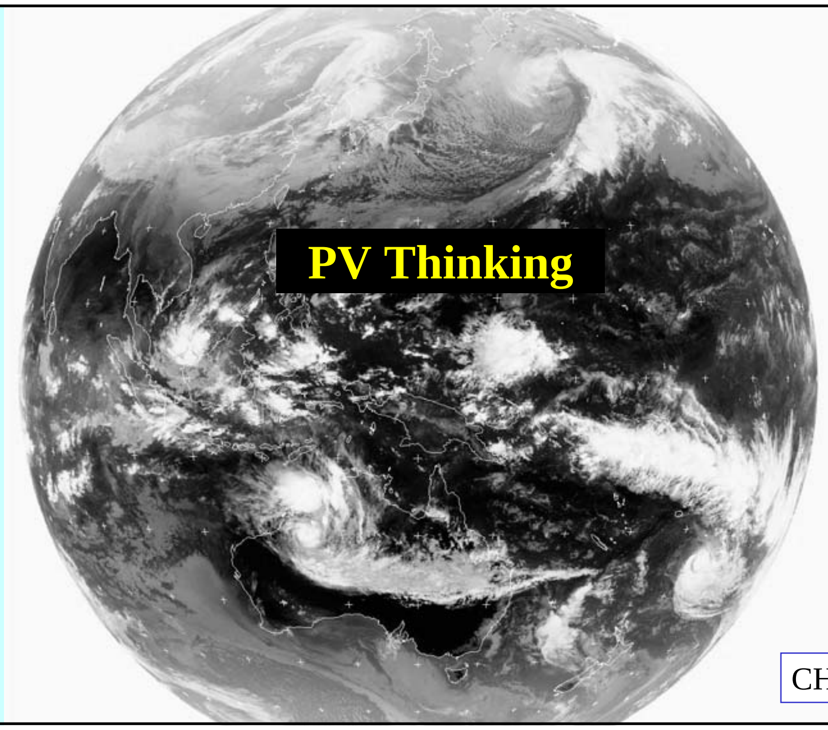
PV Thinking



CH 06

What is PV thinking?

- **Two main approaches to solving fluid flow problems:**
  1. **We can integrate the momentum, continuity and thermodynamic equations (the primitive equations) directly.**
  2. **In certain cases we can use the vorticity – streamfunction formulation**
- **Often (2) is more insightful than (1).**
- **The vorticity – streamfunction approach can provide a neat conceptual framework in which to understand the dynamics.**
- **In certain circumstances the approach can be generalized.**

How can PV thinking help the forecaster?

- Davies and Emanuel (1991)
  - "… a proper integration of the equations of motion is not synonymous with a conceptual grasp of the phenomena being predicted"
- Answer
  - "PV thinking" can provide the forecaster with a sound conceptual framework in which to interpret numerical analyses and prognoses.

Vorticity – Streamfunction method

Assumptions: homogeneous, two-dimensional, inviscid, non-rotating flow.

conservation $$\frac{D\zeta}{Dt} = 0$$ (1)

vorticity $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

material derivative $$\frac{D}{DT} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

continuity $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

invertibility $$\nabla^2 \psi = \zeta$$ (2)

Rotating flow on a β-plane

Assumptions: homogeneous, two-dimensional, inviscid, rotating flow on a β-plane.

conservation

$$\frac{Dq}{Dt} = 0 \qquad (1)$$

absolute vorticity

$$q = \zeta + f$$

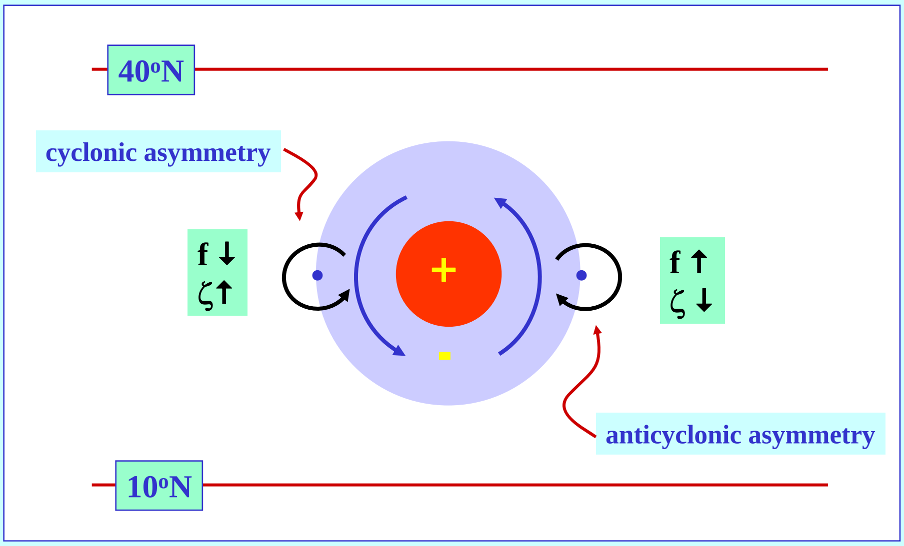
continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \;\Rightarrow\; u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

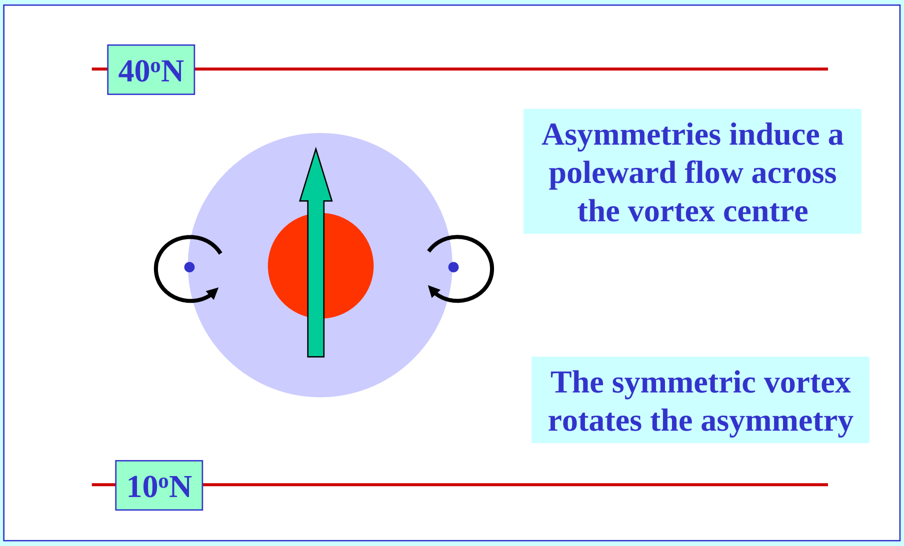
invertibility

$$\nabla^2 \psi = \zeta \qquad (2)$$

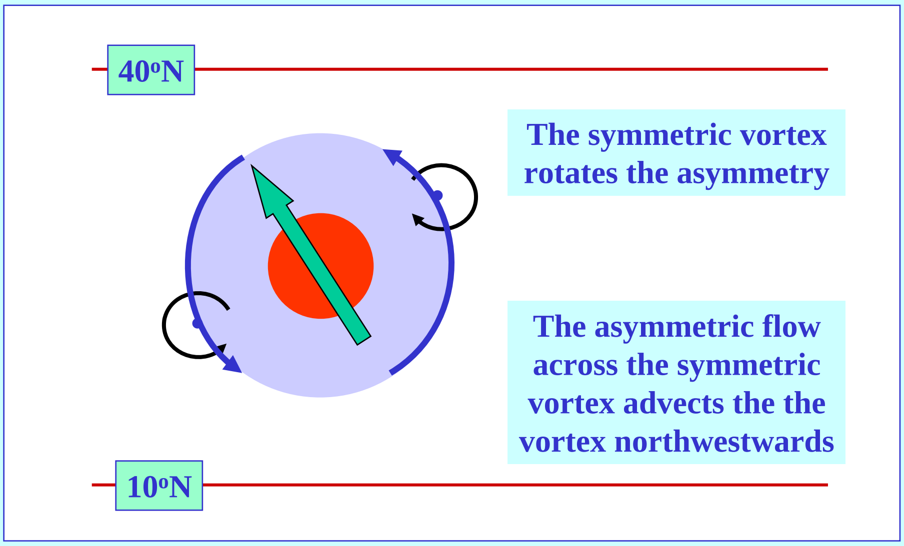
Tropical cyclone thought experiment



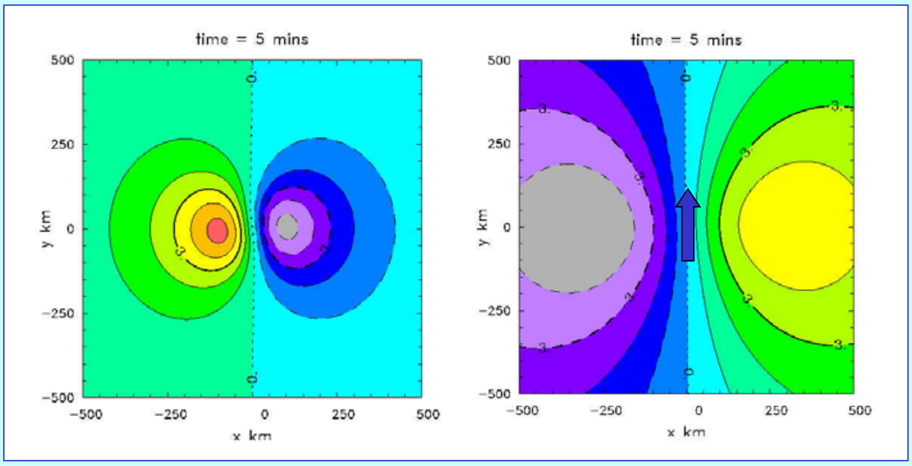
Tropical cyclone thought experiment



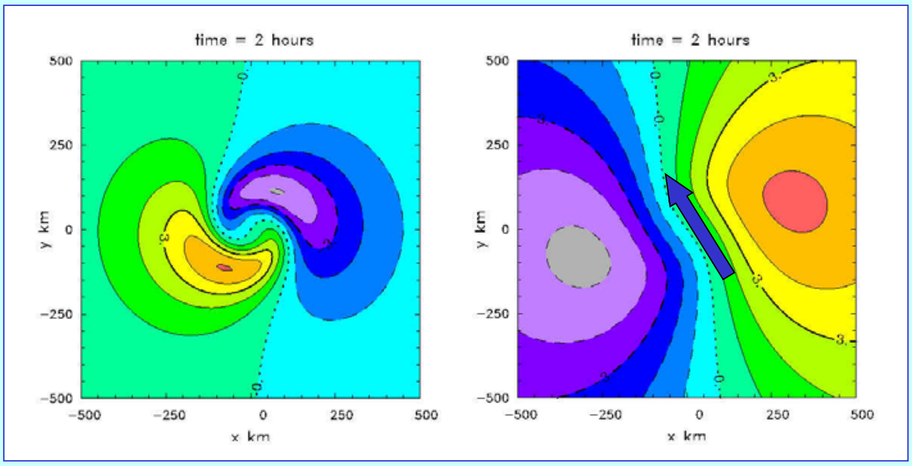
Tropical cyclone thought experiment



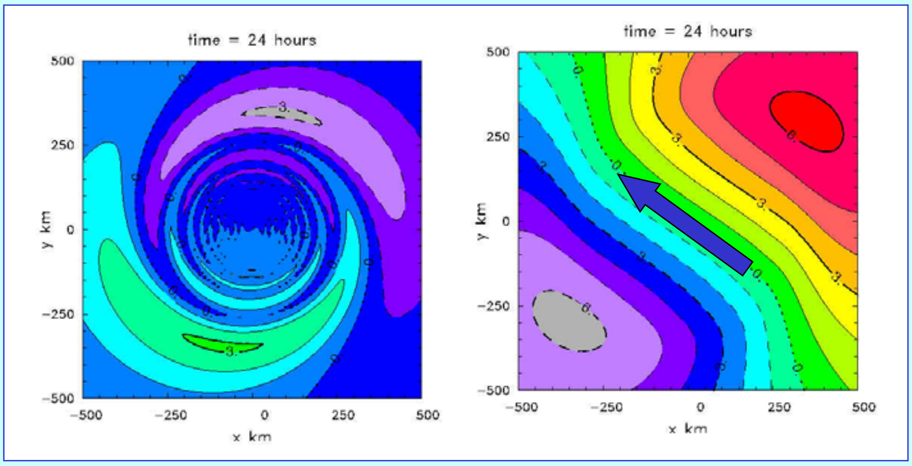
Relative vorticity **Streamfunction**



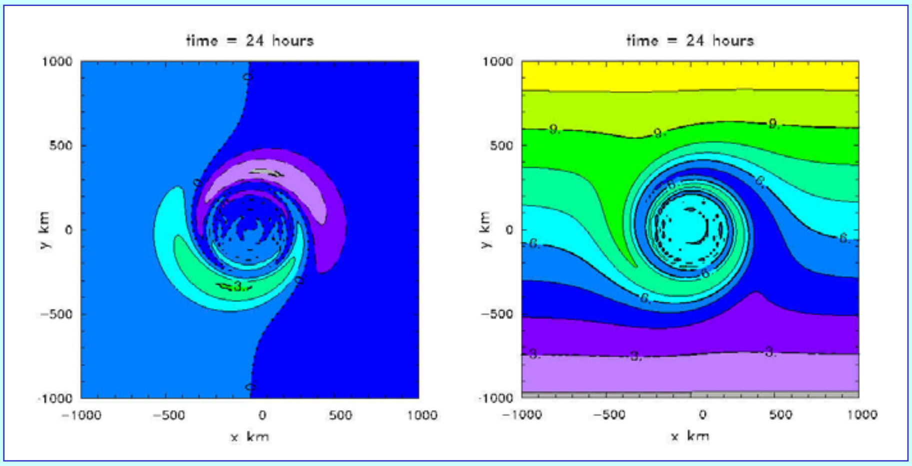
Relative vorticity **Streamfunction**



Relative vorticity **Streamfunction**

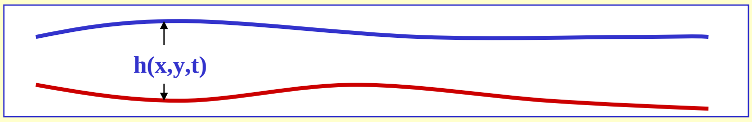


Relative vorticity **Absolute vorticity**



Divergent flow, variable depth

Assumptions: homogeneous, divergent, inviscid, variable depth h(x,y,t).



Conservation

$$\frac{Dq}{Dt} = 0$$ (1)

absolute vorticity

$$q = \frac{\zeta + f}{h}$$

continuity

$$\frac{Dh}{Dt} = -h\nabla \cdot \mathbf{u}$$

No invertibility!

Except for QG-flow

Quasi-geostrophic motion

Assumptions: stratified (Boussinesq), rotating, three-dimensional, adiabatic, inviscid, flow.

Conservation

$$\frac{Dq}{Dt} = 0$$ (1)

potential vorticity

$$q = \zeta + f + \varepsilon \frac{\partial^2 \psi}{\partial z^2}$$

material derivative

$$\frac{D}{DT} \equiv \frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla$$

continuity

$$\nabla \cdot \mathbf{u}_g = 0 \quad \Rightarrow \quad \mathbf{u}_g = \frac{1}{\rho f_o} \mathbf{k} \wedge \nabla p = \mathbf{k} \wedge \nabla \psi$$

Invertibility

$$\nabla_h^2 \psi + f + \varepsilon \frac{\partial^2 \psi}{\partial z^2} = q$$ (2)

$$\varepsilon = \frac{f^2}{N^2}$$

General adiabatic motion of a rotating stratified fluid

Assumptions: compressible, stratified, rotating, three-dimensional, adiabatic, inviscid, flow.

Conservation

$$\frac{DP}{Dt} = 0$$

1

Ertel potential vorticity

$$P = \frac{1}{\rho}(\omega + \mathbf{f}) \cdot \nabla\theta$$

material derivative

$$\frac{D}{DT} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

No invertibility! **unless …flow balanced**

Isentropic coordinates

Assumptions: compressible, stratified, rotating, three-dimensional, adiabatic, inviscid, flow.

Conservation

$$\frac{DP}{Dt} = 0$$

Ertel potential vorticity

$$P = -g(\zeta + f)\frac{\partial \theta}{\partial p} = \frac{1}{\rho}(\zeta + f)\frac{\partial \theta}{\partial z}$$

material derivative

$$\frac{D}{DT} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Formulation of an invertibility principle

- **To formulate an invertibility principle one must:**
  1. **Specify some kind of balance condition, the simplest, but least accurate option being ordinary quasi-geostrophic balance,**
  2. **Specify some sort of reference state, expressing the mass distribution of θ, and**
  3. **Solve the inversion principle globally, with proper attention to boundary conditions.**

Diabatic and frictional effects

PV is no longer materially conserved:

$$\frac{DP}{Dt} = \frac{1}{\rho}\zeta_a \cdot \nabla\dot{\theta} + \frac{1}{\rho}\mathbf{K} \cdot \nabla\theta$$

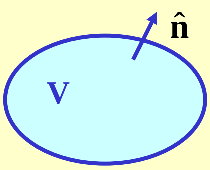
$$= -\frac{1}{\rho}\nabla \cdot \mathbf{Y}$$

ζ_a = absolute vorticity

$\dot{\theta}$ = absolute vorticity

$\mathbf{K} = \nabla \wedge \mathbf{F}$

$\mathbf{Y} = -\dot{\theta}\zeta_a + \nabla\theta \wedge \mathbf{K}$



$$\frac{d}{dt}\int_V \rho P dV = \int_S (\dot{\theta}\zeta_a + \theta\mathbf{K}) \cdot \hat{\mathbf{n}} dS$$

Frictionless, adiabatic $\quad \frac{d}{dt}\int_V \rho P dV = 0$

Physical interpretation

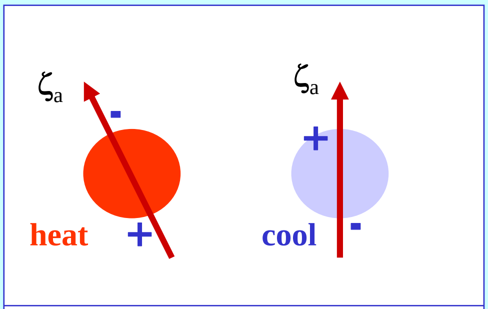
Diabatic term

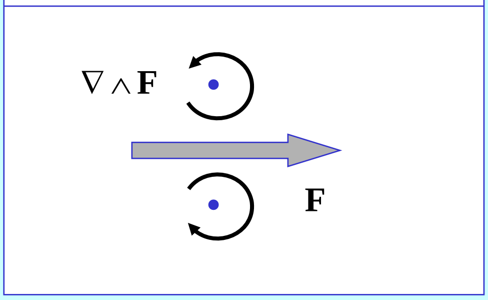
$$\frac{1}{\rho}\zeta_a \cdot \nabla\dot{\theta}$$

Friction term

$$\frac{1}{\rho}\mathbf{K}\cdot\nabla\theta$$

$$\mathbf{K} = \nabla \wedge \mathbf{F}$$





Example

- **Consider the case of an axisymmetric vortex with tangential velocity distribution** $v(r)$**.**
- **Assume that a linear frictional force** $\mathbf{F} = -\mu v(r)$ **acts at the ground** $z = 0$**.**
- **Then** $\mathbf{K} = -\mu\zeta\mathbf{k}$**, where** ζ **is the vertical component of relative vorticity.**
- **Then PV is destroyed at the rate**

$$\frac{1}{\rho}\mathbf{K}\cdot\nabla\theta = -\frac{1}{\rho}\mu\zeta\left(\frac{\partial\theta}{\partial z}\right)_{z=0}$$