The Equations of Motion in a Rotating Coordinate System

Chapter 3

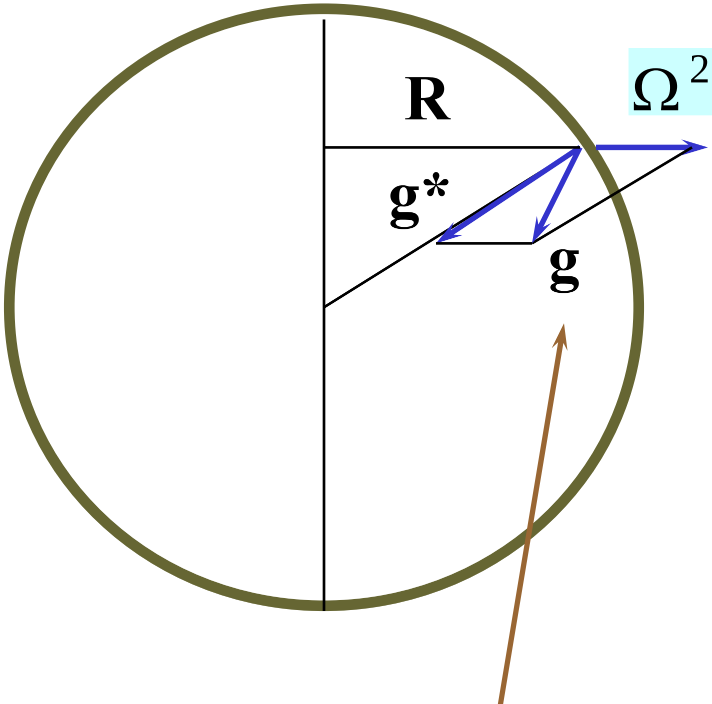
Since the earth is rotating about its axis and since it is convenient to adopt a frame of reference fixed in the earth, we need to study the equations of motion in a rotating coordinate system.

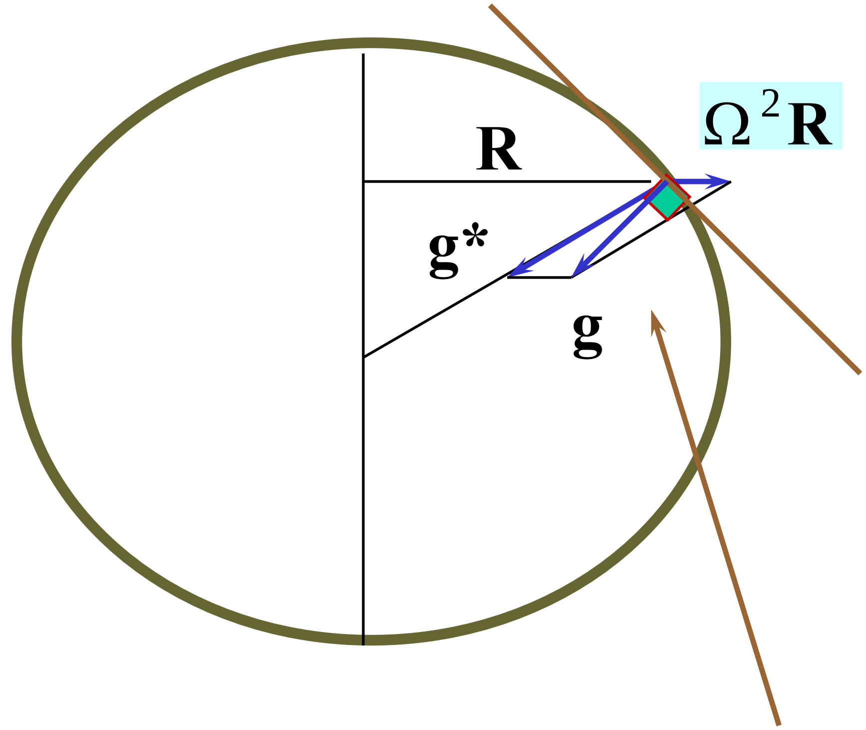
Before proceeding to the formal derivation, we consider briefly two concepts which arise therein:

Effective gravity and Coriolis force

Effective Gravity

g is everywhere normal to the earth's surface





effective gravity **g** on a spherical earth

effective gravity on an earth with a slight equatorial bulge

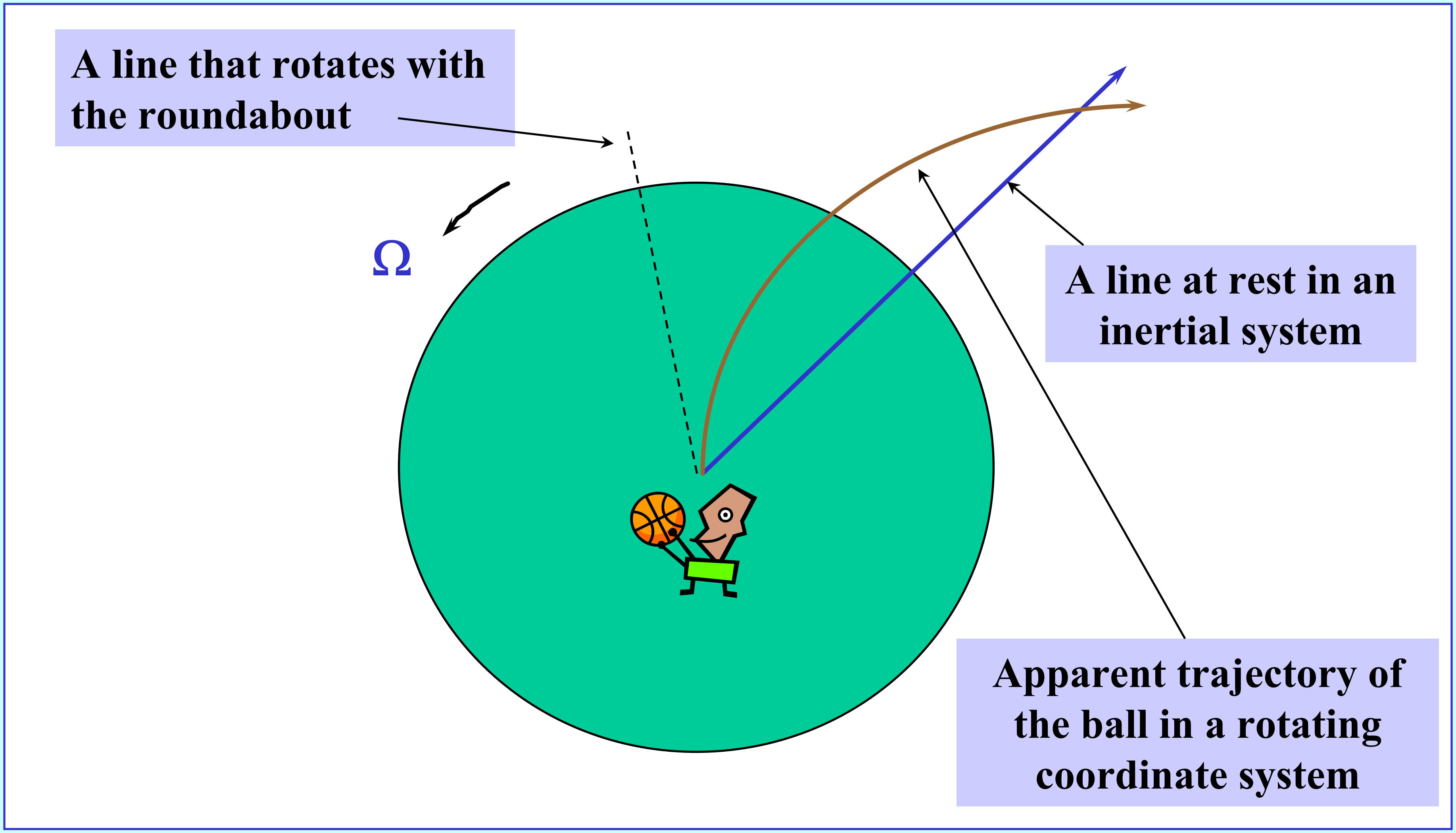
Effective Gravity

If the earth were a perfect sphere and not rotating, the only gravitational component g* would be radial.

Because the earth has a bulge and is rotating, the effective gravitational force g is the vector sum of the normal gravity to the mass distribution g*, together with a centrifugal force $\Omega^2 R$**, and this has no tangential component at the earth's surface.**

$$\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$$

The Coriolis force

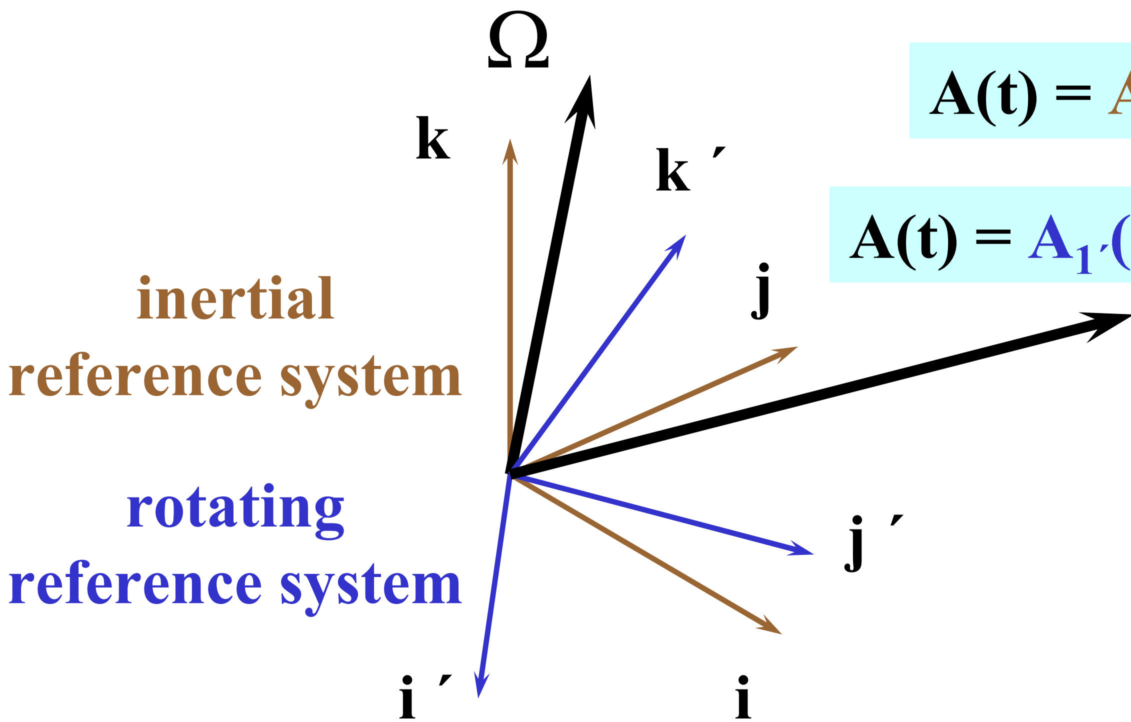


Mathematical derivation of the Coriolis force

Representation of an arbitrary vector A(t)

$$\mathbf{A}(t) = A_1(t)\mathbf{i} + A_2(t)\mathbf{j} + A_3(t)\mathbf{k}$$

$$\mathbf{A}(t) = A_{1'}(t)\mathbf{i}' + A_{2'}(t)\mathbf{j}' + A_{3'}(t)\mathbf{k}'$$



The time derivative of an arbitrary vector $\mathbf{A}(t)$

The derivative of $\mathbf{A}(t)$ with respect to time

$$\frac{d_a\mathbf{A}}{dt} = \mathbf{i}\frac{dA_1}{dt} + \mathbf{j}\frac{dA_2}{dt} + \mathbf{k}\frac{dA_3}{dt}$$

the subscript "a" denotes the derivative in an inertial reference frame

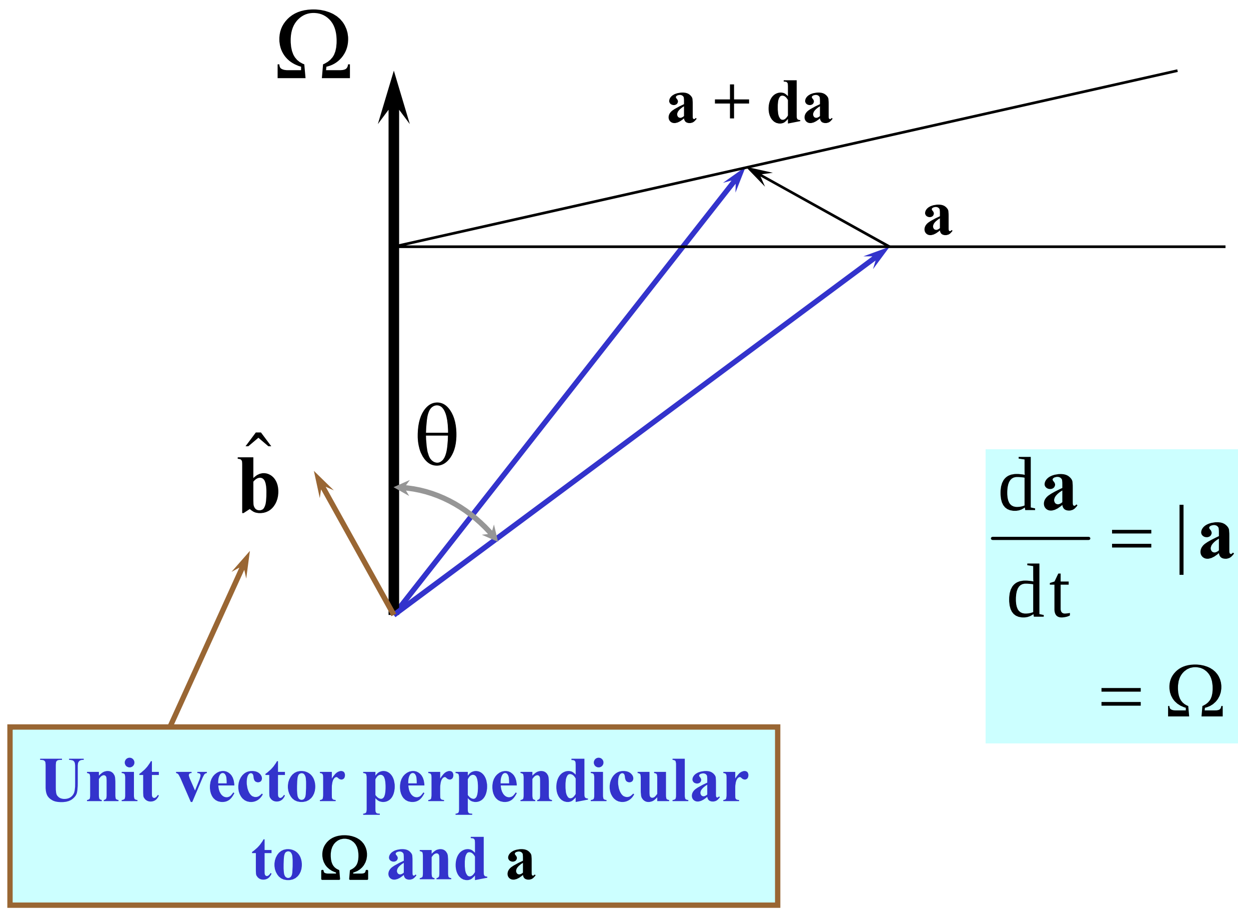
In the rotating frame of reference

$$\frac{d_a\mathbf{A}}{dt} = \mathbf{i}'\frac{dA_1'}{dt} + A_1'\frac{d\mathbf{i}'}{dt} + \ldots$$

$$= \mathbf{i}'\frac{dA_1'}{dt} + A_1'\left(\boldsymbol{\Omega}\wedge\mathbf{i}'\right) + \ldots$$

$$= \left[\frac{d}{dt} + \boldsymbol{\Omega}\wedge\right]\left(A_1'\mathbf{i}' + \ldots\right)$$

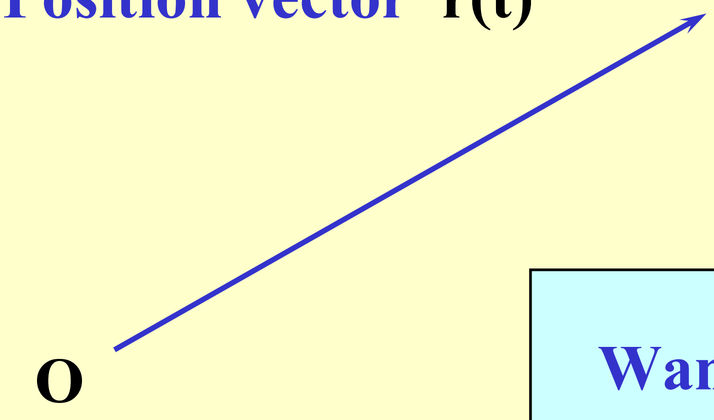
Let $\mathbf{a}$ be any vector rotating with angular velocity $\boldsymbol{\Omega}$



$$\frac{d\mathbf{a}}{dt} = |\mathbf{a}||\Omega|\sin\theta\,\hat{\mathbf{b}}$$

$$= \Omega \wedge \mathbf{a}$$

Position vector $\mathbf{r}(t)$



Want to calculate $\mathbf{u_a} = \dfrac{d_a \mathbf{r}}{dt}$

the absolute velocity of an air parcel

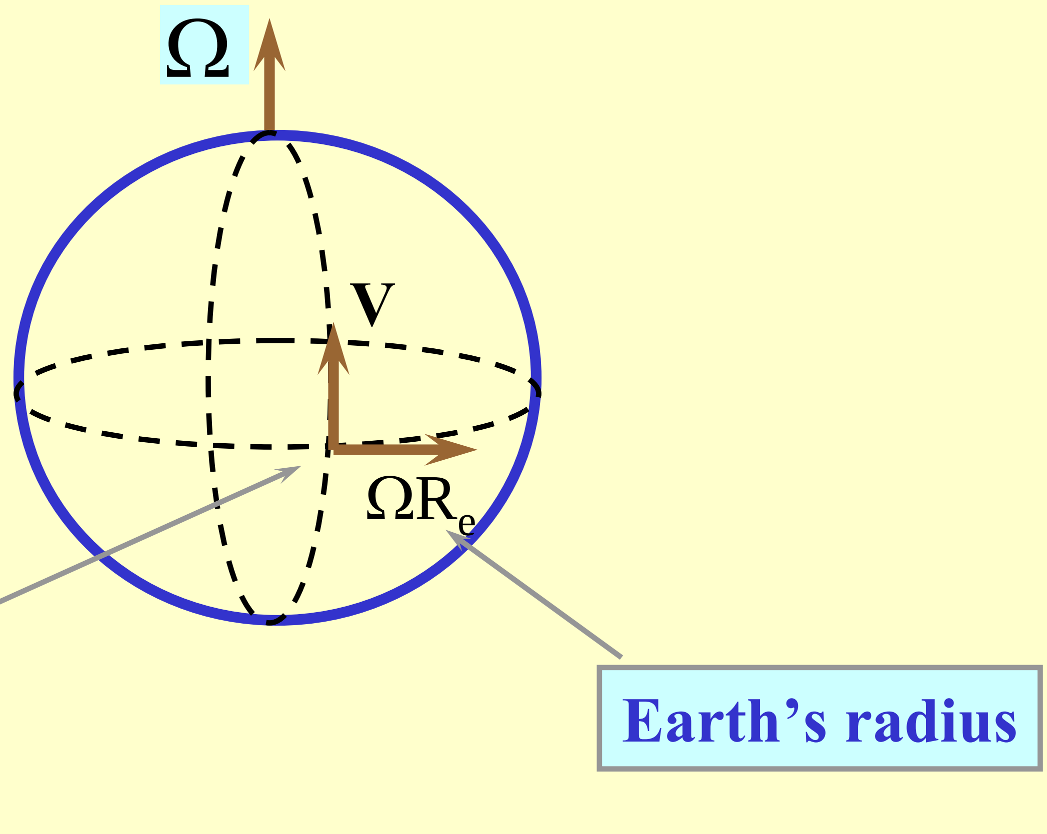
The relative velocity in a rotating frame of reference is

$$\mathbf{u} = \frac{d\mathbf{r}}{dt} = \frac{dr_{1'}}{dt}\mathbf{i}' + \frac{dr_{2'}}{dt}\mathbf{j}' + \frac{dr_{3'}}{dt}\mathbf{k}'$$

and $\mathbf{u_a} = \mathbf{u} + \Omega \wedge \mathbf{r}$

Example

$$\mathbf{u_a} = \mathbf{u} + \Omega \wedge \mathbf{r}$$



This air parcel starts relative to the earth with a poleward velocity V.

It begins relative to space with an additional eastwards velocity component ΩR_e .

We need to calculate the absolute acceleration if we wish to apply Newton's second law

$$\Rightarrow \quad \frac{d_a \mathbf{u_a}}{dt}$$

Measurements on the earth give only the relative velocity u and therefore the relative acceleration

$$\Rightarrow \quad \frac{d\mathbf{u}}{dt}$$

$$\mathbf{u_a} = \mathbf{u} + \Omega \wedge \mathbf{r}$$

With $\mathbf{A} = \mathbf{u_a}$

$$\frac{d_a \mathbf{u}_a}{dt} = \frac{d\mathbf{u}_a}{dt} + \Omega \wedge \mathbf{u}_a$$

$$\frac{d_a \mathbf{u}_a}{dt} = \frac{d\mathbf{u}}{dt} + 2\Omega \wedge \mathbf{u} + \Omega \wedge (\Omega \wedge \mathbf{r})$$

absolute acceleration

$$\frac{d_a \mathbf{u}_a}{dt} = \frac{d\mathbf{u}}{dt} + 2\Omega \wedge \mathbf{u} + \Omega \wedge (\Omega \wedge \mathbf{r})$$

relative acceleration

Coriolis-acceleration

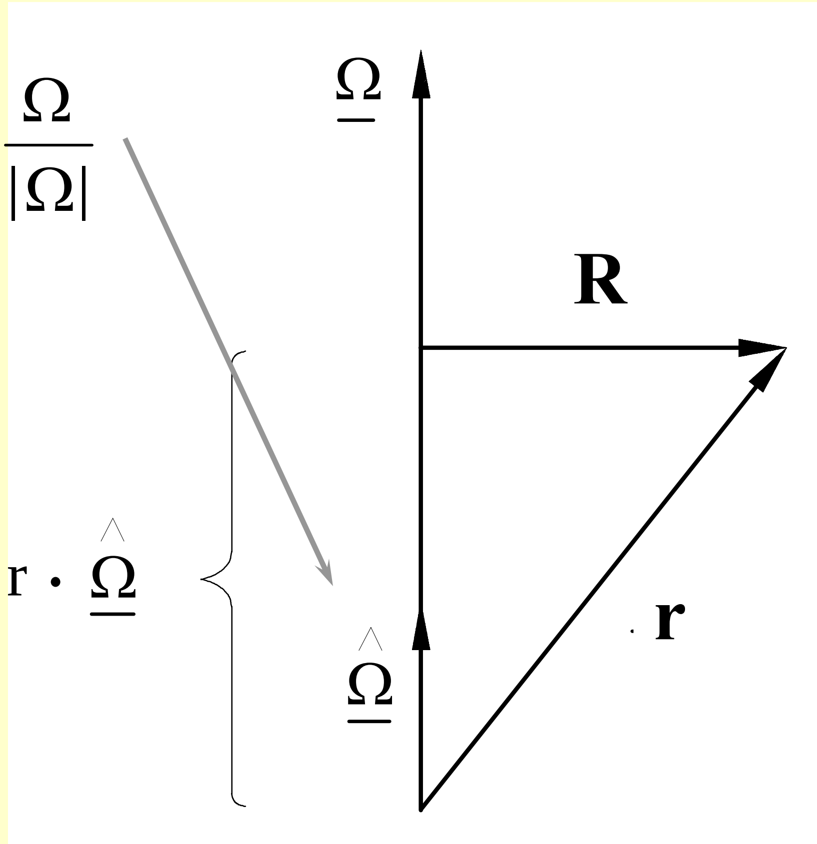
Centripetal acceleration

Centripetal acceleration

Position vector **r** is split up

$$\mathbf{r} = (\mathbf{r}.\hat{\Omega})\hat{\Omega} + \mathbf{R}$$

$$\Omega \wedge (\Omega \wedge \mathbf{r}) = \Omega \wedge (\Omega \wedge \mathbf{R})$$
$$= -|\Omega|^2\,\mathbf{R}$$



The Centripetal acceleration is directed inwards towards the axis of rotation and has magnitude $|\Omega|^2\mathbf{R}$.

Newton's second law in a rotating frame of reference

In an inertial frame:

$$\rho \frac{d_a \mathbf{u}_a}{dt} = \mathbf{F}$$

density

force per unit mass

In a rotating frame:

$$\rho\left[\frac{d\mathbf{u}}{dt} + 2\Omega \wedge \mathbf{u} + \Omega \wedge (\Omega \wedge \mathbf{r})\right] = \mathbf{F}$$

Alternative form

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{F} - 2\rho\Omega \wedge \mathbf{u} - \rho\Omega \wedge (\Omega \wedge \mathbf{r})$$

Coriolis force

Centrifugal force

Let the total force F = g* + F′ be split up

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{F}' + \mathbf{g}^* - 2\rho\Omega \wedge \mathbf{u} - \rho\Omega \wedge (\Omega \wedge \mathbf{r})$$

$$\Omega \wedge (\Omega \wedge \mathbf{r}) = -|\Omega|^2 \mathbf{R}$$

With $\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{F}' + \rho\mathbf{g} - 2\rho\Omega \wedge \mathbf{u}$$

The centrifugal force associated with the earth's rotation no longer appears explicitly in the equation; it is contained in the effective gravity.

When frictional forces can be neglected, F' is the pressure gradient force

total pressure

$$\mathbf{F}' = -\nabla p_T$$ **per unit volume**

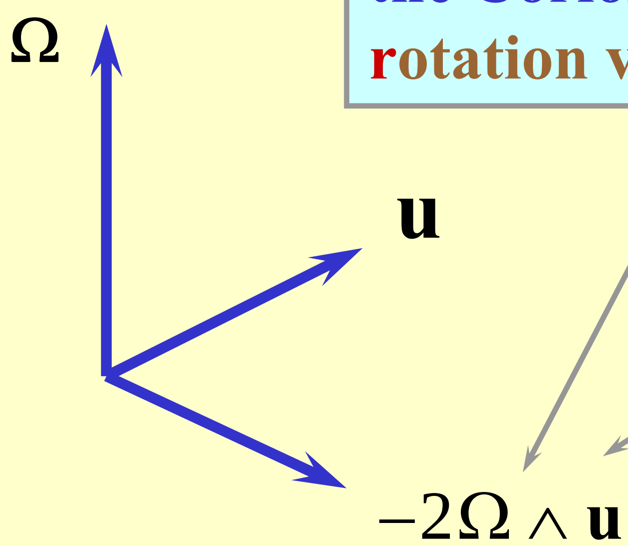
$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{F}' + \rho\mathbf{g} - 2\rho\Omega \wedge \mathbf{u}$$

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p_T + \mathbf{g} - 2\Omega \wedge \mathbf{u}$$

per unit mass

This is the Euler equation in a rotating frame of reference.

The Coriolis force does no work



the Coriolis force acts normal to the rotation vector and normal to the velocity.

is directly proportional to the magnitude of $\mathbf{u}$ and Ω.

Note: the Coriolis force does no work because $\mathbf{u} \cdot (2\Omega \wedge \mathbf{u}) \equiv 0$

Perturbation pressure, buoyancy force

Define $p_T = p_0(z) + p$ **where** $\dfrac{dp_0}{dz} = -g\rho_0$

$p_0(z)$ **and** $\rho_0(z)$ **are the reference pressure and density fields**

p **is the perturbation pressure**

Important: $p_0(z)$ **and** $\rho_0(z)$ **are not uniquely defined**

$\mathbf{g} = (0, 0, -g)$

Euler's equation becomes

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p + \underbrace{\mathbf{g}\left[\frac{\rho - \rho_0}{\rho}\right]}_{\text{the buoyancy force}}$$

Important: **the perturbation pressure gradient** $-\frac{1}{\rho}\nabla p$ **and buoyancy force** $\mathbf{g}\left(\frac{\rho-\rho_0}{\rho}\right)$ **are not uniquely defined.**

But the total force $-\frac{1}{\rho}\nabla p+\mathbf{g}\left(\frac{\rho-\rho_0}{\rho}\right)$ **is uniquely defined.**

Indeed

$$-\frac{1}{\rho}\nabla p+\mathbf{g}\left(\frac{\rho-\rho_0}{\rho}\right)=-\frac{1}{\rho}\nabla p_T+\mathbf{g}$$

A mathematical demonstration

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p' + b\hat{\mathbf{k}} \qquad \nabla \cdot \mathbf{u} = 0$$

Momentum equation **Continuity equation**

The divergence of the momentum equation gives:

$$\nabla^2 p' = -\left[\nabla\cdot\left(\rho\mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\left(\rho b\hat{\mathbf{k}}\right)\right]$$

This is a diagnostic equation!

Newton's 2nd law

mass × acceleration = force

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho$$

buoyancy form

Put $p = p_o(z) + p'$ **where** $\dfrac{dp_o}{dz} = -g\rho_o$

$\rho = \rho_o(z) + \rho'$

Then $\dfrac{Dw}{Dt} = -\dfrac{1}{\rho}\dfrac{\partial p'}{\partial z} + b$ **where** $b = -g\left(\dfrac{\rho - \rho_o}{\rho}\right)$

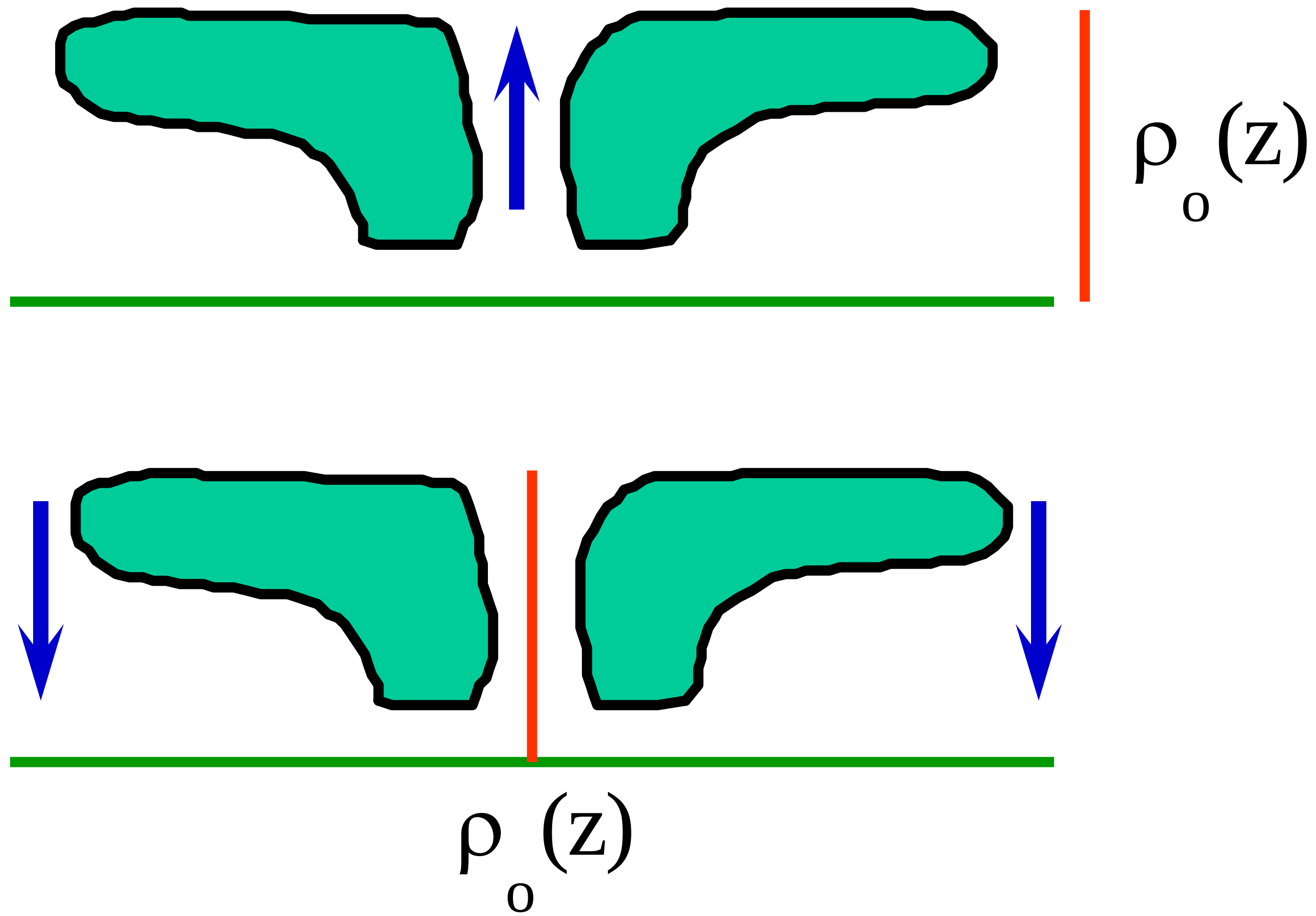
buoyancy force is NOT unique

$$b = -g\left(\frac{\rho - \rho_o}{\rho}\right)$$

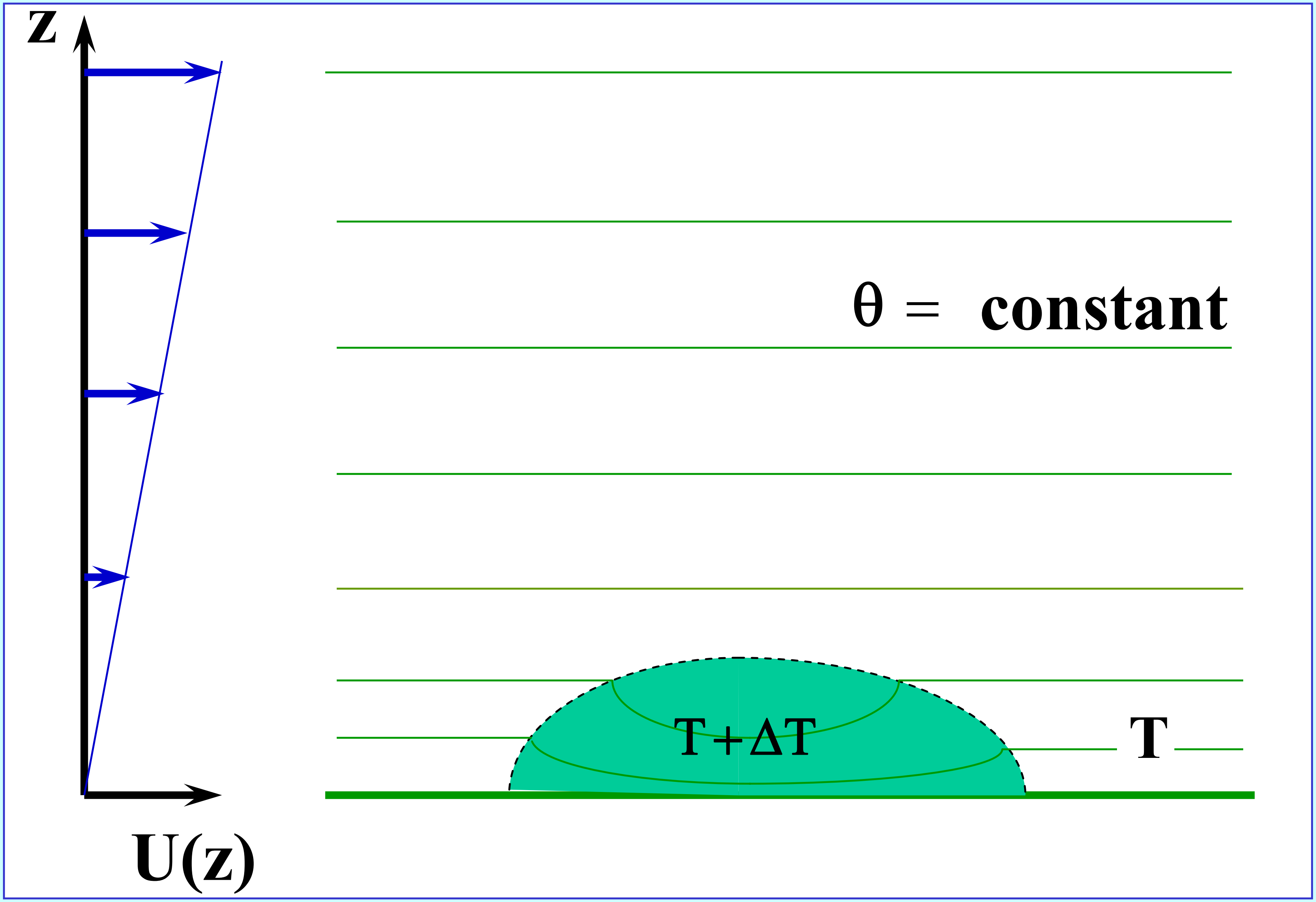
it depends on choice of reference density $\rho_o(z)$

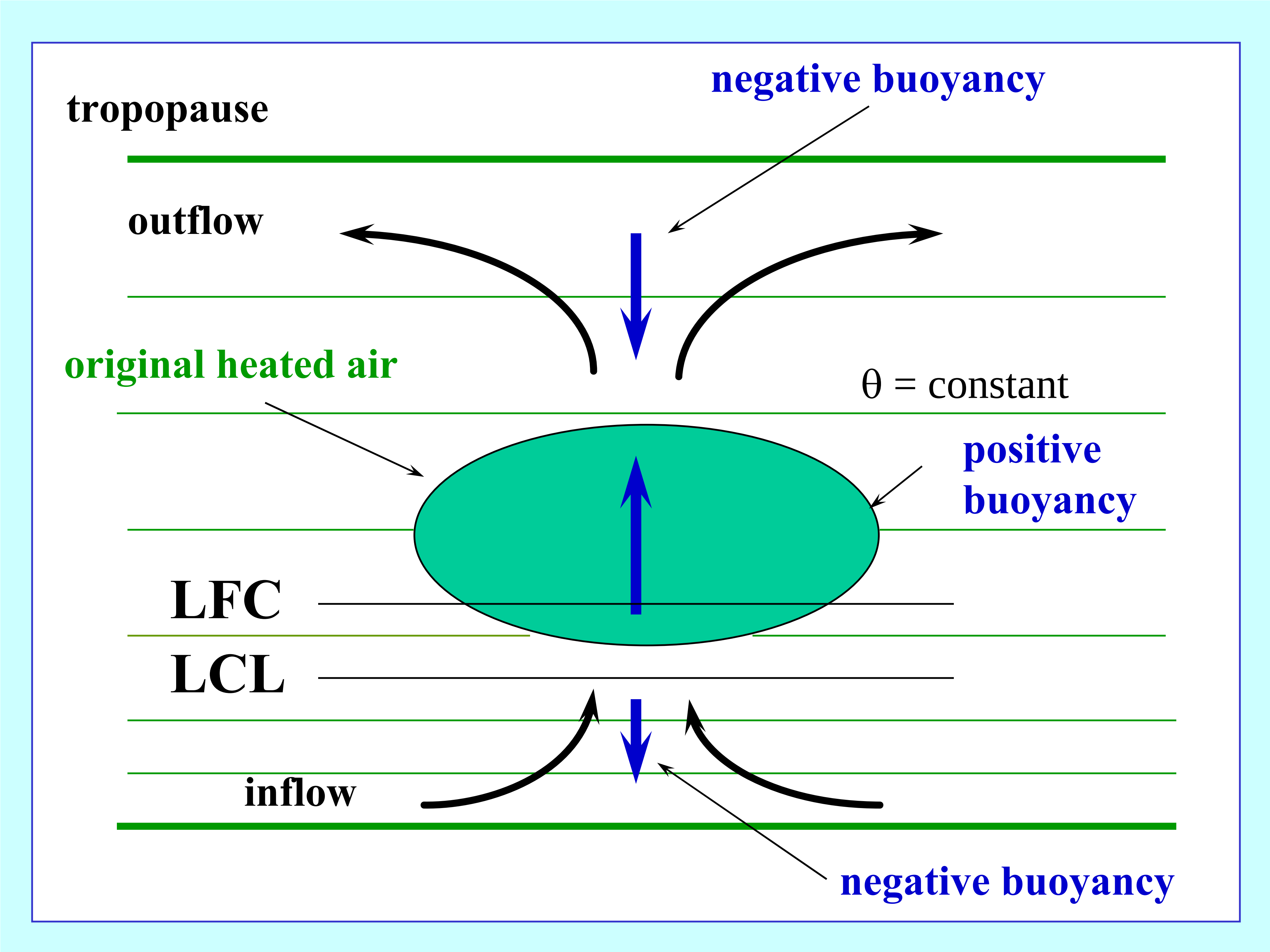
but $$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b$$ **is unique**

Buoyancy force in a hurricane



Initiation of a thunderstorm

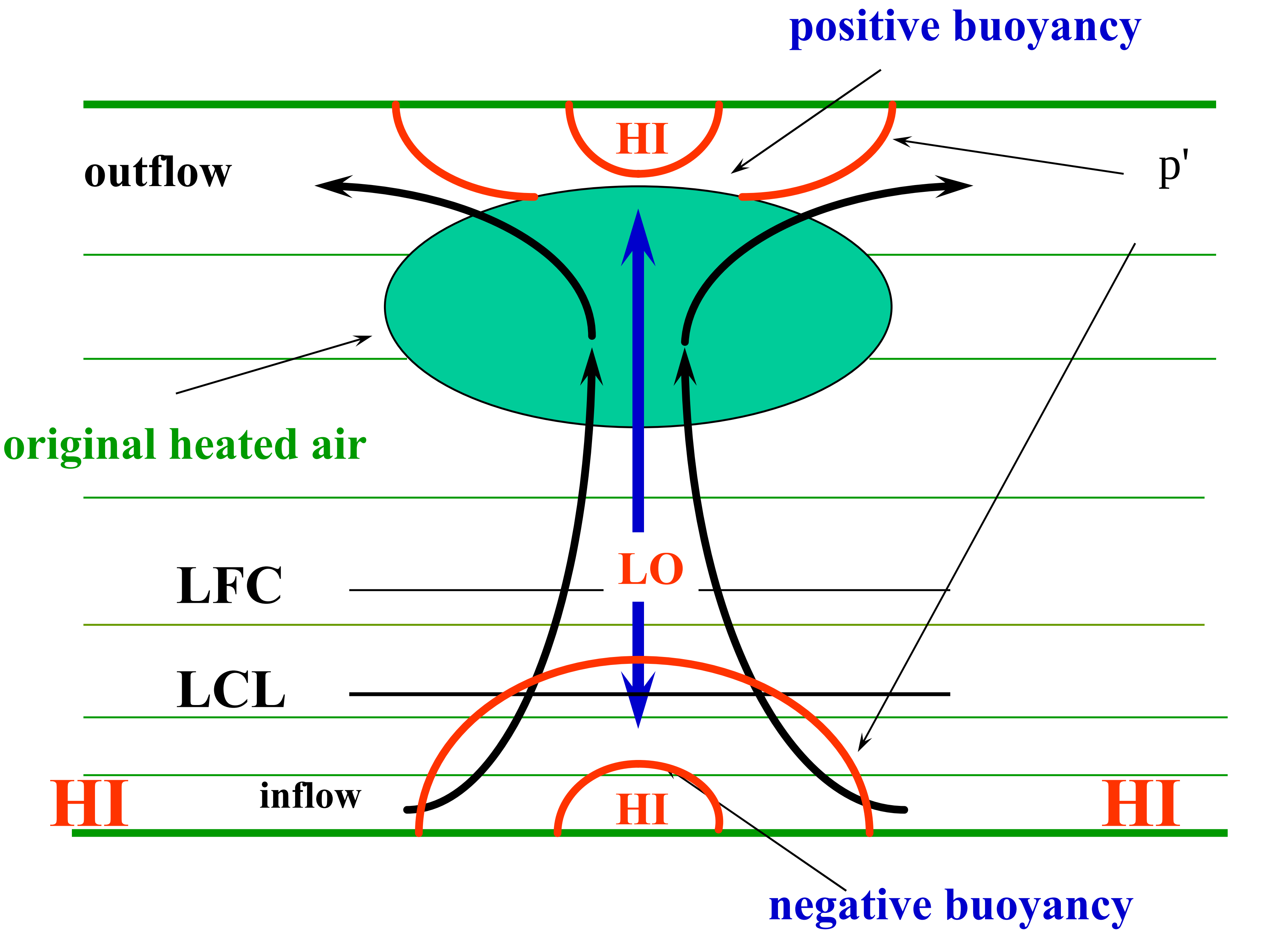




negative buoyancy
tropopause
outflow
original heated air
θ = constant
positive
buoyancy
LFC
LCL
inflow
negative buoyancy

Some questions

- How does the flow evolve after the original thermal has reached the upper troposphere?
- What drives the updraught at low levels?
  - Observation in severe thunderstorms: the updraught at cloud base is negatively buoyant!
  - Answer: - the perturbation pressure gradient


positive buoyancy
HI
outflow
p'
original heated air
LO
LFC
LCL
HI
inflow
HI
HI
negative buoyancy

Scale analysis

- **Assume a homogeneous fluid** ρ = constant.
- **Euler's equation becomes:**

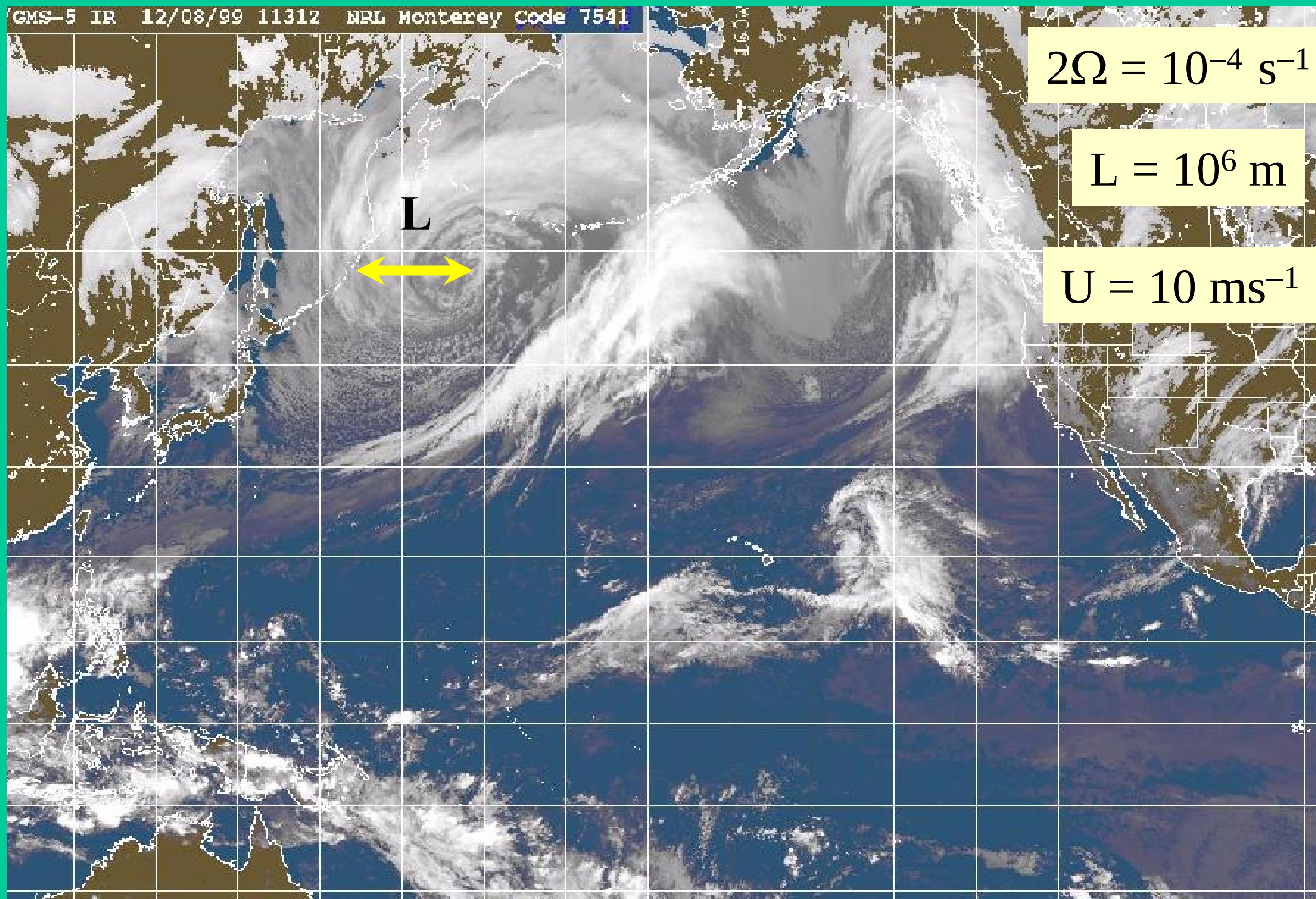
$$\frac{D\mathbf{u}}{Dt} + 2\Omega \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

scales: $\qquad \dfrac{U^2}{L} \qquad 2\Omega U \qquad \dfrac{\Delta P}{\rho L}$

Then $\qquad \dfrac{|D\mathbf{u}/Dt|}{|2\Omega \wedge \mathbf{u}|} \sim \dfrac{U^2/L}{2\Omega U} = \dfrac{U}{2\Omega L} = Ro$

Rossby number

Extratropical cyclone

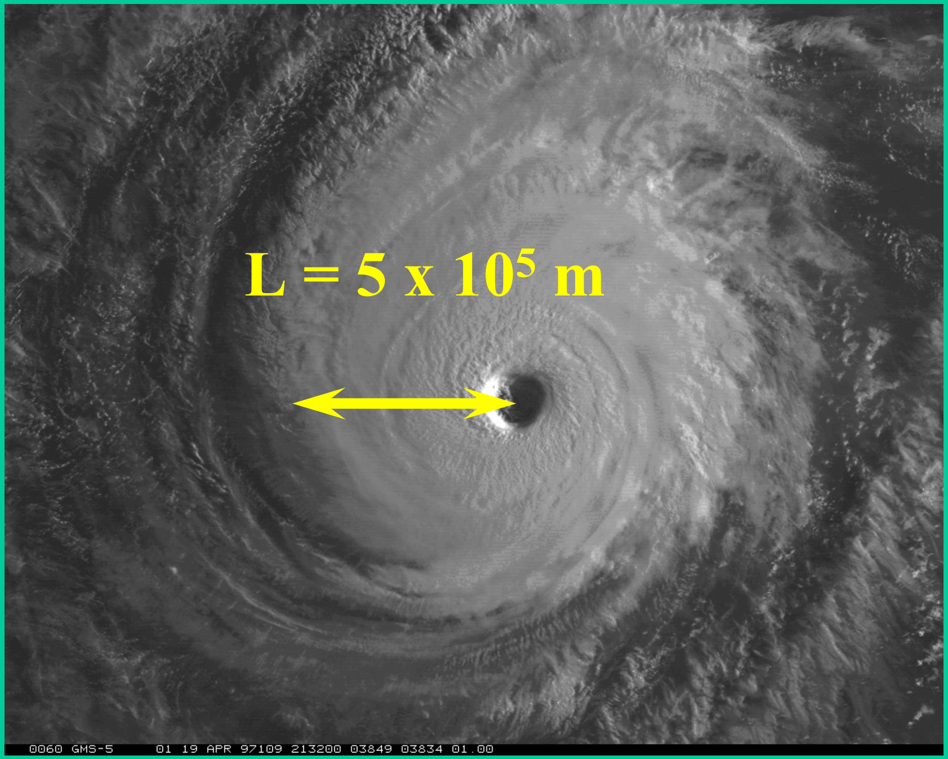


$$Ro = \frac{U}{2\Omega L} = \frac{10}{10^{-4} \times 10^{6}} = 10^{-1}$$

Tropical cyclone

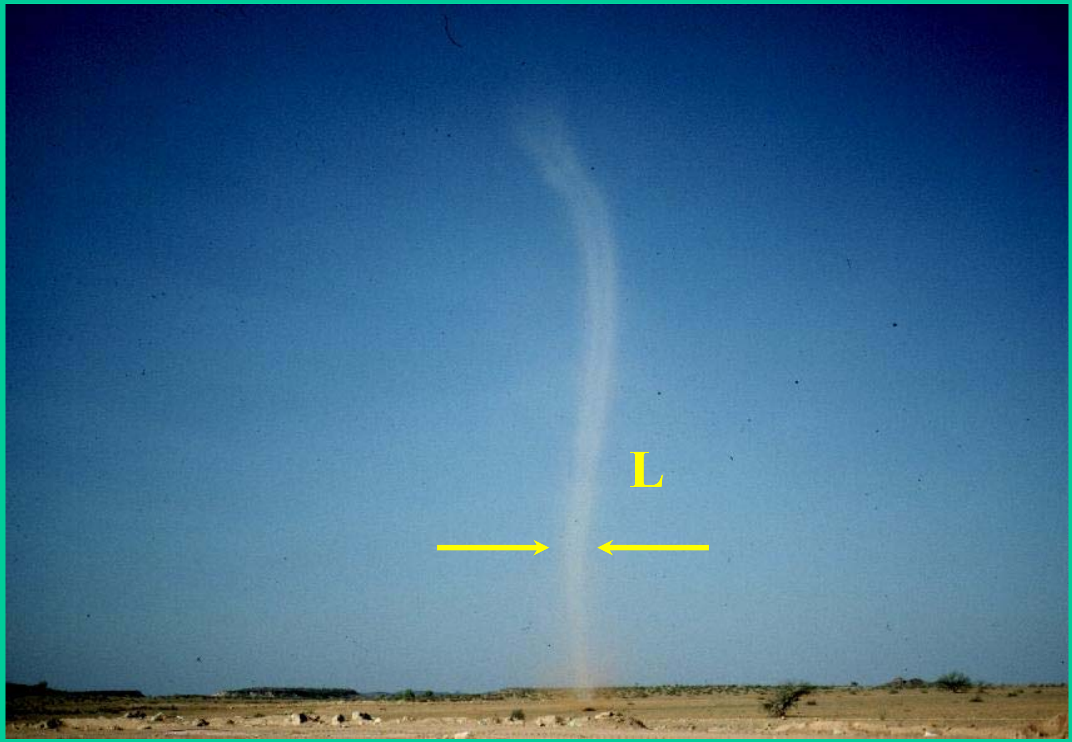
U = 50 m/sec

$2\Omega = 5 \text{ X } 10^{-5}$



$$Ro = \frac{U}{2\Omega L} = \frac{50}{5 \times 10^{-5} \times 5 \times 10^{5}} = 2$$

Dust devil



$L = 10 - 100\ \mathrm{m} \qquad U = 10\ \mathrm{ms^{-1}} \qquad 2\Omega = 10^{-4}\mathrm{s^{-1}}$

$$Ro = \frac{U}{2\Omega L} = \frac{10}{10^{-4} \times (10{,}100)} = 10^{-3} \rightarrow 10^{-4}$$

Waterspout

$L = 100$ m

$U = 50\ \mathrm{ms}^{-1}$

$2\Omega = 10^{-4}\mathrm{s}^{-1}$

$Ro = 5 \times 10^{3}$

Aeroplane wing

$L = 10$ m $\quad U = 200$ m s^{-1} $\quad 2\Omega = 10^{-4} s^{-1}$

$Ro = 2 \times 10^5$

The Rossby number

Flow system	L	U m s^{-1}	Ro
Ocean circulation	10^3 - 5×10^3 km	1 (or less)	10^{-2} - 10^{-3}
Extra-tropical cyclone	10^3 km	1-10	10^{-2} - 10^{-1}
Tropical cyclone	500 km	50 (or >)	1
Tornado	100 m	100	10^4
Dust devil	10-100 m	10	10^3 - 10^4
Cumulonimbus cloud	1 km	10	10^2
Aerodynamic	1-10 m	1-100	10^3 - 10^6
Bath tub vortex	1 m	10^{-1}	10^3

Summary

(i) Large scale meteorological and oceanic flows are strongly constrained by rotation ($Ro \ll 1$), except possibly in equatorial regions.

(ii) Tropical cyclones are always cyclonic and appear to derive their rotation from the background rotation of the earth. They never occur within 5 deg. of the equator where the normal component of the earth's rotation is small.

(iii) Most tornadoes are cyclonic, but **why**?

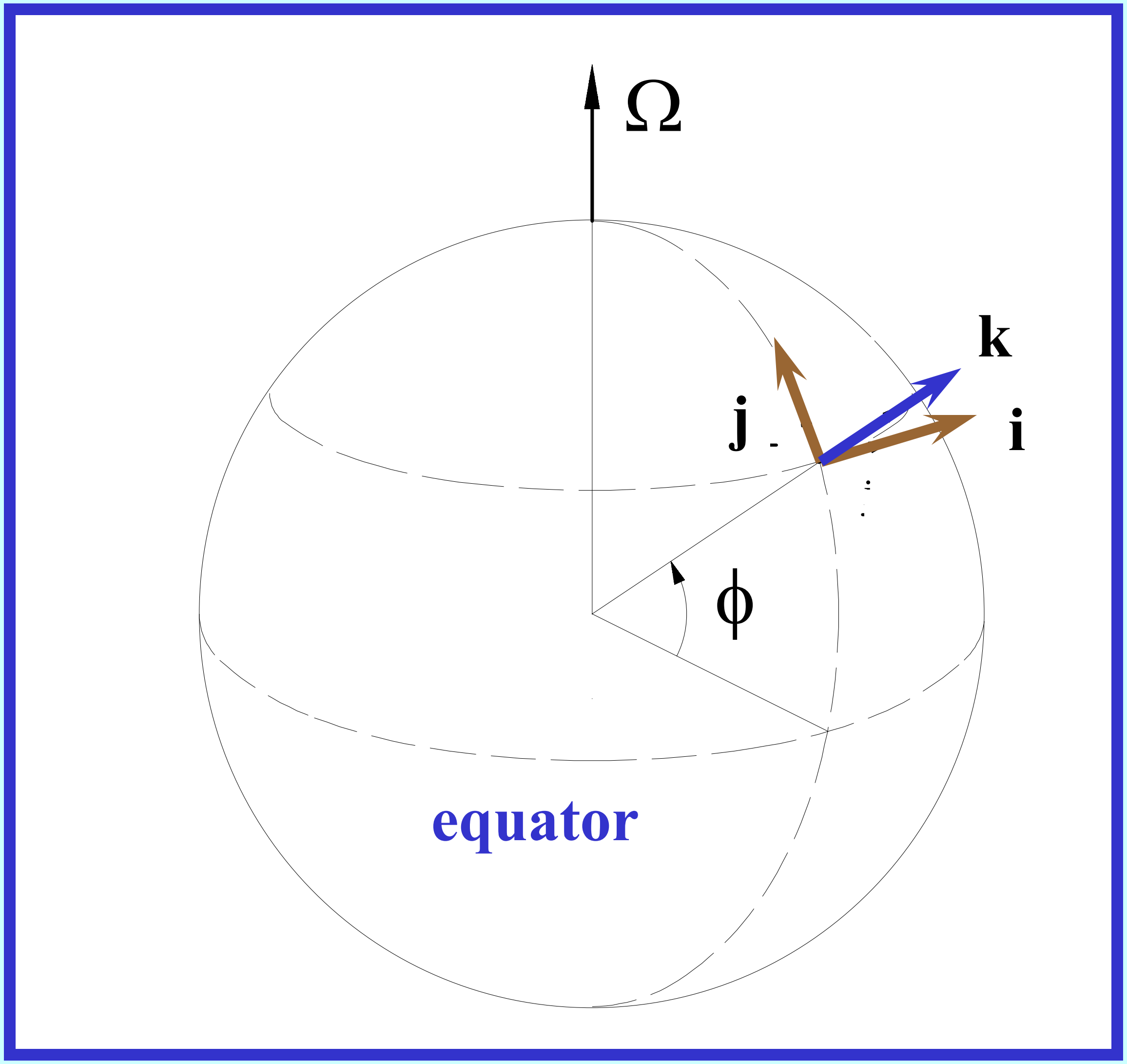
(iv) Dust devils do not have a preferred sense of rotation as expected.

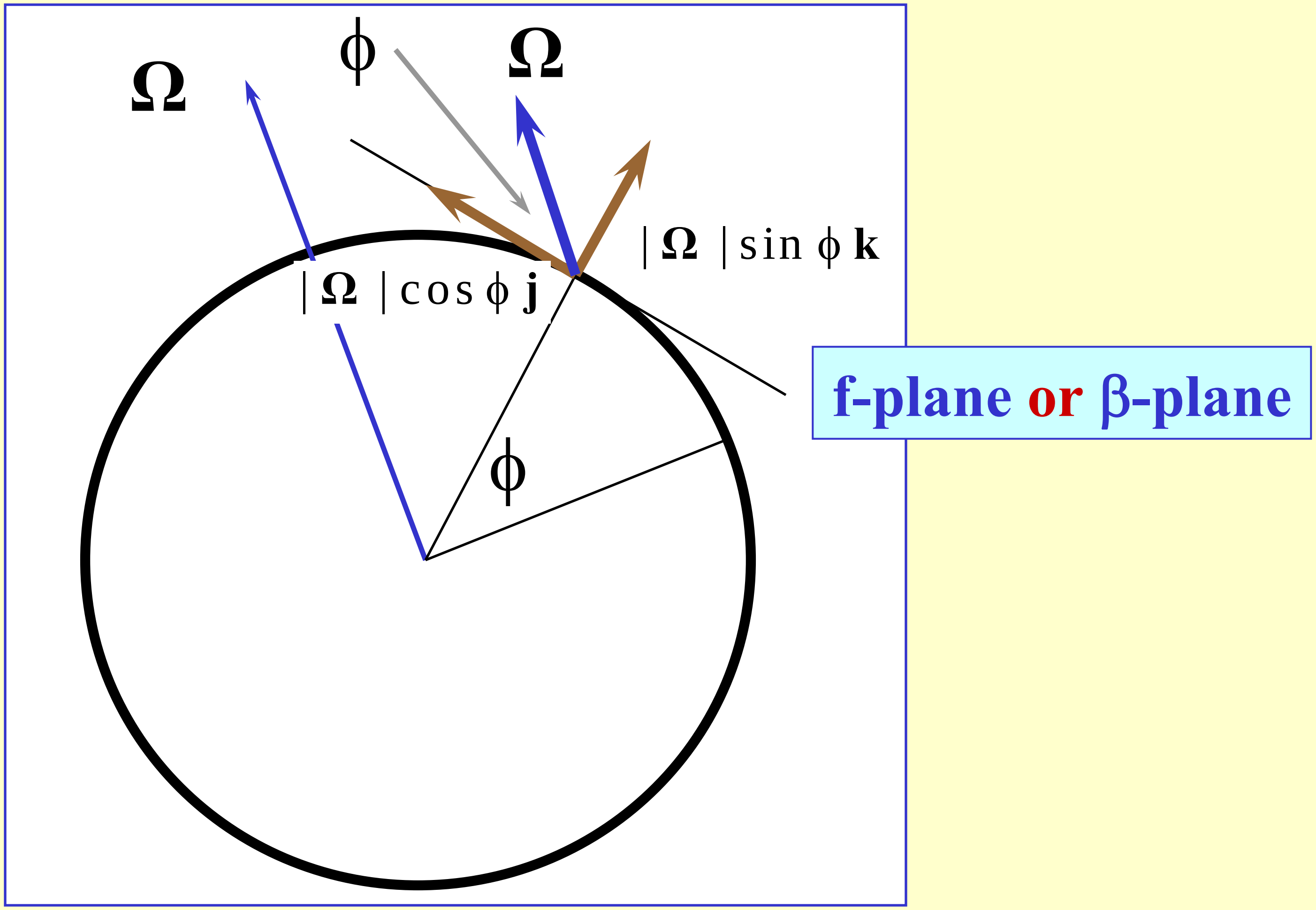
(v) In aerodynamic flows, and in the bath (!), the effect of the earth's rotation may be ignored.

Coordinate systems and the earth's sphericity

- **Many of the flows we shall consider have horizontal dimensions which are small compared with the earth's radius.**
- **In studying these, it is both legitimate and a great simplification to assume that the earth is locally flat and to use a rectangular coordinate system with z pointing vertically upwards.**
- **Holton (§2.3, pp31-35) shows the precise circumstances under which such an approximation is valid.**
- **In general, the use of spherical coordinates merely refines the theory, but does not lead to a deeper understanding of the phenomena.**

Take **rectangular coordinates** fixed relative to the earth and centred at a point on the surface at latitude.





$$\mathbf{\Omega} = |\mathbf{\Omega}|\cos\phi\,\mathbf{j} + |\mathbf{\Omega}|\sin\phi\,\mathbf{k}$$

$$2\mathbf{\Omega} \wedge \mathbf{u} = |\mathbf{\Omega}|\cos\phi\,\mathbf{j} \wedge \mathbf{u} + |\mathbf{\Omega}|\sin\phi\,\mathbf{k} \wedge \mathbf{u}$$

In component form

$$2\Omega \wedge \mathbf{u} = \begin{bmatrix} -2\Omega v \sin\phi + \cancel{2\Omega w \cos\phi} \\ 2\,\Omega\, u\, \sin\phi \\ \cancel{-2\,\Omega\, u \cos\phi} \end{bmatrix}$$

I will show that for middle latitude, synoptic-scale weather systems such as extra-tropical cyclones, the terms involving $\cos\phi$ **may be neglected**.

$$2\boldsymbol{\Omega} \wedge \mathbf{u} = 2\,|\boldsymbol{\Omega}|\cos\phi\, \mathbf{j} \wedge \mathbf{u} + \underbrace{2\,|\boldsymbol{\Omega}|\sin\phi\, \mathbf{k} \wedge \mathbf{u}}$$

The important term for large-scale motions

To a good approximation

$$2\Omega \wedge \mathbf{u} = \underline{2\,|\Omega|\sin\phi}\;\mathbf{k} \wedge \mathbf{u} = \underline{\mathbf{f}} \wedge \mathbf{u}$$

$$f = 2\,|\Omega|\sin\phi \qquad\qquad \mathbf{f} = f\,\mathbf{k}$$

Coriolis parameter

Scale analysis of the equations for middle latitude synoptic systems

- **Much of the significant weather in middle latitudes is associated with extra-tropical cyclones, or depressions.**
- **We shall base our scaling on such systems.**
- **Let** L, H, T, U, W, P **and** R **be scales for the horizontal size, vertical extent, time,** $|\mathbf{u_h}|$, w, **perturbation pressure, and density in an extra-tropical cyclone, say at** 45° **latitude, where** f **(**= $2\Omega \sin\phi$ **) and** $2\Omega\cos\phi$ **are both of order** 10^{-4}**.**

and

$$U = 10\,\mathrm{ms}^{-1};\; W = 10^{-2}\,\mathrm{ms}^{-1};$$

$$L = 10^{6}\,\mathrm{m}\ (10^{3}\,\mathrm{km});\; H = 10^{4}\,\mathrm{m}\ (10\,\mathrm{km});$$

$$T = L / U \sim 10^{5}\,\mathrm{s}\ (\sim 1\,\mathrm{day});\; \delta P = 10^{3}\,\mathrm{Pa}\ (10\,\mathrm{mb})$$

$$R = 1\,\mathrm{kg\ m}^{-3}.$$

horizontal momentum equations

$$\frac{Du}{Dt} - 2\Omega v \sin\phi + 2\Omega w \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + 2\Omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

scales	U^2/L	$2\,\Omega U \sin\phi$	$2\Omega W \cos\phi$	$\delta P/\rho L$
orders	10^{-4}	10^{-3}	10^{-6}	10^{-3}

$$\frac{D\mathbf{u}_h}{Dt} + f\mathbf{k} \wedge \mathbf{u}_h = -\frac{1}{\rho}\nabla_h p$$

vertical momentum equations

$$\frac{Dw}{Dt} - 2\Omega\ u\ \cos\phi = -\frac{1}{\rho}\frac{\partial p_T}{\partial z} - g$$

scales	UW/L	$2\Omega U\cos\phi$	$\delta P_T/\rho H$	g
orders	10^{-7}	10^{-3}	10	10

negligible

the atmosphere is strongly hydrostatic on the synoptic scale.

But are the disturbances themselves hydrostatic?

Question: when we subtract the reference pressure p_0 from p_T, is it still legitimate to neglect Dw/Dt etc.?

vertical momentum equations

$$\frac{Dw}{Dt} - 2\Omega\, u \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$$

scales	UW/L	$2\Omega U \cos\phi$	$\delta P/\rho H*$	$g\delta T/T_0$
orders	10^{-7}	10^{-3}	$\geq 10^{-1}$	10^{-1}

still negligible

assume that H* ≤ H

assume that δT ≈ 3°K/300 km

H* **= height scale for a perturbation pressure difference** δp **of** 10 mb

In summary

$$\frac{Dw}{Dt} - 2\Omega\, u \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$$

orders 10^{-7} $\quad 10^{-3}$ $\quad \geq 10^{-1}$ $\quad 10^{-1}$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$$

In synoptic-scale disturbances, the perturbations are in close hydrostatic balance

Remember: it is small departures from this equation which drive the weak vertical motion in systems of this scale.

The hydrostatic approximation

The hydrostatic approximation permits enormous simplifications in dynamical studies of large-scale motions in the atmosphere and oceans.

End of
Chapter 3