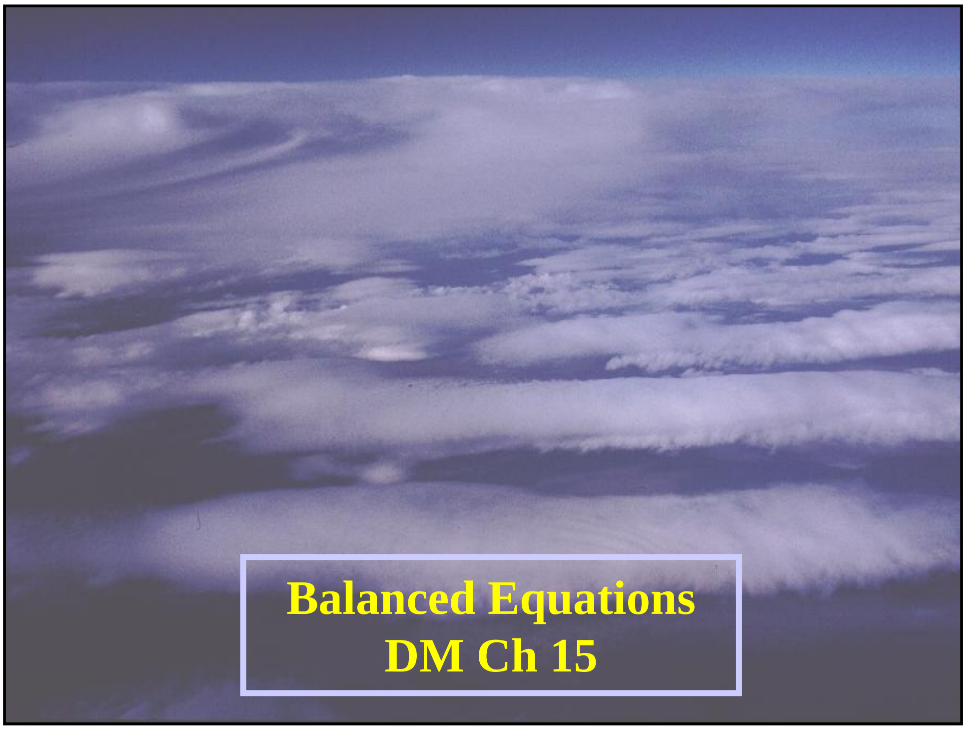
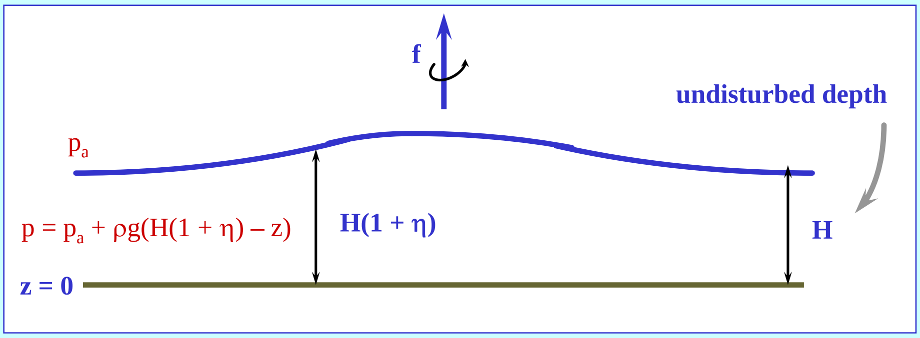


Reminder: Inertia-gravity waves



Shallow water model configuration

- In **pure inertial wave motion**, horizontal pressure gradients are zero.
- Consider now **waves** in a layer of rotating fluid with a **free surface** where horizontal pressure gradients are associated with free surface displacements.

Consider hydrostatic motions - then

$$p(x,y,z,t) = p_a + \rho g[H\{1+\eta(x,y,t)\} - z]$$

$$\Rightarrow \quad \frac{1}{\rho}\nabla_h p = gH\nabla_h \eta$$

independent of z

$\Rightarrow$ **The fluid acceleration is independent of** z.

$\Rightarrow$ **If the velocities are initially independent of** z, **then they will remain so.**

Linearized equations - no basic flow

$$\frac{\partial u}{\partial t} - fv = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -gH\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \eta}{\partial t}$$

Consider wave motions which are independent of y.

A solution exists of the form

$$u = \hat{u}\cos(kx - \omega t),$$

$$v = \hat{v}\sin(kx - \omega t),$$

$$\eta = \hat{\eta}\cos(kx - \omega t),$$

if $\Rightarrow$

$\hat{u}$, $\hat{v}$ and $\hat{\eta}$ constants

$$\omega\hat{u} - f\hat{v} - gHk\hat{\eta} = 0,$$
$$f\hat{u} - \omega\hat{v} = 0,$$
$$-k\hat{u} + \omega\hat{\eta} = 0,$$

These algebraic equations for $\hat{u}$, $\hat{v}$ and $\hat{\eta}$ **have solutions only if**

$$\begin{vmatrix} \omega & -f & -gHk \\ f & -\omega & 0 \\ -k & 0 & \omega \end{vmatrix} = -\omega^3 + \omega(f^2 + gHk^2) = 0$$

$$\omega = 0 \text{ or } \omega^2 = f^2 + gHk^2$$

The solution with $\omega = 0$ **corresponds with the steady solution (**$\partial/\partial t = 0$**) of the equations and represents a steady current in strict geostrophic balance in which**

$$v = \frac{gH}{f}\frac{\partial\eta}{\partial x}$$

The other two solutions correspond with so-called inertia-gravity waves, with the dispersion relation

$$\omega^2 = f^2 + gHk^2$$

The phase speed of these is

$$c_p = \omega / k = \pm\sqrt{[gH + f^2 / k^2]}$$

The waves are dispersive

Inclusion of the beta effect

- **Replace the v-momentum equation by the vorticity equation =>**

$$\frac{\partial u}{\partial t} - f_0 v = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \zeta}{\partial t} + \beta v = f_0 \frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Assume that f can be approximated by its value f_0 at a particular latitude, except when differentiated with respect to y in the vorticity equation. This is justified provided that meridional particle displacements are small.

- **Again assume that $\partial/\partial y \equiv 0$ and consider travelling-wave solutions of the form:**

$$u = \hat{u}\cos(kx - \omega t),$$

$$v = \hat{v}\sin(kx - \omega t),$$

$$\eta = \hat{\eta}\cos(kx - \omega t),$$

$$\omega\hat{u} - f_0\hat{v} - gHk\hat{\eta} = 0\ ,$$

$$(k\omega + \beta)\hat{v} - f_0\omega\hat{\eta} = 0\ ,$$

$$-k\hat{u} + \omega\hat{\eta} = 0\ ,$$

These are consistent only if the determinant is zero

Now

expanding by the second row

$$\begin{vmatrix} \omega & -f_0 & -gHk \\ 0 & +(k\omega+\beta) & -\omega f_0 \\ -k & 0 & \omega \end{vmatrix} = (\omega^2 - gHk^2)(\omega k + \beta) - f_0^2\,\omega k = 0$$

A cubic equation for ω with three real roots.

When $\omega >> \beta/k$, the two non-zero roots are given approximately by the formula

$$\omega^2 = gHk^2 + f_0^2$$

This is precisely the dispersion relation for inertia-gravity waves.

When $\omega^2 << gHk^2$, there is one root given approximately by

$$\omega = -\beta k/(k^2 + f_0^2/gH)$$

Filtering

Equations:

$$\frac{\partial u}{\partial t} - f_0 v = -gH\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \zeta}{\partial t} + \beta v = f_0\frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v = \frac{gH}{f_0}\frac{\partial \eta}{\partial x}$$

If $\partial/\partial y \equiv 0$ as before, the vorticity equation reduces to an equation for η, and since $\zeta = \partial v/\partial x$ =>

$$\frac{\partial}{\partial t}\left(\frac{gH}{f_0}\frac{\partial^2 \eta}{\partial x^2} - f_0\eta\right) + \beta\frac{gH}{f_0}\frac{\partial \eta}{\partial x} = 0$$

This has the solution $\eta = \hat{\eta}\cos(kx - \omega t)$, where

Dispersion relation for a divergent planetary wave

$$\omega = -\beta k / (k^2 + f_0^2 / gH)$$

- There is no other solution for ω as there was before.
- In other words, making the geostrophic approximation when calculating v has filtered out in the inertia-gravity wave modes from the equation set, leaving only the low frequency planetary wave mode.
- This is not too surprising since the inertia-gravity waves, by their very essence, are not geostrophically-balanced motions.

Filtered equations more generally

- The idea of filtering sets of equations is an important one in geophysical applications.
- The quasi-geostrophic equations are often referred to as 'filtered equations' since, as in the above analysis, the consequence of computing the horizontal velocity geostrophically from the pressure or stream-function suppresses the high frequency inertia-gravity waves which would otherwise be supported by the Boussinesq equations.
- Furthermore, the Boussinesq equations themselves form a filtered system in the sense that the approximations which lead to them filter out compressible, or acoustic waves.

The Balance Equations

- **The full nonlinear form of the shallow-water equations in a layer of fluid of variable depth** h(x,y,t) **is**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$\zeta = \partial_x v - \partial_y u$ **relative vorticity**

$D = \partial_x u + \partial_y v$ **horizontal divergence**

- **The** u- **and** v-**equations can be replaced by the vorticity and divergence equations: =>**

vorticity equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v + (\zeta + f)D = 0$$

divergence equation

$$\frac{\partial D}{\partial t} + u\frac{\partial D}{\partial x} + v\frac{\partial D}{\partial y} + D^2 - f\zeta - 2J(u,v)$$

$$+\beta u + g\nabla^2 h = 0,$$

$\beta = df/dy$

Combine the vorticity equation and the continuity equation to form the potential vorticity equation:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0,$$

potential vorticity is $q = (\zeta + f)/h$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v + (\zeta + f)D = 0$$

$$\frac{\partial D}{\partial t} + u\frac{\partial D}{\partial x} + v\frac{\partial D}{\partial y} + D^2 - f\zeta - 2J(u, v) + \beta u + g\nabla^2 h = 0$$

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0$$

Equivalent to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Scale analysis

- **Choose representative scales:**
  - U **for the horizontal velocity components** u, v**,**
  - L **for the horizontal length scale of the motion,**
  - H **for the undisturbed fluid depth,**
  - δH **for depth departures** $h - H$**,**
  - f_0 **for the Coriolis parameter, and**
  - T = L/U **- an advective time scale.**
- **Define two nondimensional parameters:**
  - **the Rossby number,** $Ro = U/(f_0 L)$**, and**
  - **the Froude number,** $Fr = U^2/(gH)$**.**

Treatment of f

Let $f = f_0(1 + Ro\beta' y)$,

where y **is nondimensional and** $\beta' = \beta L^2 / U$ **.**

For middle latitude systems

$$U \sim 10\ \mathrm{m\,s^{-1}},\ \beta \sim 10^{-11}\ \mathrm{m^{-1}\,s^{-1}},\ L \sim 10^6\ \mathrm{m}$$

➡ **β' is of order unity.**

The nondimensional forms of the u-**momentum and continuity equations are:**

$$Ro\left(\frac{\partial u}{\partial t} + \ldots - \beta' yv\right) - v = -\frac{g\,\delta H}{f_0 UL}\frac{\partial h}{\partial x},$$

$$\frac{\delta H}{H}\left(\frac{\partial h}{\partial t} + \ldots\right) + D = 0,$$

- **Let us examine first the quasi-geostrophic scaling:**

➡

$$\mathrm{Ro}[\partial_t u + \ldots - \beta' y v] - v = -(g\,\delta H / f_0 UL)\,\partial_x h,$$

$$(\delta H / H)[\partial_t h + \ldots] + D = 0,$$

- In quasi-geostrophic motion ($\mathrm{Ro} \ll 1$), the term proportional to Ro can be neglected.
- Then $g\delta H/f_0 UL \approx O(1)$.
- We could choose the scale H so that this quantity is exactly unity, i.e. $\delta H = f_0 UL/g$. Then the term $\delta H/H$ may be written

$$\delta H / H = \mathrm{Ro}^{-1}\,\mathrm{Fr}.$$

The momentum equations are

$$\mathrm{Ro}\left(\frac{\partial u}{\partial t} + \ldots - \beta' y v\right) - v = -\frac{g\,\delta H}{f_0 UL}\frac{\partial h}{\partial x}$$

$$\mathrm{Ro}\left(\frac{\partial v}{\partial t} + \ldots - \beta' y v\right) + u = -\frac{g\,\delta H}{f_0 UL}\frac{\partial h}{\partial y}$$

$\Rightarrow$ to lowest order in Ro

$$v = \frac{\partial h}{\partial x} \qquad u = -\frac{\partial h}{\partial y}$$

$D = 0$ (or, more generally, $D \le O(\mathrm{Ro})$).

$$\frac{\delta H}{H}\left(\frac{\partial h}{\partial t} + \ldots\right) + D = 0,$$

a consistent scaling for quasi-geostrophic motion is that $\delta H/H = \mathrm{Ro}^{-1}\,\mathrm{Fr} = O(\mathrm{Ro})$ so that the term proportional to $\delta H/H \to 0$ as $\mathrm{Ro} \to 0$.

The vorticity and divergence equations

- **In nondimensional form, the vorticity and divergence equations are**

$$Ro\left(\frac{\partial \zeta}{\partial t} + \ldots + \beta' v\right) + D + Ro(\beta' y + \zeta) D = 0$$

and

$$Ro\left(\frac{\partial D}{\partial t} + \ldots\right) + D^2 - 2J(u, v)] - \zeta(1 + \beta' Ro\, y) + Ro\beta' u + \nabla^2 h = 0,$$

where J(u,v) **is the Jacobian** $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$

The quasi-geostrophic approximation

$$Ro\left(\frac{\partial \zeta}{\partial t} + \ldots + \beta' v\right) + D + Ro(\beta' y + \zeta) D = 0$$

$$Ro\left(\frac{\partial D}{\partial t} + \ldots + D^2 - 2J(u, v)\right) - \zeta(1 + \beta' Ro\, y) + Ro\beta' u + \nabla^2 h = 0$$

At lowest order in Rossby number

$$Ro\left(\frac{\partial \zeta}{\partial t} + \ldots + \beta' v + D_1\right) = 0$$ **where** $D = RoD_1$

$$\nabla^2 h = \zeta$$

When $\delta H/H = Ro^{-1}Fr$**, the continuity equation** $\Rightarrow$

$$\frac{\partial h}{\partial t} + \ldots \frac{1}{\mu} D_1 = 0$$

where $\mu = Ro^{-2}Fr$
and $D = Ro\ D_1$**.**

$$Ro\left(\frac{\partial\zeta}{\partial t}+\ldots+\beta' v+D_1\right)=0$$

$$\nabla^2 h=\zeta$$

$$\frac{\partial h}{\partial t}+\ldots\frac{1}{\mu}D_1=0$$

- These are the quasi-geostrophic forms of the vorticity, divergence and continuity equations.
- Note that the divergence equation has reduced to a diagnostic one relating the fluid depth to the vorticity which is consistent with ζ obtained from

$$\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}+\beta v+(\zeta+f)D=0$$

- $D = 0$ at $O(Ro^0)$ => there exists a streamfunction ψ such that

$$u=-\frac{\partial\psi}{\partial y}\qquad v=\frac{\partial\psi}{\partial x}$$

and from

$$Ro\left(\frac{\partial u}{\partial t}+\ldots-\beta' yv\right)-v=-\frac{g\,\delta H}{f_0 UL}\frac{\partial h}{\partial x}$$

$\psi = h$ +constant (e.g. $h - H$).

The potential vorticity equation

- The **potential vorticity equation** in nondimensional form can be obtained by eliminating D_1 between

$$Ro\left(\frac{\partial \zeta}{\partial t} + \ldots + \beta' v + D_1\right) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t} + \ldots \frac{1}{\mu} D_1 = 0$$

⇒ a single equation for ψ (or h) =>

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\nabla^2\psi + \beta' y - \mu\psi) = 0$$

analogous to the form for a stably-stratified fluid.

Note: in the case with a continuous vertical stratification, the ψ term would be replaced by a second-order vertical derivative of ψ.

- The **quasi-geostrophic approximation** leads to an elegant mathematical theory, but calculations based upon it tend to be inaccurate in many atmospheric situations such as when the isobars are strongly curved.
- In the latter case, we know that centrifugal forces are important and gradient wind balance gives a more accurate approximation.
- We try to improve the quasi-geostrophic theory by including terms of **higher order** in Ro in the equations.
- We decompose the horizontal velocity into **rotational** and **divergent** components $\mathbf{u} = \mathbf{u}_\psi + \mathbf{u}_\chi$

$$\mathbf{u}_\psi = k \wedge \nabla\psi \qquad \mathbf{u}_\chi = \nabla\chi$$

ψ is the **streamfunction**.

χ the velocity potential

It follows that

$$\zeta = \nabla^2\psi \quad \text{and} \quad D = \nabla^2\chi$$

- This decomposition is general (see Holton, 1972, Appendix), but is **not unique** : ⇒
- One can add equal and opposite flows with zero vorticity and divergence to the two components without affecting the total velocity **u**.
- Consistent with the quasi-geostrophic scaling, where χ is zero to O(Ro^0), we scale χ with ULRo so that in nondimensional form

$$\mathbf{u} = \mathbf{u}_\psi + \mathbf{u}_\chi \quad \Rightarrow \quad \mathbf{u} = \mathbf{u}_\psi + Ro\,\mathbf{u}_\chi$$

Equations

$$Ro[\partial_t\zeta + \ldots + \beta' v] + D + Ro(\beta' y + \zeta) D = 0,$$

$$Ro[\partial_t D + \ldots + D^2 - 2J(u, v)] - \zeta(1 + \beta' Ro\, y) + Ro\beta' u + \nabla^2 h = 0,$$

⇒

$$[\partial_t \nabla^2\psi + (\mathbf{u}_\psi + Ro.\mathbf{u}_\chi)\cdot\nabla(\nabla^2\psi) + \beta'\partial_x\psi] + \nabla^2\chi + (\beta' y + \nabla^2\psi)\nabla^2\chi = 0,$$

and

$$Ro^2[\partial_t\nabla^2\chi + J(\psi, \nabla^2\chi)] + Ro^3[\mathbf{u}_\chi\cdot\nabla(\nabla^2\chi) + (\nabla^2\chi)^2] - 2Ro\, J(u_\psi, v_\psi) - 2Ro^2[J(u_\chi, v_\psi) + J(u_\psi, v_\chi)] - 2Ro^3 J(u_\chi, v_\chi) - \nabla^2\psi(1 + \beta' Ro\, y) + \beta' Ro\, u_\psi + \beta' Ro^2 u_\chi + \nabla^2 h = 0.$$

The idea is to neglect terms of order Ro^2 and Ro^3 in these equations, together with the equivalent approximation in the continuity equation

$$\partial_t h + u\partial_x h + v\partial_y h + h(\partial_x u + \partial_y v) = 0\,,$$

In dimensional form they may be written

$$[\partial_t + \mathbf{u}\cdot\nabla](\nabla^2\psi + f) + (\nabla^2\psi + f)\nabla^2\chi = 0$$

$$2[(\partial_{xx}\psi)(\partial_{yy}\psi) - (\partial_{xy}\psi)^2] + \nabla\cdot(f\nabla\psi) - \nabla^2 h = 0$$

$$\partial_t h + \mathbf{u}\cdot\nabla h + h\nabla^2\chi = 0$$

$\mathbf{u} = \mathbf{u}_\psi + \mathbf{u}_\chi$.

Note: the divergence equation has been reduced to a diagnostic one relating ψ to h.

Moreover the advection of $\nabla^2\psi + f$ in

$$[\partial_t + \mathbf{u}\cdot\nabla](\nabla^2\psi + f) + (\nabla^2\psi + f)\nabla^2\chi = 0$$

and of h in $\partial_t h + \mathbf{u}\cdot\nabla h + h\nabla^2\chi = 0$

is by the total wind **u** and not just the nondivergent component of **u** as in the quasi-geostrophic approximation.

The Balance Equations

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)(\nabla^2\psi+f)+(\nabla^2\psi+f)\nabla^2\chi=0$$

$$2\left(\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right)+\nabla\cdot(f\nabla\psi)-g\nabla^2 h=0$$

$$\frac{\partial h}{\partial t}+\mathbf{u}\cdot\nabla h+h\nabla^2\chi=0$$

- **These equations are called the balance equations.**
- **They were first discussed by Charney (1955) and Bolin (1955).**

The Balance Equations

- **It can be shown that for a steady axisymmetric flow on an f-plane, the equation:**

$$2\left(\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right)+\nabla\cdot(f\nabla\psi)-\nabla^2 h=0$$

reduces to the gradient wind equation.

- **=> we may expect the equations to be a better approximation than the quasi-geostrophic system for strongly curved flows.**

Methods of solution

- Unfortunately it is not possible to combine the balance equations into a single equation for y as in the quasi-geostrophic case and they are rather difficult to solve.
- Methods of solution are discussed by Gent and McWilliams (1983).
- **Note:** although the balanced equations were derived by truncating terms of $O(Ro^2)$ and higher, the only equation where approximation is made is the divergence equation.
- => the equations represent an approximate system valid essentially for sufficiently small horizontal divergence.
- As long as this is the case, the Rossby number is of no importance.

- Elimination of $\nabla^2\chi$ between

$$\frac{\partial h}{\partial t} + \mathbf{u}\cdot\nabla h + h\nabla^2\chi = 0$$

and

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0$$

gives the potential vorticity equation:

Thus an alternative form of the balance system is

$$q = \frac{\zeta + f}{h}$$

$$2\left(\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2\right) + \nabla\cdot(f\nabla\psi) - g\nabla^2 h = 0$$

➔ ψ and h

$$\frac{\partial h}{\partial t} + \mathbf{u}\cdot\nabla h + h\nabla^2\chi = 0$$

Given q**, the first two equations can be regarded as a pair of simultaneous equations for diagnosing** ψ **and** h**, subject to appropriate boundary conditions.**

Equation $\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0$ **enables the prediction of** q**,**

while $\frac{\partial h}{\partial t} + \mathbf{u}\cdot\nabla h + h\nabla^2\chi = 0$ **may be used to diagnose** χ**.**

The Linear Balance Equations

Under certain circumstances [e.g. $Ro \ll 1$, $\beta' > O(1)$**], the nonlinear terms in Eq.(11.47) may be neglected in which case the equation becomes**

$$\nabla\cdot(f\nabla\psi) - g\nabla^2 h = 0$$

With this approximation, the system (11.46), (11.48) and (11.49) or (11.30), (11.31), (11.48), (11.49) constitute the linear balance equations.
These systems are considerably easier to solve than the balance equations.