The Equations of Motion in a Rotating Coordinate System

Chapter 3
Since the earth is rotating about its axis and since it is convenient to adopt a frame of reference fixed in the earth, we need to study the equations of motion in a rotating coordinate system.

Before proceeding to the formal derivation, we consider briefly two concepts which arise therein:

**Effective gravity and Coriolis force**
Effective Gravity

Effective gravity $g$ is everywhere normal to the earth’s surface.

- **Effective gravity on a spherical earth**
- **Effective gravity on an earth with a slight equatorial bulge**
Effective Gravity

If the earth were a perfect sphere and not rotating, the only gravitational component $g^*$ would be radial.

Because the earth has a bulge and is rotating, the effective gravitational force $g$ is the vector sum of the normal gravity to the mass distribution $g^*$, together with a centrifugal force $\Omega^2 R$, and this has no tangential component at the earth’s surface.

$$g = g^* + \Omega^2 R$$
A line at rest in an inertial system

A line that rotates with the roundabout

Apparent trajectory of the ball in a rotating coordinate system
Mathematical derivation of the Coriolis force

Representation of an arbitrary vector $\mathbf{A}(t)$

- **Inertial reference system**
  - $\mathbf{A}(t) = A_1(t)i + A_2(t)j + A_3(t)k$

- **Rotating reference system**
  - $\mathbf{A}(t) = A_1'(t)i' + A_2'(t)j' + A_3'(t)k'$
The time derivative of an arbitrary vector $A(t)$

The derivative of $A(t)$ with respect to time

$$\frac{d_a A}{dt} = i \frac{dA_1}{dt} + j \frac{dA_2}{dt} + k \frac{dA_3}{dt}$$

the subscript “a” denotes the derivative in an inertial reference frame

In the rotating frame of reference

$$\frac{d_a A}{dt} = i' \frac{dA_1'}{dt} + A'_1 \frac{di'}{dt} + ...$$

$$= i' \frac{dA_1'}{dt} + A'_1 (\Omega \land i') + ...$$

$$= \left[ \frac{d}{dt} + \Omega \land \right] \left( A'_1 i' + ... \right)$$
Let $a$ be any vector rotating with angular velocity $\Omega$

Unit vector perpendicular to $\Omega$ and $a$

\[
\frac{da}{dt} = |a| |\Omega| \sin \theta \hat{b} = \Omega \wedge a
\]
Want to calculate $u_a = \frac{d_a r}{dt}$

the absolute velocity of an air parcel.

The relative velocity in a rotating frame of reference is

$$u = \frac{dr}{dt} = \frac{dr_1'}{dt} i' + \frac{dr_2'}{dt} j' + \frac{dr_3'}{dt} k'$$

and $u_a = u + \Omega \wedge r$
This air parcel starts relative to the earth with a poleward velocity $V$.

It begins relative to space with an additional eastwards velocity component $\Omega R_e$. 

Example

$$u_a = u + \Omega \wedge r$$
We need to calculate the absolute acceleration if we wish to apply Newton's second law

\[ \frac{d_a u_a}{dt} \]

Measurements on the earth give only the relative velocity \( u \) and therefore the relative acceleration

\[ \frac{du}{dt} \]

With \( A = u_a \)

\[ \frac{d_a u_a}{dt} = \frac{du}{dt} + \Omega \wedge u_a \]

\[ \frac{d_a u_a}{dt} = \frac{du}{dt} + 2\Omega \wedge u + \Omega \wedge (\Omega \wedge r) \]
\[
\frac{d_a u_a}{dt} = \frac{du}{dt} + 2\Omega \wedge u + \Omega \wedge (\Omega \wedge r)
\]
Position vector $\mathbf{r}$ is split up

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\mathbf{\Omega}})\hat{\mathbf{\Omega}} + \mathbf{R}$$

$$\mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}) = \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{R})$$

$$= -|\mathbf{\Omega}|^2 \mathbf{R}$$

The Centripetal acceleration is directed inwards towards the axis of rotation and has magnitude $|\mathbf{\Omega}|^2 \mathbf{R}$. 
Newton’s second law in a rotating frame of reference

In an inertial frame:

\[ \rho \frac{d}{dt} \mathbf{u}_a = \mathbf{F} \]

In a rotating frame:

\[ \rho \left[ \frac{d\mathbf{u}}{dt} + 2\Omega \times \mathbf{u} + \Omega \times (\Omega \times \mathbf{r}) \right] = \mathbf{F} \]
Alternative form

\[ \rho \frac{du}{dt} = F - 2\rho \Omega \wedge u - \rho \Omega \wedge (\Omega \wedge r) \]

Coriolis force

Centrifugal force
Let the total force \( F = g^* + F' \) be split up

\[
\rho \frac{du}{dt} = F' + g^* - 2 \rho \Omega \wedge u - \rho \Omega \wedge (\Omega \wedge r)
\]

With \( g = g^* + \Omega^2 R \)

\[
\rho \frac{du}{dt} = F' + \rho g - 2 \rho \Omega \wedge u
\]

The centrifugal force associated with the earth’s rotation no longer appears explicitly in the equation; it is contained in the effective gravity.
When frictional forces can be neglected, $F'$ is the pressure gradient force

$$ F' = -\nabla p_T \quad \text{per unit volume} $$

$$ \rho \frac{du}{dt} = F' + \rho \mathbf{g} - 2\rho \Omega \wedge \mathbf{u} $$

$$ \frac{du}{dt} = -\frac{1}{\rho} \nabla p_T + \mathbf{g} - 2\Omega \wedge \mathbf{u} \quad \text{per unit mass} $$

This is the Euler equation in a rotating frame of reference.
The Coriolis force does no work

The Coriolis force acts normal to the rotation vector and normal to the velocity.

is directly proportional to the magnitude of u and \( \Omega \).

Note: the Coriolis force does no work because \( u \cdot (2\Omega \wedge u) \equiv 0 \)
Define \( p_T = p_0(z) + p \) where \( \frac{dp_0}{dz} = -g \rho_0 \).

\( p_0(z) \) and \( \rho_0(z) \) are the reference pressure and density fields.

\( p \) is the perturbation pressure.

Important: \( p_0(z) \) and \( \rho_0(z) \) are not uniquely defined.

Euler’s equation becomes:

\[
\frac{Du}{Dt} + 2\Omega \wedge u = -\frac{1}{\rho} \nabla p + g \left[ \frac{\rho - \rho_0}{\rho} \right]
\]

the buoyancy force
Important: the perturbation pressure gradient \(-\frac{1}{\rho} \nabla p\) and buoyancy force \(g \left( \frac{\rho - \rho_0}{\rho} \right)\) are not uniquely defined.

But the total force \(-\frac{1}{\rho} \nabla p + g \left( \frac{\rho - \rho_0}{\rho} \right)\) is uniquely defined.

Indeed \(-\frac{1}{\rho} \nabla p + g \left( \frac{\rho - \rho_0}{\rho} \right) = -\frac{1}{\rho} \nabla p_T + g\)
A mathematical demonstration

\[
\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p' + b\hat{k} \quad \nabla \cdot u = 0
\]

Momentum equation \hspace{1cm} Continuity equation

The divergence of the momentum equation gives:

\[
\nabla^2 p' = -\left[ \nabla \cdot (\rho u \cdot \nabla u) - \nabla \cdot (\rho b\hat{k}) \right]
\]

This is a diagnostic equation!
Newton’s 2nd law

mass × acceleration = force

\[ \rho \frac{Dw}{Dt} = - \frac{\partial p}{\partial z} - g \rho \]
buoyancy form

Put

\[ p = p_o(z) + p' \]
\[ \rho = \rho_o(z) + \rho' \]

where

\[ \frac{dp}{dz} = -g \rho_o \]

Then

\[ \frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p'}{\partial z} + b \]

where

\[ b = -g \left( \frac{\rho - \rho_o}{\rho} \right) \]
buoyancy force is NOT unique

\[ b = -g \left( \frac{\rho - \rho_o}{\rho} \right) \]

it depends on choice of reference density \( \rho_o(z) \)

but

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial z} - g = -\frac{1}{\rho} \frac{\partial \rho'}{\partial z} + b \]

is unique
Buoyancy force in a hurricane

\[ \rho_0(z) \]
Initiation of a thunderstorm

\[ \theta = \text{constant} \]
\[ \theta = \text{constant} \]

- **tropopause**
- **negative buoyancy**
- **outflow**
- **original heated air**
- **\( \theta = \text{constant} \)**
- **positive buoyancy**
- **LFC**
- **LCL**
- **inflow**
- **negative buoyancy**
Some questions

- How does the flow evolve after the original thermal has reached the upper troposphere?

- What drives the updraught at low levels?
  - Observation in severe thunderstorms: the updraught at cloud base is negatively buoyant!
  - **Answer:** - the perturbation pressure gradient
outflow

original heated air

LFC

LCL

HI

inflow

positive buoyancy

p'

negative buoyancy

HI
Assume a homogeneous fluid $\rho = \text{constant}$.

Euler’s equation becomes:

\[
\frac{Du}{Dt} + 2\Omega \wedge u = -\frac{1}{\rho} \nabla p
\]

Scales:

\[
\frac{U^2}{L}, \quad 2\Omega U, \quad \frac{\Delta P}{\rho L}
\]

Then

\[
\left| \frac{Du /Dt}{2\Omega \wedge u} \right| \sim \frac{U^2 / L}{2\Omega U} = \frac{U}{2\Omega L} = \frac{U}{2\Omega L} = \text{Ro}
\]

Rossby number
Extratropical cyclone

\[ 2\Omega = 10^{-4} \text{ s}^{-1} \]

\[ L = 10^6 \text{ m} \]

\[ U = 10 \text{ ms}^{-1} \]

\[ Ro = \frac{U}{2\Omega L} = \frac{10}{10^{-4} \times 10^6} = 10^{-1} \]
U = 50 m/sec

2Ω = 5 \times 10^{-5}

L = 5 \times 10^5 m

Ro = \frac{U}{2\Omega L} = \frac{50}{5 \times 10^{-5} \times 5 \times 10^5} = 2
Dust devil

$L = 10 - 100 \text{ m}$ \quad $U = 10 \text{ ms}^{-1}$ \quad $2\Omega = 10^{-4}\text{s}^{-1}$

$$\text{Ro} = \frac{U}{2\Omega L} = \frac{10}{10^{-4} \times (10,100)} = 10^{-3} \rightarrow 10^{-4}$$
Waterspout

$L = 100 \text{ m}$

$U = 50 \text{ ms}^{-1}$

$2\Omega = 10^{-4}\text{s}^{-1}$

$Ro = 5 \times 10^{3}$
Aeroplane wing

$L = 10 \text{ m}$  $U = 200 \text{ m s}^{-1}$  $2\Omega = 10^{-4}\text{s}^{-1}$  $Ro = 2 \times 10^5$
The Rossby number

<table>
<thead>
<tr>
<th>Flow system</th>
<th>L</th>
<th>U m s$^{-1}$</th>
<th>Ro</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ocean circulation</td>
<td>$10^3 - 5 \times 10^3$ km</td>
<td>1 (or less)</td>
<td>$10^{-2} - 10^{-3}$</td>
</tr>
<tr>
<td>Extra-tropical cyclone</td>
<td>$10^3$ km</td>
<td>1-10</td>
<td>$10^{-2} - 10^{-1}$</td>
</tr>
<tr>
<td>Tropical cyclone</td>
<td>500 km</td>
<td>50 (or &gt;)</td>
<td>1</td>
</tr>
<tr>
<td>Tornado</td>
<td>100 m</td>
<td>100</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Dust devil</td>
<td>10-100 m</td>
<td>10</td>
<td>$10^3 - 10^4$</td>
</tr>
<tr>
<td>Cumulonimbus cloud</td>
<td>1 km</td>
<td>10</td>
<td>$10^2$</td>
</tr>
<tr>
<td>Aerodynamic</td>
<td>1-10 m</td>
<td>1-100</td>
<td>$10^3 - 10^6$</td>
</tr>
<tr>
<td>Bath tub vortex</td>
<td>1 m</td>
<td>$10^{-1}$</td>
<td>$10^3$</td>
</tr>
</tbody>
</table>
(i) Large scale meteorological and oceanic flows are strongly constrained by rotation ($Ro << 1$), except possibly in equatorial regions.

(ii) Tropical cyclones are always cyclonic and appear to derive their rotation from the background rotation of the earth. They never occur within 5 deg. of the equator where the normal component of the earth's rotation is small.

(iii) Most tornadoes are cyclonic, but why?

(iv) Dust devils do not have a preferred sense of rotation as expected.

(v) In aerodynamic flows, and in the bath (!), the effect of the earth's rotation may be ignored.
Many of the flows we shall consider have horizontal dimensions which are small compared with the earth's radius.

In studying these, it is both legitimate and a great simplification to assume that the earth is locally flat and to use a rectangular coordinate system with $z$ pointing vertically upwards.

Holton (§2.3, pp31-35) shows the precise circumstances under which such an approximation is valid.

In general, the use of spherical coordinates merely refines the theory, but does not lead to a deeper understanding of the phenomena.
Take rectangular coordinates fixed relative to the earth and centred at a point on the surface at latitude.
\[ \Omega = |\Omega| \cos \phi \, j + |\Omega| \sin \phi \, k \]

\[ 2 \Omega \wedge u = |\Omega| \cos \phi \, j \wedge u + |\Omega| \sin \phi \, k \wedge u \]
In component form

\[ 2\Omega \wedge u = \begin{bmatrix} -2\Omega v \sin \phi + 2\Omega w \cos \phi \\ 2\Omega u \sin \phi \\ -2\Omega u \cos \phi \end{bmatrix} \]

I will show that for middle latitude, synoptic-scale weather systems such as extra-tropical cyclones, the terms involving \( \cos \phi \) may be neglected.

\[ 2\Omega \wedge u = 2 |\Omega| \cos \phi j \wedge u + 2 |\Omega| \sin \phi k \wedge u \]

The important term for large-scale motions is
To a good approximation

\[ 2 \Omega \wedge u = 2 |\Omega| \sin \phi \mathbf{k} \wedge u = \mathbf{f} \wedge u \]

\[ f = 2 |\Omega| \sin \phi \quad f = \mathbf{f} \mathbf{k} \]

Coriolis parameter
Much of the significant weather in middle latitudes is associated with extra-tropical cyclones, or depressions.

We shall base our scaling on such systems.

Let $L$, $H$, $T$, $U$, $W$, $P$ and $R$ be scales for the horizontal size, vertical extent, time, $|u_h|$, $w$, perturbation pressure, and density in an extra-tropical cyclone, say at 45° latitude, where $f = 2\Omega \sin \phi$ and $2\Omega \cos \phi$ are both of order $10^{-4}$.

Scale analysis of the equations for middle latitude synoptic systems

<table>
<thead>
<tr>
<th>Scale</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$10 \text{ ms}^{-1}$</td>
</tr>
<tr>
<td>$W$</td>
<td>$10^{-2} \text{ ms}^{-1}$</td>
</tr>
<tr>
<td>$L$</td>
<td>$10^6 \text{ m (10^3 km)}$</td>
</tr>
<tr>
<td>$H$</td>
<td>$10^4 \text{ m (10 km)}$</td>
</tr>
<tr>
<td>$T$</td>
<td>$L / U \sim 10^5 \text{ s (1 day)}$</td>
</tr>
<tr>
<td>$\delta P$</td>
<td>$10^3 \text{ Pa (10 mb)}$</td>
</tr>
<tr>
<td>$R$</td>
<td>$1 \text{ kg m}^{-3}$</td>
</tr>
</tbody>
</table>
horizontal momentum equations

\[
\frac{Du}{Dt} - 2\Omega v \sin \phi + 2\Omega w \cos \phi = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{Dv}{Dt} + 2\Omega u \sin \phi = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

scales \hspace{1cm} U^2/L, 2\Omega U \sin \phi, 2\Omega W \cos \phi, \delta P/\rho L

orders \hspace{1cm} 10^{-4}, 10^{-3}, 10^{-6}, 10^{-3}

\[
\frac{Du_h}{Dt} + f k \wedge u_h = -\frac{1}{\rho} \nabla_h p
\]
### vertical momentum equations

\[
\frac{Dw}{Dt} - 2\Omega u \cos \phi = -\frac{1}{\rho} \frac{\partial p_T}{\partial z} - g
\]

**scales**  
UW/L  \quad 2\Omega U \cos \phi  \quad \delta p_T/\rho H  \quad g

**orders**  
10^{-7}  \quad 10^{-3}  \quad 10  \quad 10

**negligible**

The atmosphere is **strongly hydrostatic** on the synoptic scale.

But are the disturbances themselves hydrostatic?
Question: when we subtract the reference pressure $p_0$ from $p_T$, is it still legitimate to neglect $Dw/Dt$ etc.?

**vertical momentum equations**

$$\frac{Dw}{Dt} - 2\Omega u \cos \phi = -\frac{1}{\rho} \frac{\partial p}{\partial z} + b$$

**scales**

- $UW/L$
- $2\Omega U \cos \phi$
- $\delta P/\rho H^*$
- $g\delta T/T_0$

**orders**

- $10^{-7}$
- $10^{-3}$
- $\geq 10^{-1}$
- $10^{-1}$

still negligible

assume that $H^* \leq H$

assume that $\delta T \approx 3^\circ K/300$ km

$H^*$ = height scale for a perturbation pressure difference $\delta p$ of 10 mb
In summary

\[
\frac{Dw}{Dt} - 2\Omega u \cos \phi = \frac{1}{\rho} \frac{\partial p}{\partial z} + b
\]

orders

\[
10^{-7} \quad 10^{-3} \quad \geq 10^{-1} \quad 10^{-1}
\]

\[
0 = \frac{1}{\rho} \frac{\partial p}{\partial z} + b
\]

In synoptic-scale disturbances, the perturbations are in close hydrostatic balance.

Remember: it is small departures from this equation which drive the weak vertical motion in systems of this scale.
The hydrostatic approximation permits enormous simplifications in dynamical studies of large-scale motions in the atmosphere and oceans.
End of Chapter 3