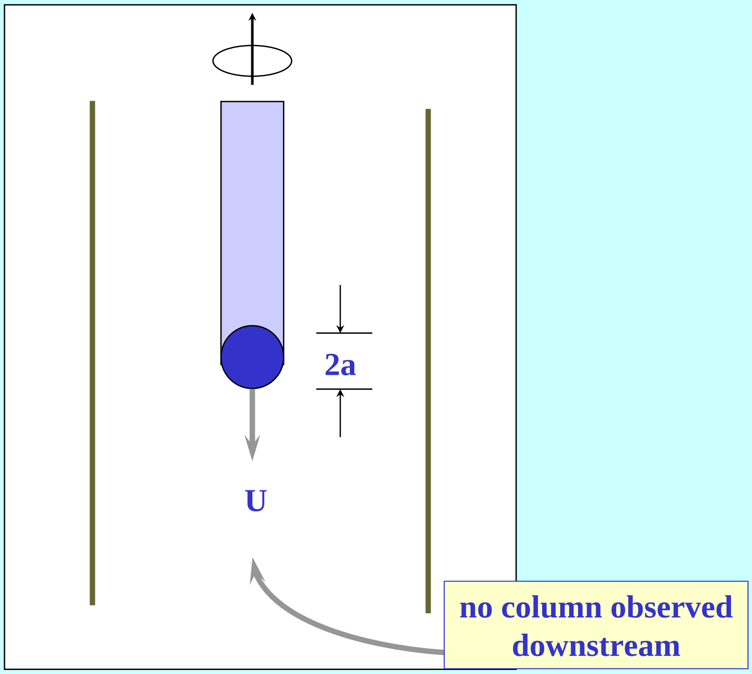
- **The implications of the Taylor-Proudman theorem are highlighted by a series of laboratory experiments performed by G. I. Taylor after whom the theorem is named.**

Taylor’s experiment



An obstacle with linear dimension a is towed with speed U along the bottom of a tank of fluid of depth greater than a in solid body rotation with angular velocity Ω.

Taylor performed also a second experiment in which a sphere was towed slowly along the axis of a rotating fluid.



- **It is worth reiterating the conditions of the Taylor-Proudman theorem:**
- **The theorem applies to slow, steady, inviscid flow in a homogeneous** (ρ = constant) **rotating fluid.**
- **If the flow becomes ageostrophic in any locality, the theorem breaks down and three-dimensional flow will occur in that locality, i.e., time dependent, nonlinear, or viscous terms may become important.**

Jupiter

- Taylor columns are not observed in the atmosphere in any recognizable form, presumably because one or more of the conditions required for their existence are violated.
- It has been suggested by **R. Hide** that the **Giant Red Spot** on the planet Jupiter may be a Taylor column which is locked to some topographical feature below the visible surface.
- Although it is not easy to test this idea, it should be remarked that Jupiter has a mean diameter $10^{1}/_{2}$ times that of the earth and **rotates once every ten hours.**

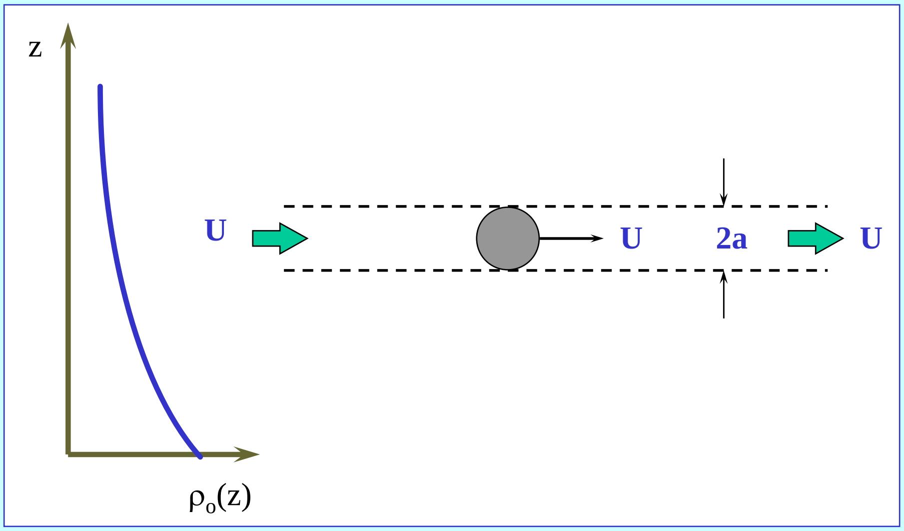
Jupiter

Jupiter's Red Spot

Blocking

- The phenomenon of blocking in a stably stratified fluid is analogous to that of Taylor column formation in a rotating fluid.
- If an obstacle with substantial lateral extent such as a long cylinder is moved horizontally with a small velocity parallel to the isopycnals (lines of constant ρ) in a stably stratified fluid, the obstacle will push ahead of it and pull behind it fluid in a layer of order the diameter of the body.

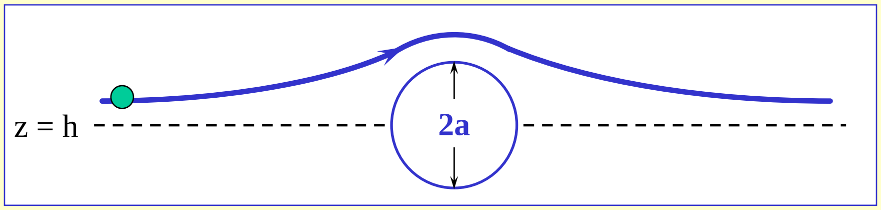
Blocking



Physical interpretation of blocking

- The restoring force on a parcel of fluid displaced vertically in a stratified fluid is approximately minus N^2 times the displacement.
- Blocking occurs when parcels of fluid have insufficient kinetic energy to overcome the buoyancy forces which would be experienced in surmounting the obstacle.
- We can do a rough calculation to illustrate this.

- Consider a stationary obstacle symmetrical about the height $z = h$.



Suppose a fluid parcel of mass m is at a height $z = h + \frac{1}{2}a$

To surmount the obstacle, the parcel will need to rise a distance of at least $\frac{1}{2}a$.

The work it will have to do against the buoyancy forces is

$$\int_0^{\frac{1}{2}a} mN^2\xi d\xi = \frac{1}{8}mN^2a^2$$

If the fluid parcel moves with speed U, its kinetic energy is

$$\frac{1}{2}mU^2$$

Neglecting friction effects, this will have to be greater than

$$\frac{1}{2}mU^2 > \frac{1}{8}mN^2a^2$$

for the parcel to be able to surmount the obstacle, i.e.

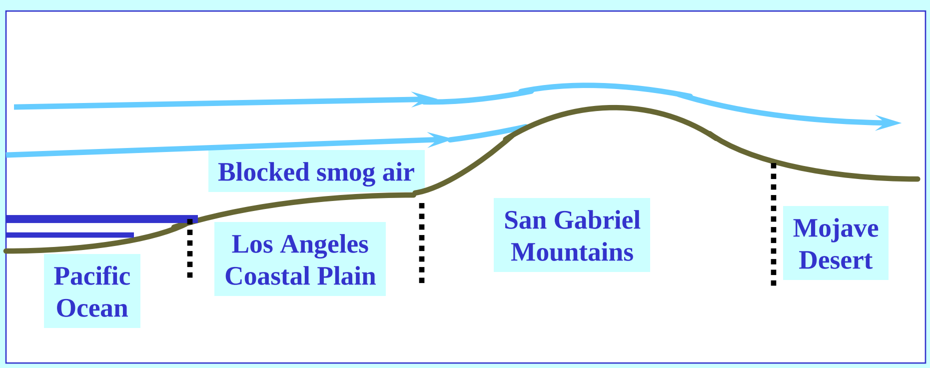
$$U > \frac{1}{2}aN$$

Alternatively, if $2U/aN < 1.0$, all fluid parcels in a layer of at least depth a centred on $z = h$ will be blocked.

Blocking in the atmosphere

- Blocking is a common occurrence in the atmosphere in the neighbourhood of hills or mountains.
- A good example is the region of Southern California.

Blocking over Southern California

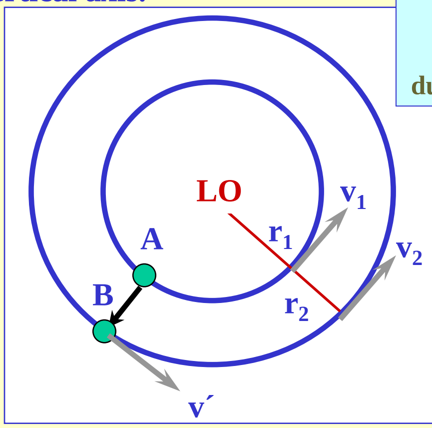


Los Angeles Smog

Analogy between blocking and axial Taylor columns

- **We can interpret the formation of Taylor columns along the axis of a rotating fluid in a similar manner to the foregoing interpretation of blocking.**
- **In the former case, fluid particles, or rings of fluid must do work against centrifugal forces to pass round the obstacle.**
- **If they have insufficient energy to do this, the flow will be "blocked" and a Taylor column will form.**
- **It is instructive to work through some details.**

Consider a fluid rotating with tangential velocity $v(r)$ about a vertical axis.



the parcel at A conserves its angular momentum during its radial displacement to B

$$r_2 v' = r_1 v_1,$$

or

$$v' = \frac{r_1}{r_2} v_1$$

The forces acting on the parcel at B are:

inwards $$\left.\frac{1}{\rho}\frac{dp}{dr}\right]_{r=r_2} = \frac{v_2^{\,2}}{r_2}$$

outwards $$\frac{v'^2}{r_2}$$

The displaced parcel experiences an outward force per unit mass,

F = centrifugal force – radial pressure gradient

$$v' = \frac{r_1}{r_2} v_1 \qquad = \frac{v'^2}{r_2} - \left.\frac{1}{\rho}\frac{\partial p}{\partial r}\right]_{r=r_2} \qquad = \frac{v_2^{\,2}}{r_2}$$

$$F = \frac{1}{r_2^{\,3}}\left[(r_1 v_1)^2 - (r_2 v_2)^2\right]$$

- Special case: solid body rotation (as in the Taylor column experiment)
- $v = \Omega r$, and for a small displacement from radius $r_1 = r$ to $r_2 = r + \xi$,

$$F \approx -4\Omega^2 \xi$$

The stability of solid body rotation

$v = \Omega r \quad \Rightarrow \quad F \approx -4\Omega^2 \xi$

- **A fluid parcel displaced outwards experiences an inwards force and one displaced inwards experiences an outward force. In both cases there is a restoring force, proportional to the displacement and to the square of the angular frequency Ω.**
- **This is in direct analogy with the restoring force experienced in a stably stratified, non-rotating fluid.**
- **The physical discussion relating to blocking carries over to explain the formation of axial Taylor columns.**

Stability of a rotating fluid

We can now establish a criterion for the stability of a general rotating flow $v(r)$ **analogous to the criterion in terms of** $\mathrm{sgn}(N^2)$ **for the stability of a density stratified fluid.**

Let $\Gamma(r) = rv(r)$ **be the circulation at radius** r.

Then for a small radial displacement ξ**, the restoring force on a displaced parcel is given by**

$$-F \approx -\frac{1}{r^3}\frac{\partial}{\partial r}\left(\Gamma^2\right)\xi$$

A general swirling flow $v(r)$ **is stable, neutrally-stable, or unstable as the square of the circulation increases, is zero, or decreases with radius.**

End of Chapter 4 Part I