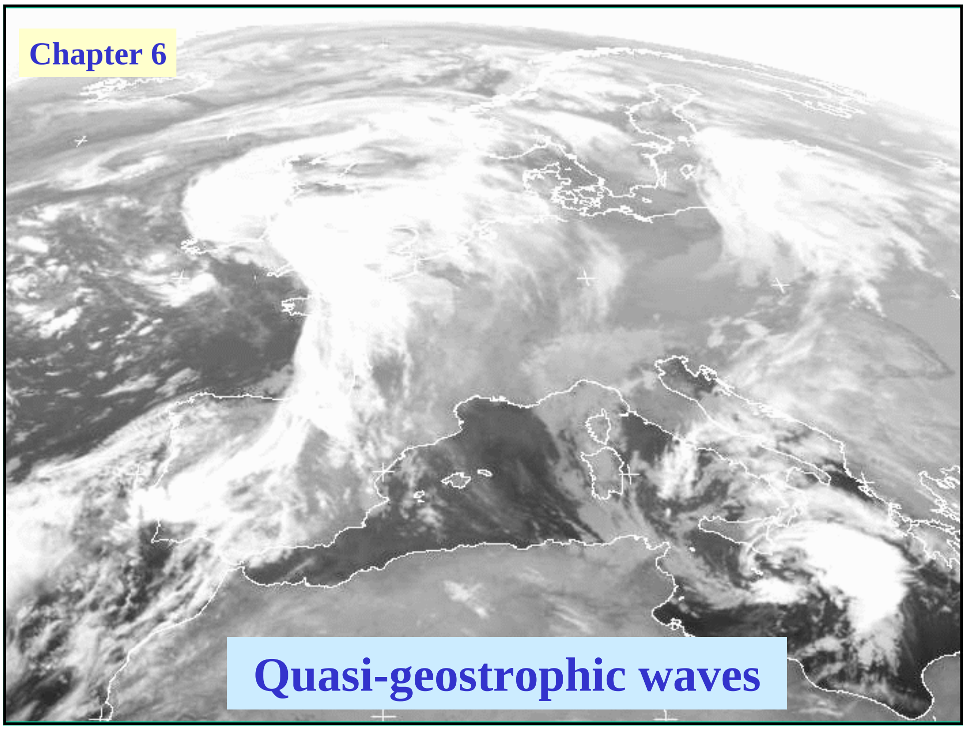
Chapter 6



Quasi-geostrophic waves

Quasi-geostrophic waves

Balanced state	Elastostatic (compressible)	Hydrostatic	Geostrophic	Sverdrup
Result of disturbing it	Acoustic oscillations	Inertia-gravity waves $\omega >> f$	Inertia or inertia-gravity waves $\omega \approx f$	Rossby waves $\omega << f$

Quasi-geostrophic waves

The quasi-geostrophic equations

Assume: a Boussinesq fluid, constant Brunt-Väisälä frequency

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] q = 0$$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \frac{\partial \psi}{\partial z} + \frac{N^2}{f_o} w = 0$$

$$\mathbf{u} = \mathbf{k} \wedge \nabla\psi$$

$$q = \nabla^2\psi + f + \frac{f_o^2}{N^2}\frac{\partial^2\psi}{\partial z^2}$$

$$\varepsilon^2$$

q **is the quasi-geostrophic potential vorticity and** $f = f_o + \beta y$

Some notes

- **Recall (DM, Chapter 8) that an important scaling assumption in the derivation of the QG-equations is that the Burger number** $B = f^2L^2/N^2H^2$**, is of order unity,** H **and** L **being vertical and horizontal length scales for the motion.**
- **It can be shown that this ratio characterizes the relative magnitude of the final term in the thermodynamic equation compared with the advective term.**
- **Hence** $B \sim 1$ **implies that there is significant coupling between the buoyancy field and the vertical motion field.**
- **A further implication is that** $L \sim L_R = NH/f$**, the Rossby radius of deformation.**

Principle behind the method of solution

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] q = 0 \qquad (1)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \frac{\partial \psi}{\partial z} + \frac{N^2}{f_o} w = 0 \qquad (2)$$

$$\mathbf{u} = \mathbf{k} \wedge \nabla\psi \qquad (3)$$

$$q = \nabla^2\psi + f + \varepsilon^2 \frac{\partial^2 \psi}{\partial z^2} \qquad (4)$$

IC : $\psi(x,y,z,0)$ **given.**

1. **Calculate** $u(x,y,z,0)$
2. **Calculate** $q(x,y,z,0)$
3. **Predict** $q(x,y,z,\Delta t)$
4. **Diagnose** $\psi(x,y,z, \Delta t)$ **using Eq. (4)**
5. **Repeat to find** $\psi(x,y,z, 2\Delta t)$ **etc.**

Eq. 2 is used to evaluate $w(x,y,z,t)$ **and to prescribe BCs**

Some notes

- **An example of the use of the thermodynamic equation for applying a boundary condition at horizontal boundaries is provided by the Eady baroclinic instability calculation in DM, Chapter 9.**
- **The ability to calculate** ψ **from (4) from a knowledge of** q **(step 3) is sometimes referred to as the invertibility principle.**

$$\nabla^2\psi + f + \varepsilon^2 \frac{\partial^2 \psi}{\partial z^2} = q \qquad (4)$$

- **The foregoing steps will be invoked in the discussion that follows shortly.**
- **I shall show that perturbations of a horizontal basic potential vorticity gradient lead to waves.**

Quasi-geostrophic perturbations

Consider a perturbation to the basic zonal flow $\bar{u}(y,z)$.

$$\left[\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right]q+\frac{\partial \psi}{\partial x}\frac{\partial \bar{q}}{\partial y}=0$$

$$\left[\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right]\frac{\partial \psi}{\partial z}+\frac{\partial \psi}{\partial x}\frac{\partial^2 \bar{\psi}}{\partial y \partial z}+\frac{f}{\varepsilon}w=0$$

$$\nabla^2\psi+\varepsilon^2\frac{\partial^2\psi}{\partial z^2}=q$$

q **and** ψ **represent perturbation quantities**

$$\frac{\partial \bar{q}}{\partial y}=\beta-\frac{\partial^2 \bar{u}}{\partial y^2}-\varepsilon^2\frac{\partial^2 \bar{u}}{\partial z^2}$$

Example 1: Rossby waves

Let $\bar{u}(y,z)=0$, $\bar{q}_y(y,z)=\beta>0$.

The physical picture is based on the conservation of total potential vorticity (here $\bar{q}+q$**) for each particle.**

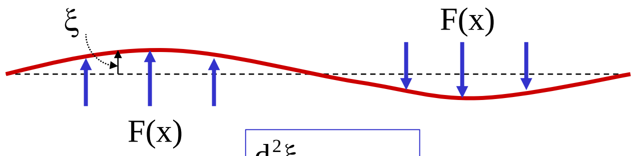
$$\frac{\partial \bar{q}}{\partial y}=\beta>0 \quad \Rightarrow$$

For a positive (northwards) displacement $\xi>0$, $q<0$
For a negative (southwards) displacement $\xi<0$, $q>0$.

Consider for simplicity motions for which $\partial_x >> \partial_y, \partial_z$

$$\Rightarrow \quad \underline{q}=\nabla^2\psi+\varepsilon^2\frac{\partial^2\psi}{\partial z^2}=\underline{\frac{\partial^2\psi}{\partial x^2}}$$

String analogy for solving $\psi_{xx} = q$



$$\frac{d^2\xi}{dx^2} = -F(x)$$

$$\frac{\partial^2\psi}{\partial x^2} = q \quad \Rightarrow \quad$$ **Interpret** $\psi(x) = \xi(x)$, $F(x) = -q(x)$

Given $q(x)$ **we can diagnose** $\psi(x)$ **using the "string analogy" and our intuition about the behaviour of a string!**

The dynamics of Rossby waves

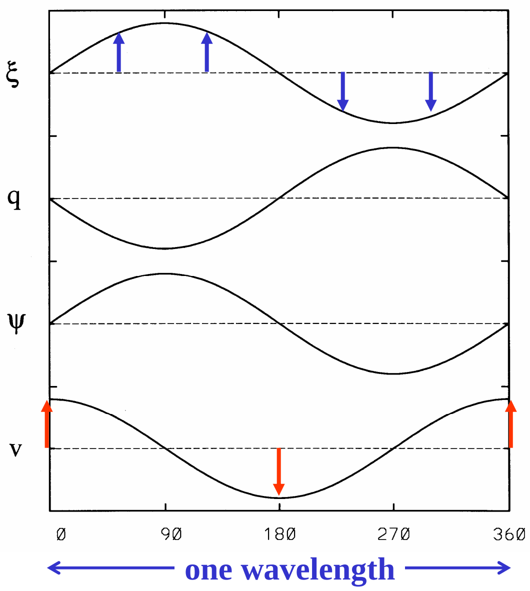
Displace a line of parcels into a sinusoidal curve

The corresponding $q(x)$ **distribution**

Invert $q(x) \Rightarrow \psi(x)$

$$\frac{\partial^2\psi}{\partial x^2} = q$$

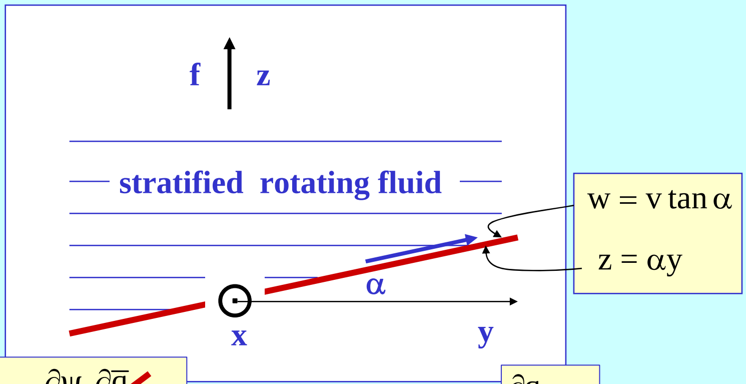
$$v = \psi_x$$



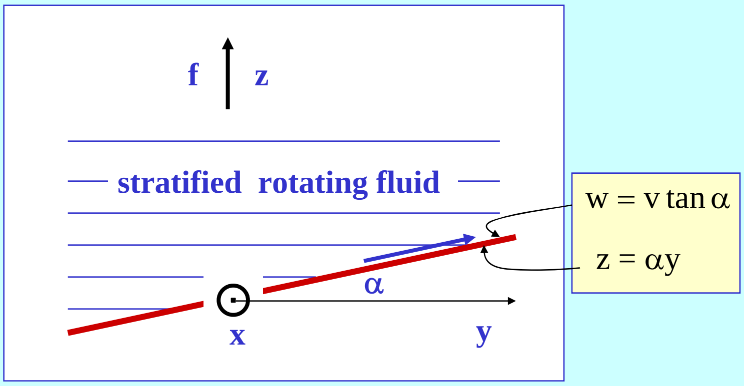
Note $\xi(x)$ **&** $v(x)$ **are** 90° **out of phase.**

Example 2: Topographic waves

Let $\bar{u}(y,z) \equiv 0$, $\bar{q}_y(y,z) = \beta \equiv 0$ **(but see later!) and a slightly sloping boundary.**



$$\left[\frac{\partial}{\partial t} + \cancel{\bar{u}\frac{\partial}{\partial x}}\right] q + \frac{\partial \psi}{\partial x}\cancel{\frac{\partial \bar{q}}{\partial y}} = 0 \quad \Longrightarrow \quad \frac{\partial q}{\partial t} = 0$$



- α **must be no larger than** $O(RoH/L)$**, otherwise the implied** w **for a given** v **would be too large to be accommodated within quasi-geostrophic theory.**
- **If** $\alpha \ll 1$, $\tan\alpha \approx \alpha$ **and can apply the boundary condition at** $z = 0$ **with sufficient accuracy** $\Rightarrow$ $w = v\alpha$ **at** $z = 0$**.**

Plane wave solutions

There exist plane wave solutions for ψ of the form

$$\psi = a \exp\left[i(kx + ly - \omega t) - (N/f)(k^2 + l^2)^{1/2} z \right]$$

$$\left[\frac{\partial}{\partial t} + \cancel{\bar{u}\frac{\partial}{\partial x}} \right] \frac{\partial \psi}{\partial z} + \frac{f}{\varepsilon} w = 0 \qquad w = \psi_x \alpha \ \textbf{at} \ z = 0$$

$$\Rightarrow \quad \boxed{\omega = -\frac{\alpha\ Nk}{(k^2 + l^2)^{1/2}}}$$

This is the dispersion relation.

Some notes

$$\boxed{\omega = -\frac{\alpha\ Nk}{(k^2 + l^2)^{1/2}}} \quad \Rightarrow$$

- **The wave propagates to the left of upslope (towards –ve** x**).**
- **Note that** ω **does not depend of** f.
- **This does not mean that** f **is unimportant; in fact for horizontal wavelength** $2\pi/\kappa$**, where** $\kappa^2 = k^2 + l^2$**, the e-folding vertical scale of the wave is** $f/(N\kappa)$.
- **Changes in relative vorticity** ζ **arise from stretching and shrinking of vortex lines at the rate** fw_z**, associated with the differences between the slope of the boundary and those of the density isopleths.**

Reformulation of the problem

Above the boundary, $\bar{q}_y \equiv 0$**, but we can say that there is a potential vorticity gradient at the boundary if we generalize the notion of potential vorticity:**

The foregoing problem can be written as

$$\partial_t[\partial_{xx}\psi + \partial_{yy}\psi + \varepsilon^2\partial_{zz}\psi] = 0$$

$$(f/N^2)\psi_{zt} + \alpha\psi_x = 0 \quad \textbf{at} \quad z = 0$$

Dirac delta function

It is mathematically equivalent to the problem:

$$\partial_t[\partial_{xx}\psi + \partial_{yy}\psi + \varepsilon^2\partial_{zz}\psi] + \bar{q}_y\psi_x = 0$$

$\psi_z = 0$, **continuous at** $z = 0_-$

$$\bar{q}_y = f\alpha\delta(z)$$

Proof of mathematical equivalence

$\delta(z) \equiv 0$ **for** $z > 0$ $\Rightarrow$

$$\left.\begin{aligned}&\partial_t[\partial_{xx}\psi + \partial_{yy}\psi + \varepsilon^2\partial_{zz}\psi] = 0\\ &\partial_t[\partial_{xx}\psi + \partial_{yy}\psi + \varepsilon^2\partial_{zz}\psi] + \bar{q}_y\psi_x = 0\end{aligned}\right\}\ \textbf{identical for } z > 0.$$

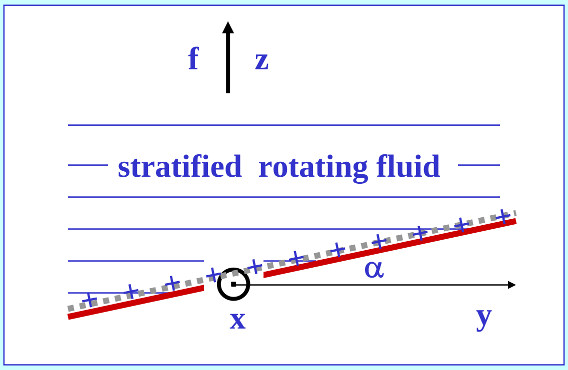
$$\underbrace{\int_{-\tau}^{\tau}\partial_t[\psi_{xx}+\psi_{yy}}_{\le 2\tau \max\limits_{-\tau<z<\tau}\left[\psi_{xxt}+\psi_{yyt}\right]} \underbrace{+\varepsilon^2\psi_{zz}]dz}_{\left[\varepsilon^2\psi_{zt}\right]_{-\tau}^{\tau}} + \underbrace{\int_{-\tau}^{\tau} f\alpha\delta(z)\psi_x dz}_{f\alpha\psi_x|_{z=0}} = 0$$

$\rightarrow 0$ as $\tau \rightarrow 0$ $\quad \rightarrow \varepsilon^2\psi_{zt}\big|_{z=0+}$ $\quad$ as $\tau \rightarrow 0$

$\psi_x = 0$ **at** $z = 0_-$

$\Rightarrow$ $(f/N^2)\psi_{zt} + \alpha\psi_x = 0$ **at** $z = 0$

Physical interpretation



$$\partial_t[\partial_{xx}\psi + \partial_{yy}\psi + \varepsilon^2\partial_{zz}\psi] + \bar{q}_y\psi_x = 0$$

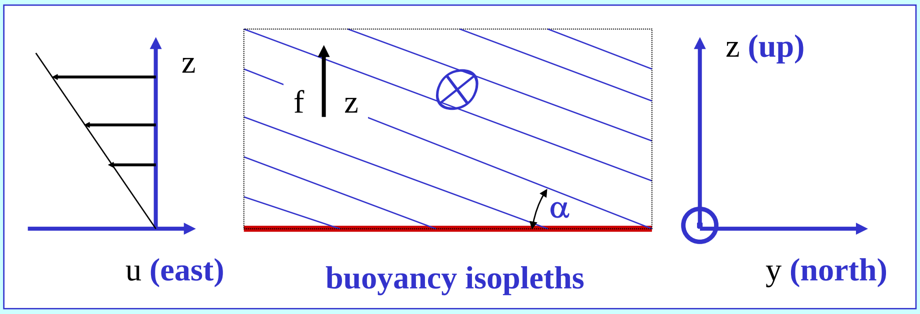
- **The alternative formulation involves a potential vorticity gradient $\bar{q}_y$ confined to a "sheet" at $z = 0$, and the wave motion can be attributed to this.**

Note

- **Note that it is of no formal consequence in the quasi-geostrophic theory whether the boundary is considered to be exactly at $z = 0$, or only approximately at $z = 0$.**
- **What matters dynamically is the slope of the isopleths relative to the boundary.**

Example 3: Waves on vertical shear

Let $\beta = 0$ and $\bar{u} = \Lambda z$, Λ constant. Then again $\bar{q}_y \equiv 0$, but now we assume a horizontal lower boundary.



When $\Lambda < 0$, the slopes of the density isopleths relative to the boundary are the same as before. Since $\bar{q}_y = 0$ for $z > 0$, the dynamics is as before within the quasi-geostrophic theory.

$$\left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right] q + \frac{\partial \psi}{\partial x}\frac{\partial \bar{q}}{\partial y} = 0$$

$$\Rightarrow \quad \left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right] q = 0 \quad \text{for} \quad z > 0$$

$q = 0$ **is a solution as before**

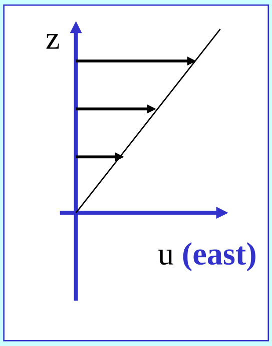
The solution is the same as in Example 2 if α is identified with $-f\Lambda/N^2$, since the slope of the density isopleths is

$$\alpha = \frac{\rho_y}{\rho_z} = \frac{(g/\bar{\rho})\rho_y}{(g/\bar{\rho})\rho_z} = \frac{\sigma_y}{N^2} = -\frac{f\bar{u}_z}{N^2} = -\frac{f\Lambda}{N^2}$$

Example 4: Waves on vertical shear

Waves at a boundary of discontinuous vertical shear ($\beta = 0$, $\bar{u} = \Lambda z H(z)$), and the flow unbounded above and below.

$H(z) = 1$ **for** $z > 0$,
$H(z) = 0$ **for** $z < 0$.



$$\bar{u}_z = \Lambda z \delta(z) + \Lambda H(z)$$

$$\bar{u}_{zz} = 2\Lambda\delta(z)$$

$$\bar{q}_y = -2\Lambda \frac{f^2}{N^2}\delta(z)$$

$$\frac{d}{dz}H(z) = \delta(z)$$

$$z\delta(z) = 0$$

There is a thin layer of negative $\bar{q}_y$ concentrated at $z = 0$.

Boundary condition at z = 0

$$\lim_{\tau\to 0}\int_{-\tau}^{\tau}\left\{\left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right]q + \frac{\partial\psi}{\partial x}\frac{\partial\bar{q}}{\partial y} = 0\right\}dz$$

$$\lim_{\tau\to 0}\int_{-\tau}^{\tau}\left\{\left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \varepsilon\frac{\partial^2\psi}{\partial z^2}\right) + \Lambda\delta(z)\frac{\partial\psi}{\partial x}\right\}dz = 0$$

$$\left[\varepsilon\psi_{zt}\right]_{0-}^{0+} = 2\Lambda\varepsilon\psi_x\big|_{z=0}$$

By inspection, the solution of the perturbation vorticity equation

$$\lim_{\tau\to 0}\left[\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\varepsilon\frac{\partial^2\psi}{\partial z^2}\right)+\Lambda\delta(z)\frac{\partial\psi}{\partial x}=0$$

subject to $\psi\to 0$ **as** $z\to\pm\infty$ **together with**

$$\left[\varepsilon\psi_{zt}\right]_{0-}^{0+}=2\Lambda\varepsilon\psi_x\big|_{z=0}$$

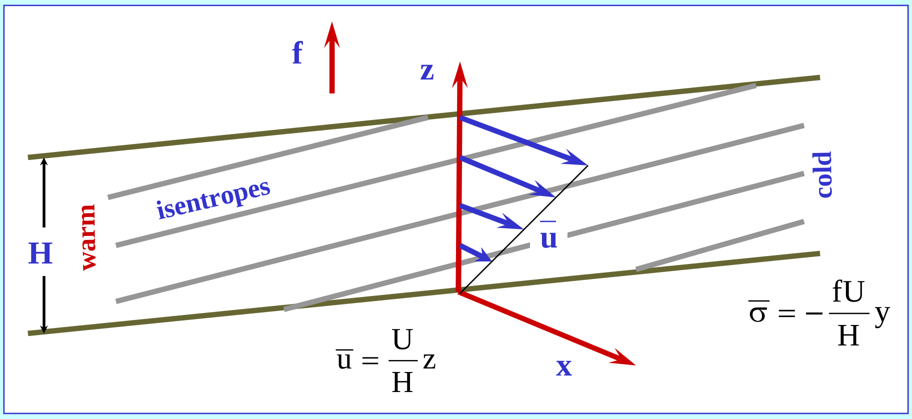
$$\Rightarrow\quad \psi=a\exp\left[i(kx+ly-\omega t)-\mathrm{sgn}(z)(N/f)(k^2+l^2)^{1/2}z\right]$$

$$\omega=\frac{\Lambda fk}{2N(k^2+l^2)^{1/2}}$$

This is the dispersion relation.

The wave is stable and has vertical scale f/(kN).

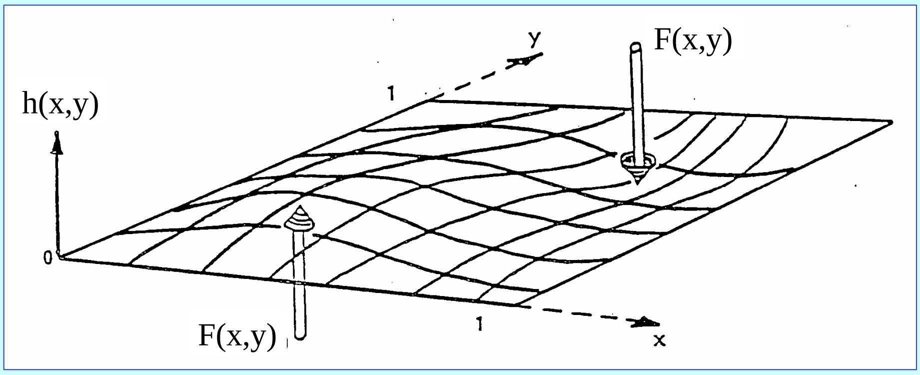
Baroclinic instability: the Eady problem



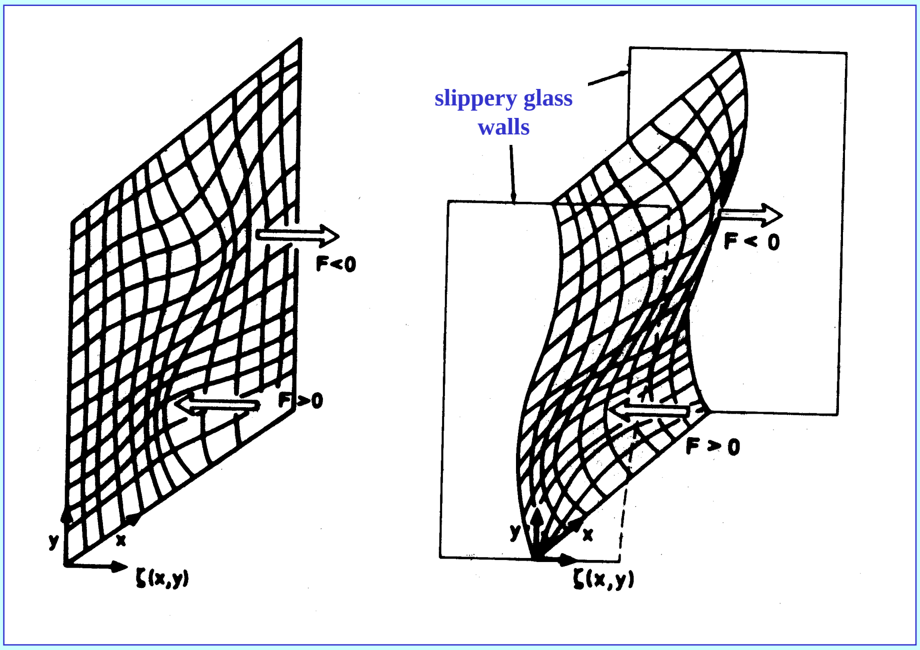
Zonal flow configuration in the Eady problem (northern hemisphere).

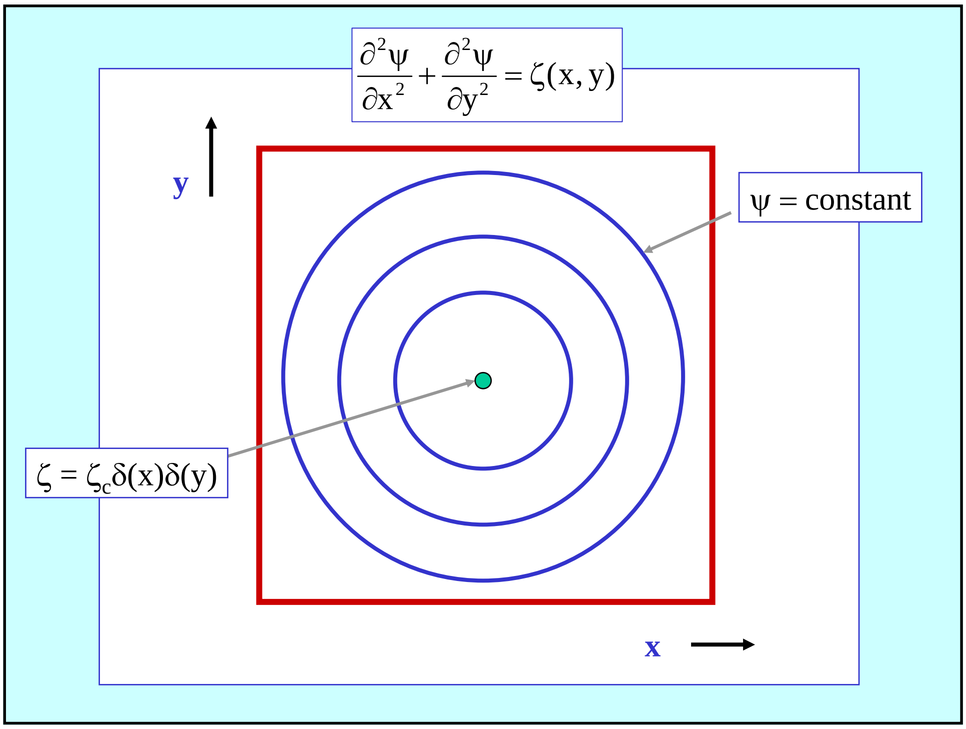
The membrane analogy

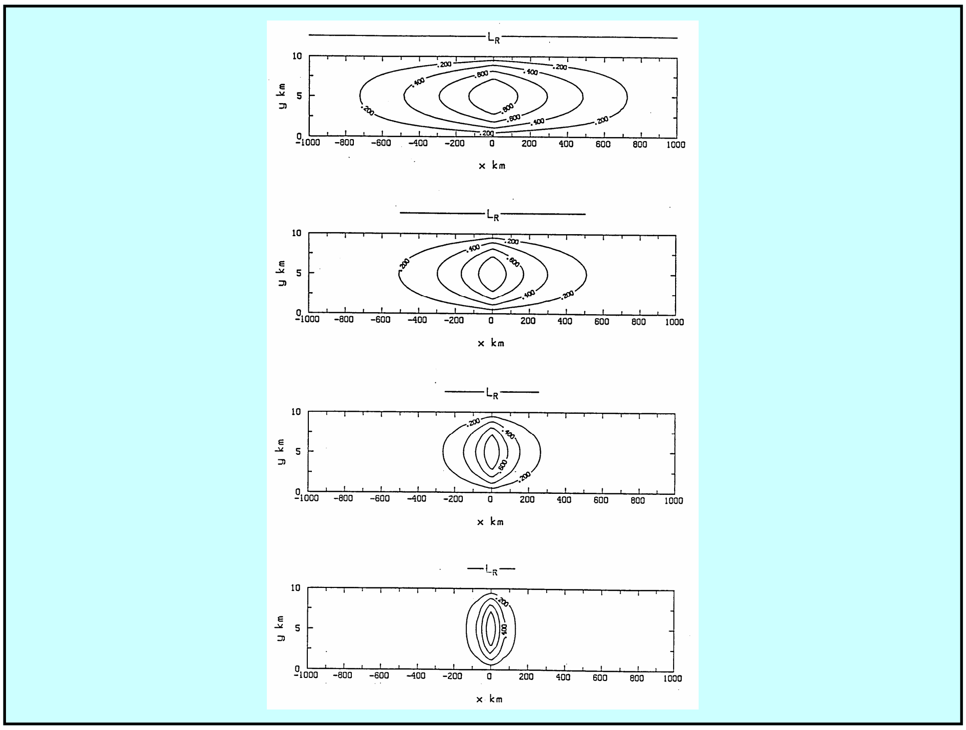
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -F(x, y)$$



Equilibrium displacement of a stretched membrane over a square under the force distribution F(x,y).

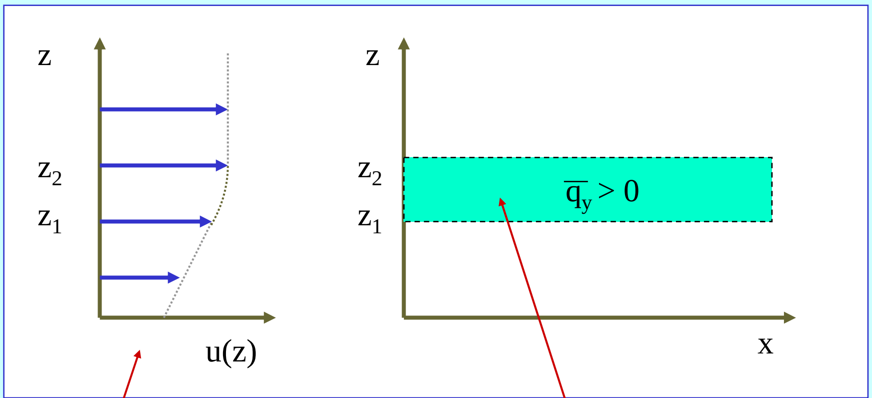



∂²ψ/∂x² + ∂²ψ/∂y² = ζ(x, y)
y
ψ = constant
ζ = ζcδ(x)δ(y)
x


LR
y km
x km

A unified theory

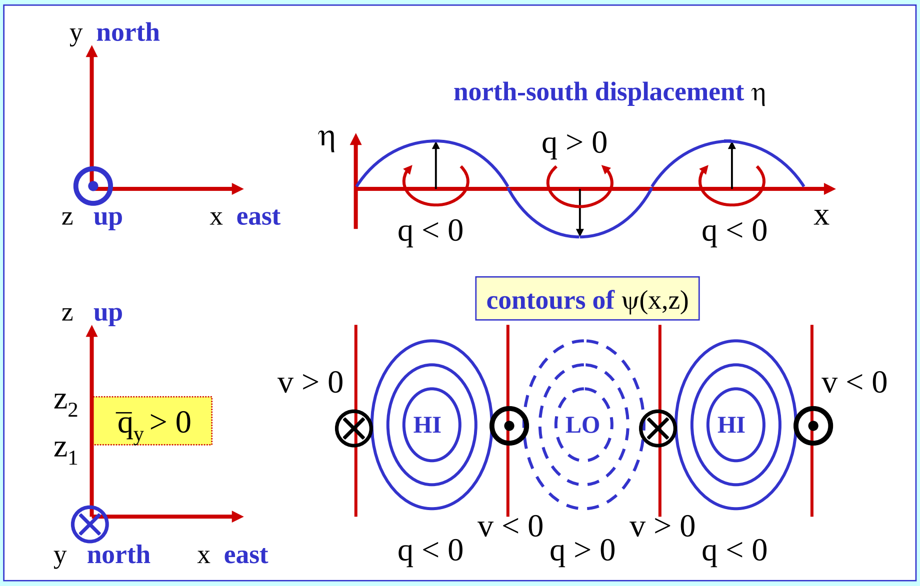
- The **generalization of** the definition of **potential vorticity gradient** to include isolated sheets of $\bar{q}_y$, either internal or at a boundary, enable a unified description of "potential vorticity to be given.
- The description is similar to that given for Example 1, but requires the motion to be viewed in two planes; a horizontal x-y plane and a vertical x-z plane.
- Consider the $\bar{q}_y$ defined by the shear flow U(z) shown in the next slide.



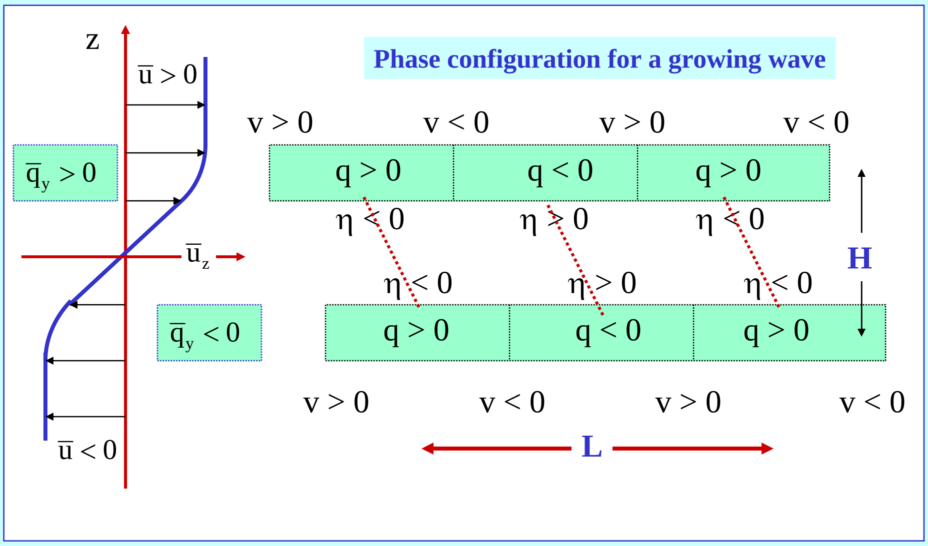
Non-uniform vertical shear flow

Layer of non-zero PV

Consider a perturbation in the form of a sinusoidal displacement in the north-south direction.



A sinusoidal displacement in the north-south direction leads to a potential vorticity perturbation in horizontal planes.



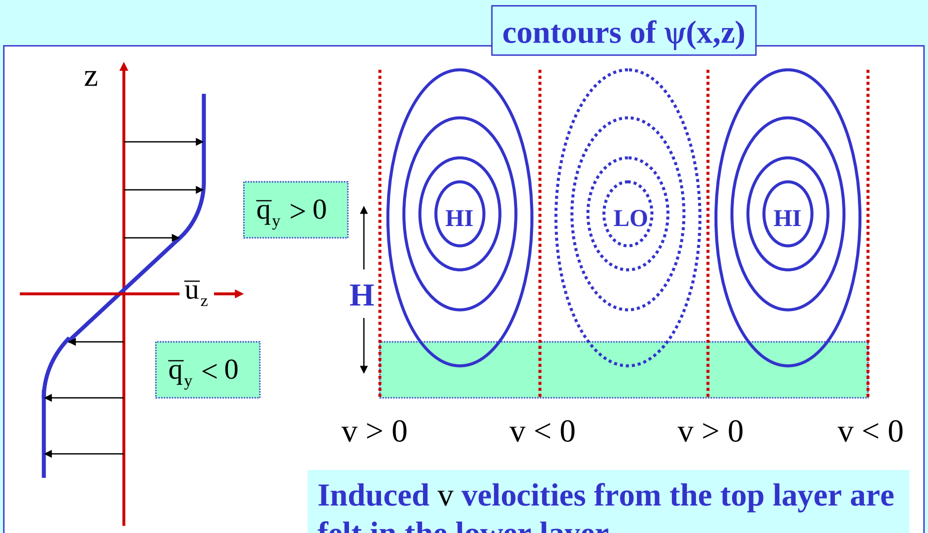
The baroclinic instability mechanism

- The foregoing ideas may be extended to provide a qualitative description of the baroclinic instability mechanism.
- We shall use the fact that a velocity field **in phase** with a displacement field corresponds to **growth of amplitude**, just as quadrature corresponds to **phase propagation.**
- Suppose that the displacement of a particle is

$$\eta = A(t)\sin nt \qquad (A, n > 0)$$

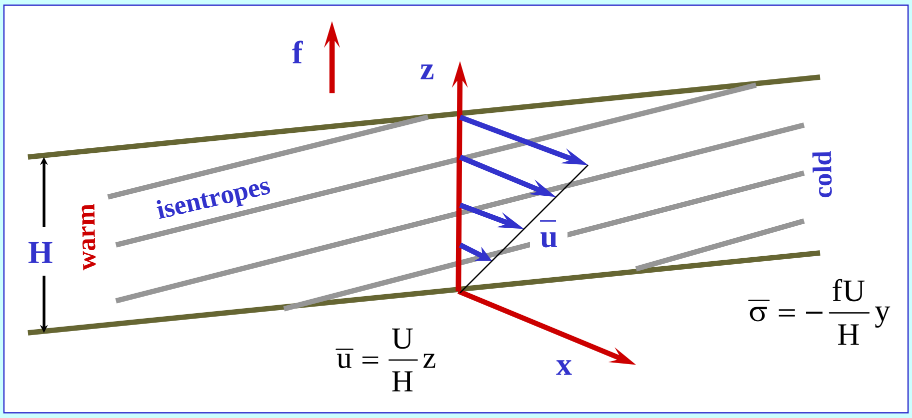
- Suppose that we know (by some independent means) that the velocity of the particle is

$$v = B(t)(\cos nt + \mu \sin nt)$$



Induced v velocities from the top layer are felt in the lower layer

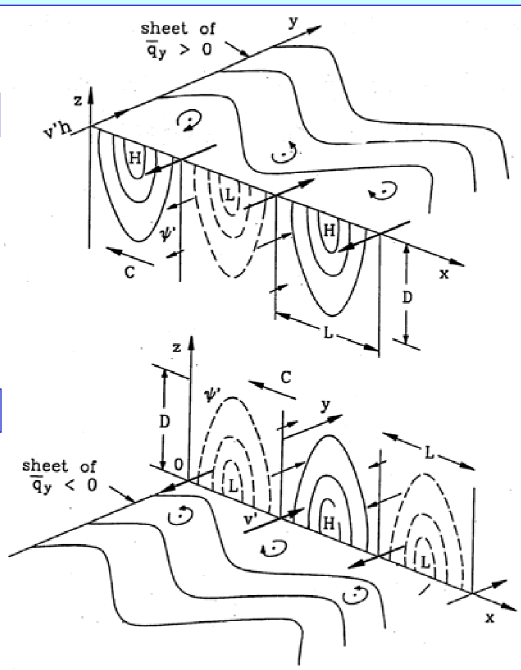
Baroclinic instability: the Eady problem



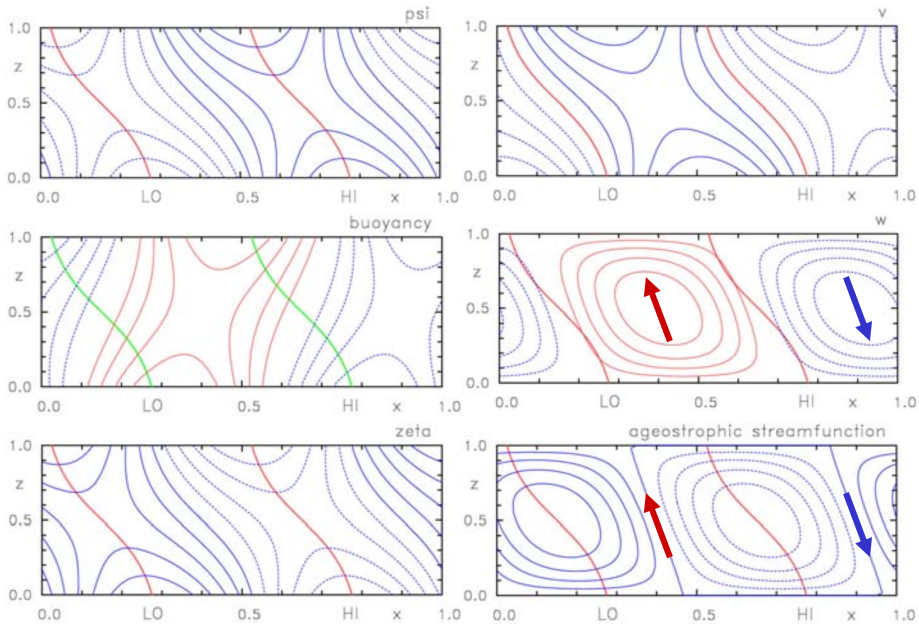
Zonal flow configuration in the Eady problem (northern hemisphere).

Upper boundary

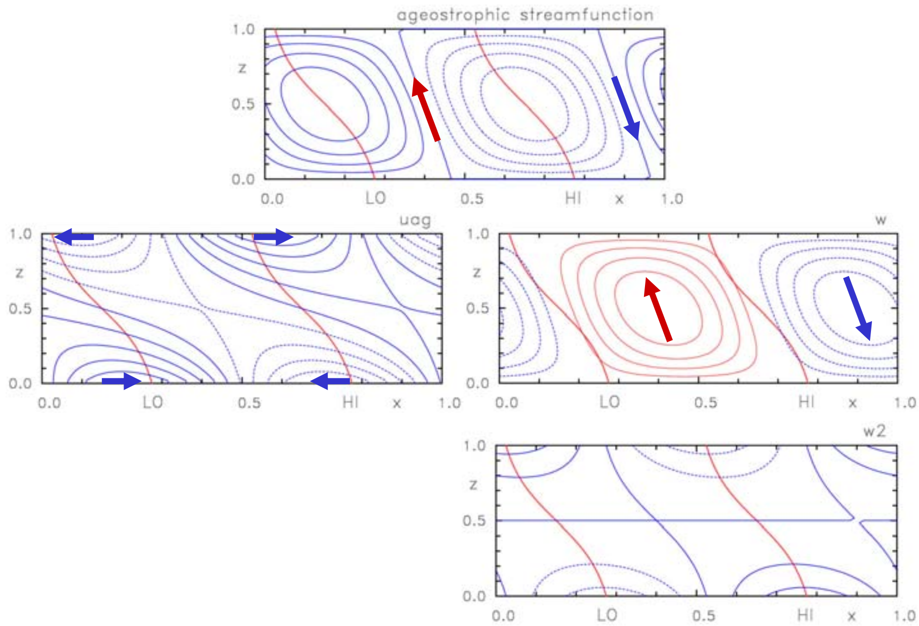
Lower boundary



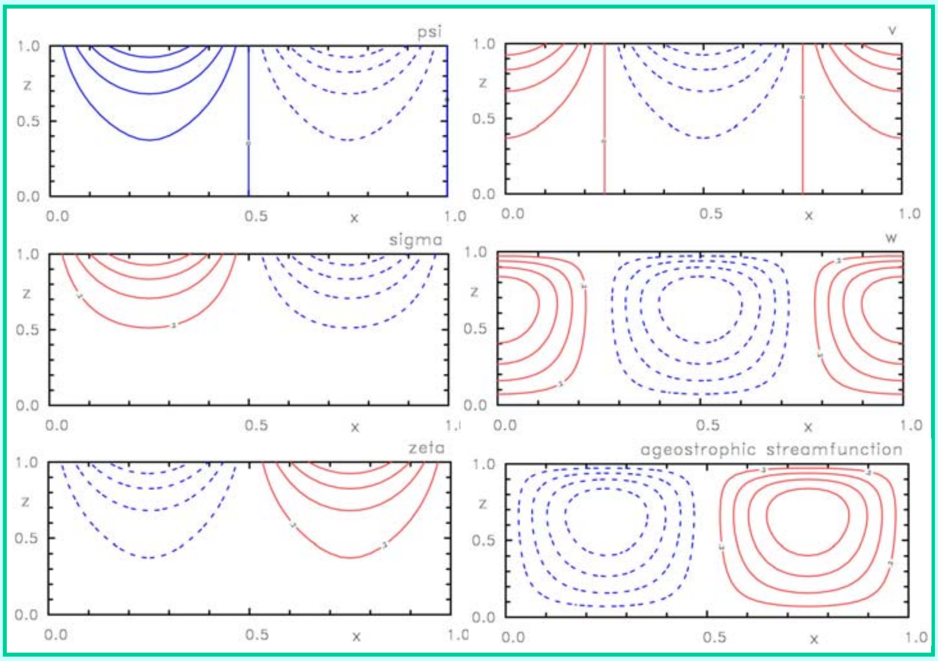
The unstable Eady wave



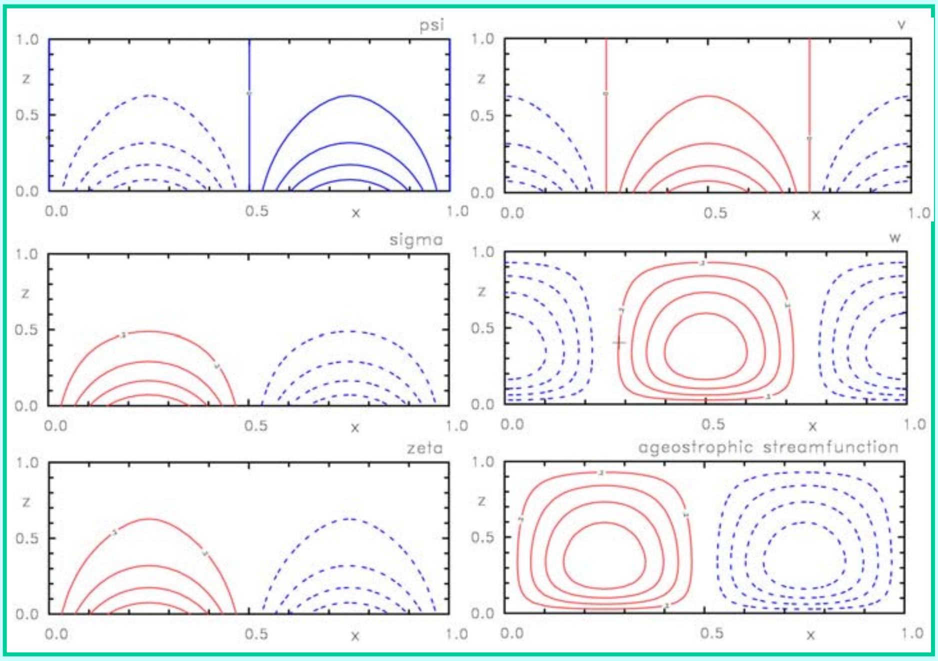
The unstable Eady wave

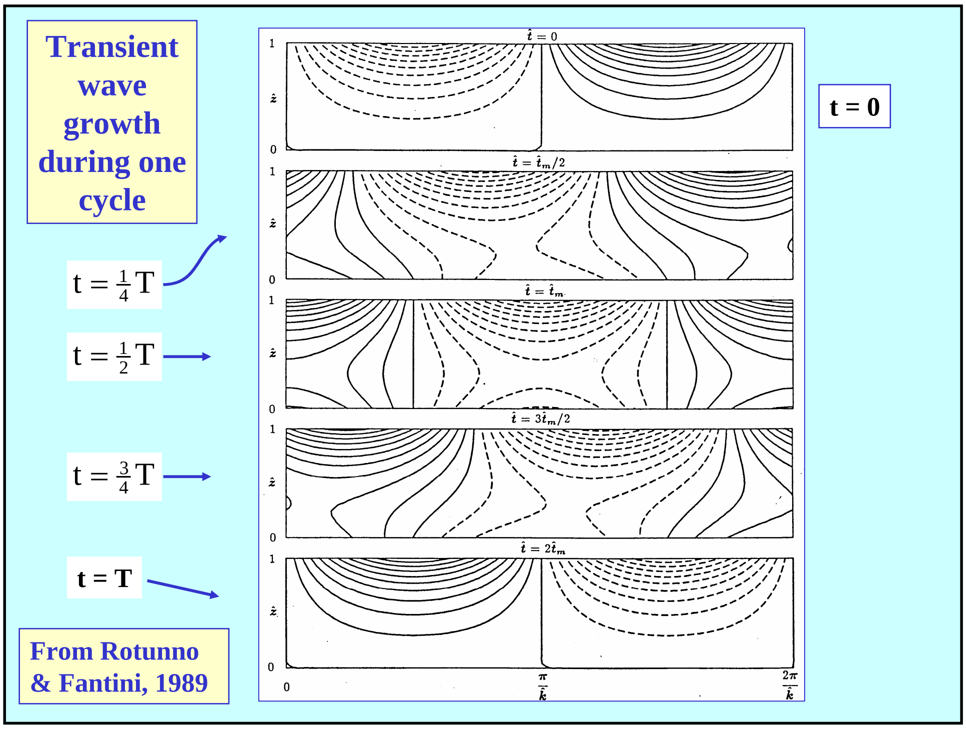


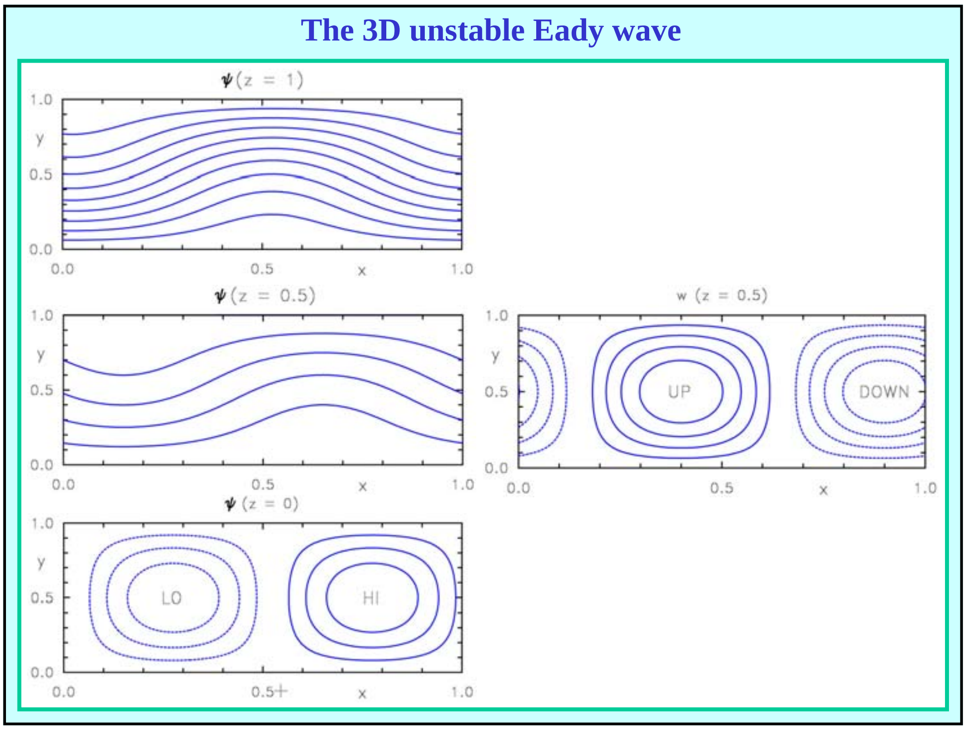
The neutral Eady wave c > 0

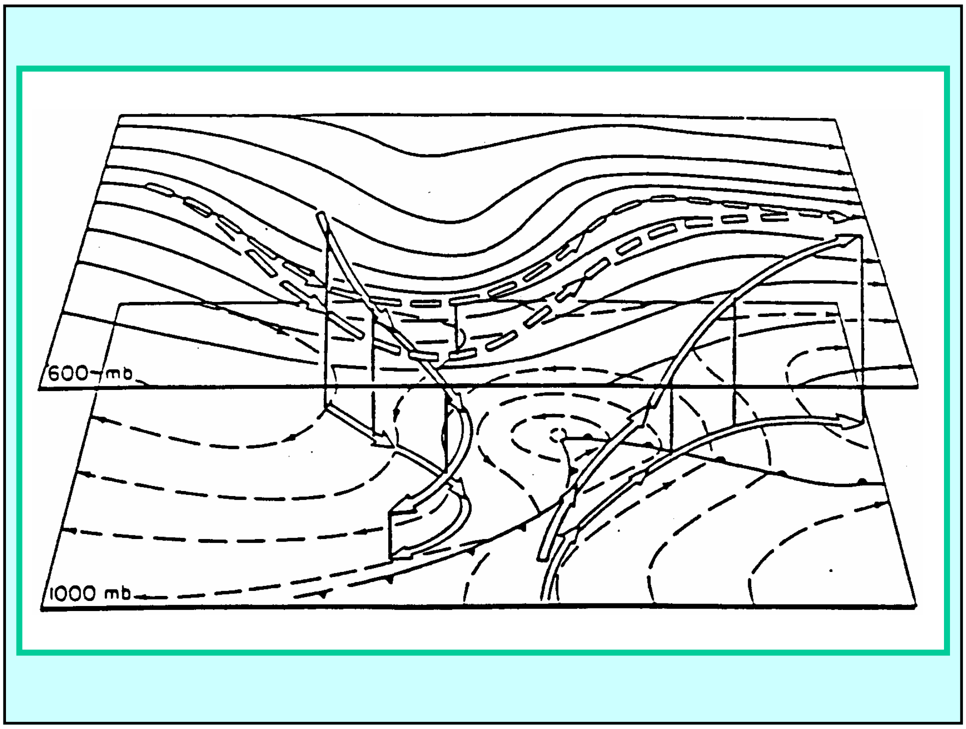


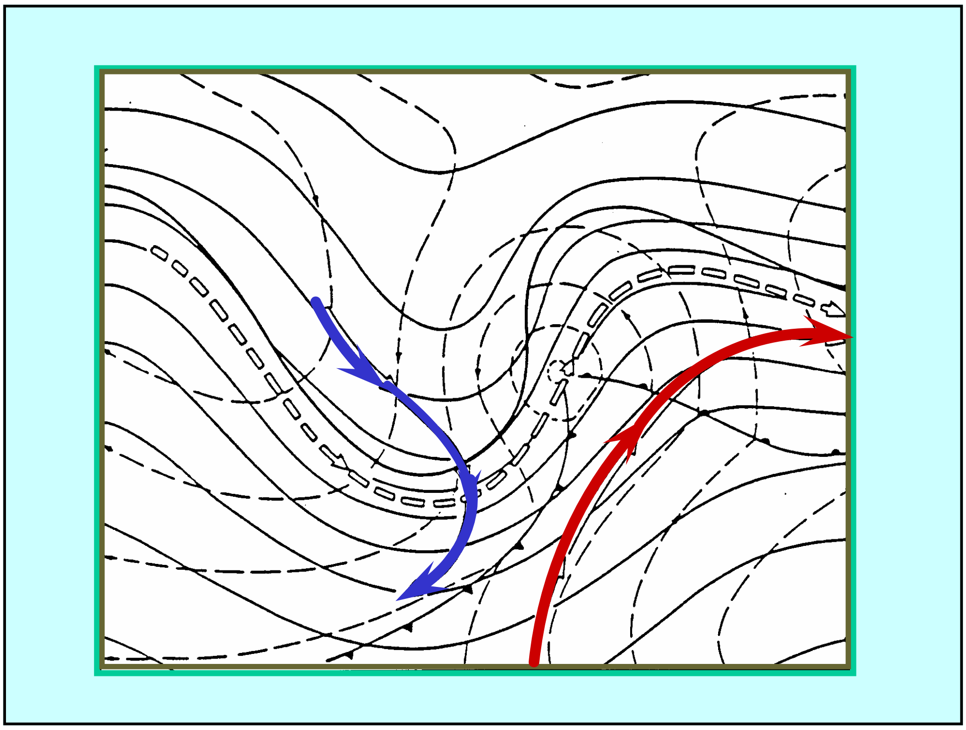
The neutral Eady wave c < 0



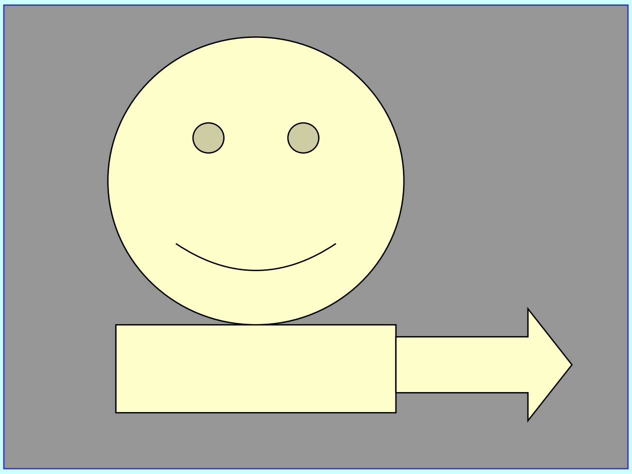

Transient wave growth during one cycle
t = ¼T
t = ½T
t = ¾T
t = T
t = 0
From Rotunno & Fantini, 1989
$\hat{t} = 0$
$\hat{t} = \hat{t}_m/2$
$\hat{t} = \hat{t}_m$
$\hat{t} = 3\hat{t}_m/2$
$\hat{t} = 2\hat{t}_m$
$\hat{z}$
$\frac{\pi}{\hat{k}}$
$\frac{2\pi}{\hat{k}}$

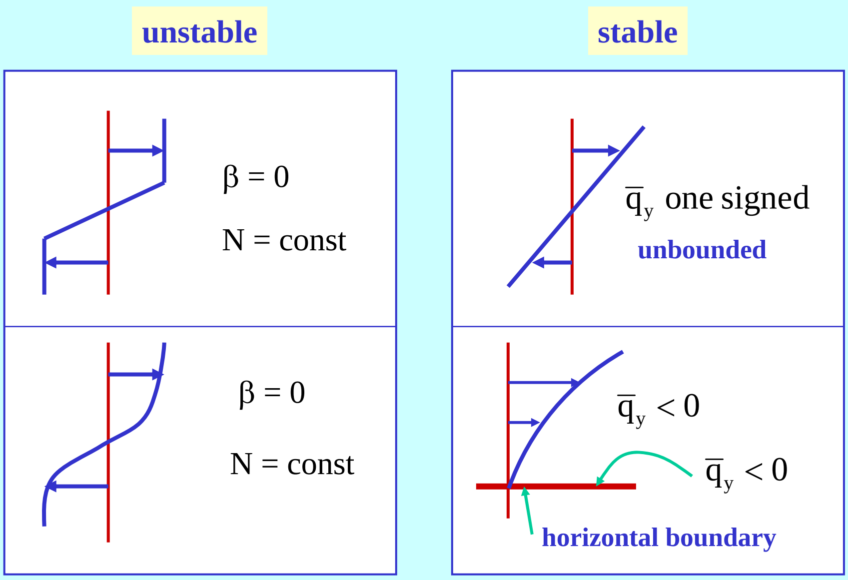

The 3D unstable Eady wave
ψ (z = 1)
ψ (z = 0.5)
w (z = 0.5)
ψ (z = 0)
UP
DOWN
LO
HI
y
x
0.0
0.5
1.0

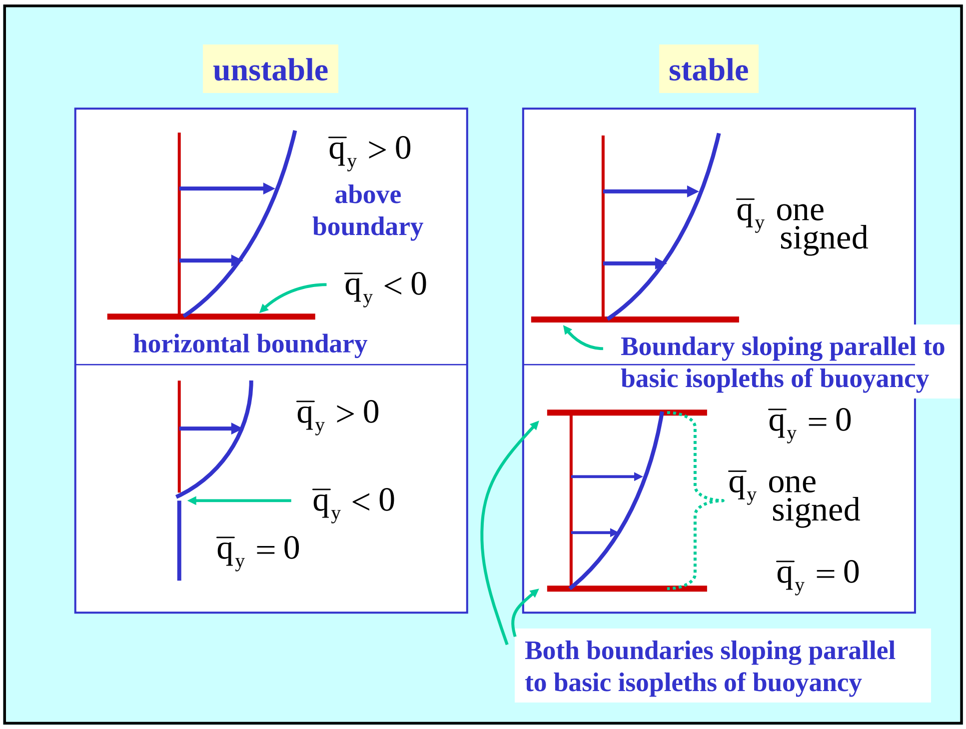

600 mb
1000 mb

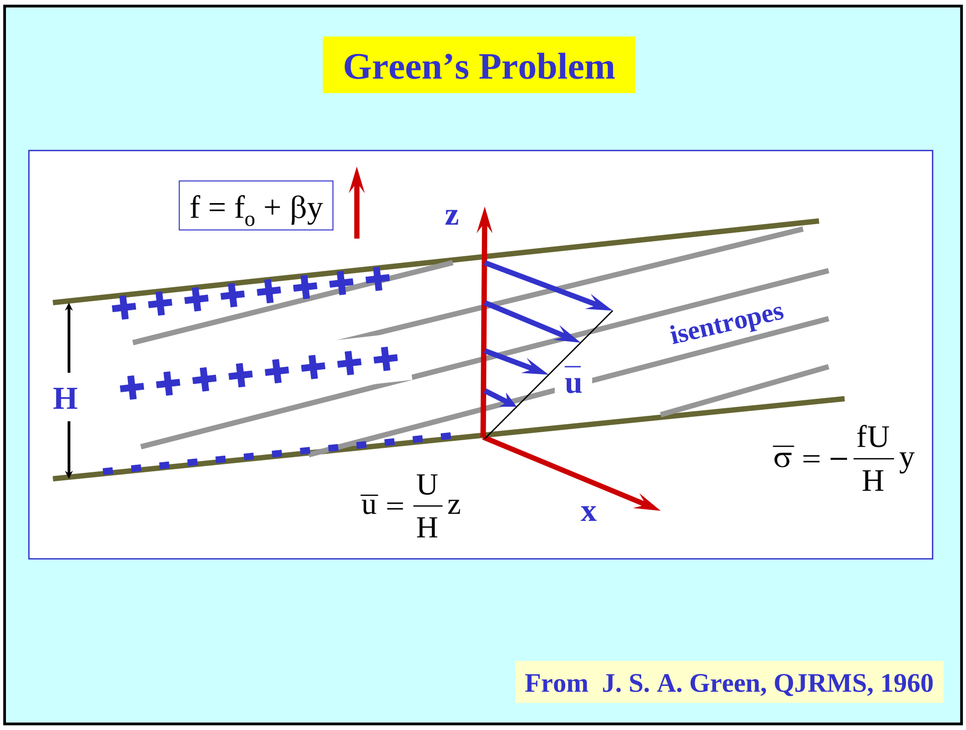


Judging the stability of various flows

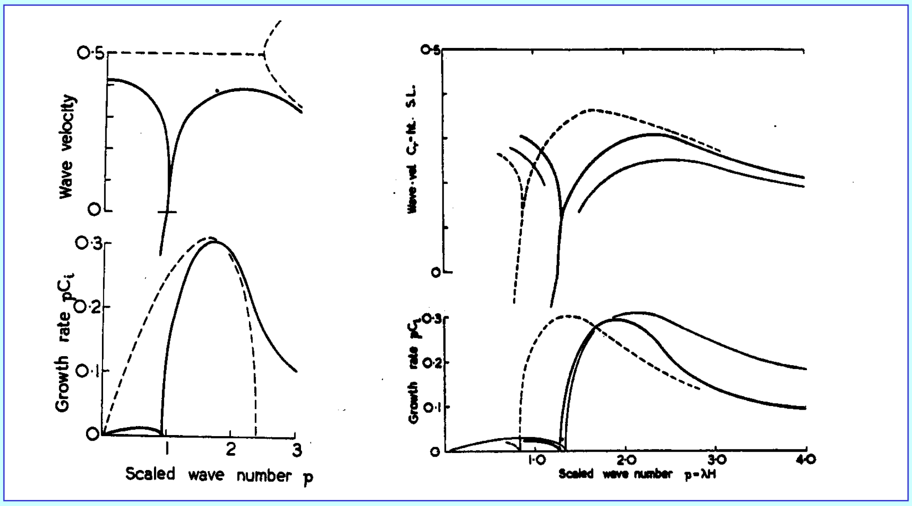





unstable
stable
$\bar{q}_y > 0$
above boundary
$\bar{q}_y < 0$
horizontal boundary
$\bar{q}_y$ one signed
Boundary sloping parallel to basic isopleths of buoyancy
$\bar{q}_y > 0$
$\bar{q}_y < 0$
$\bar{q}_y = 0$
$\bar{q}_y = 0$
$\bar{q}_y$ one signed
$\bar{q}_y = 0$
Both boundaries sloping parallel to basic isopleths of buoyancy

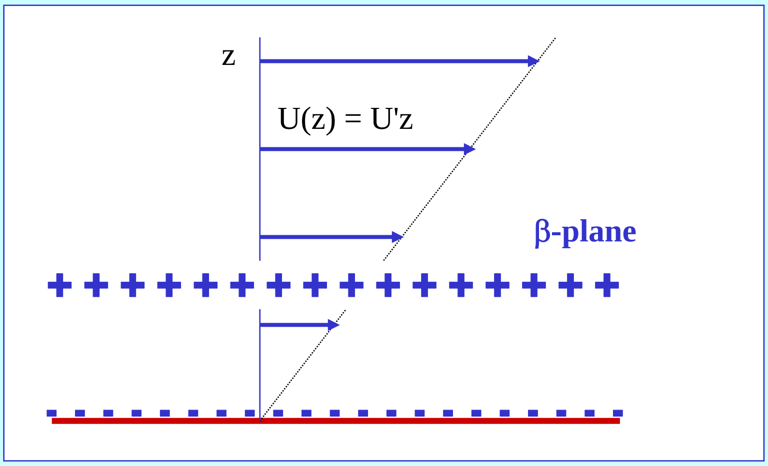

Green's Problem
$f = f_o + \beta y$
z
isentropes
$\bar{u}$
H
$\bar{\sigma} = -\frac{fU}{H}y$
$\bar{u} = \frac{U}{H}z$
x
From J. S. A. Green, QJRMS, 1960

Green's Problem



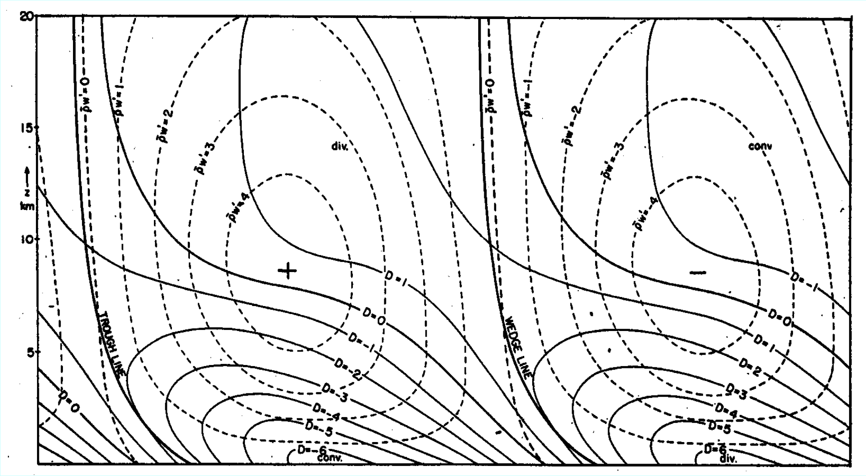
From J. S. A. Green, QJRMS, 1960

Charney's Problem



From J. G. Charney, J. Met., 1947

Charney's Problem



From J. G. Charney, J. Met., 1947

Applications to the atmosphere

- **The question now arises:**

  To what extent can we develop the foregoing ideas to understand the dynamics of synoptic-scale systems in the atmosphere?

- **We address this question in the next lecture**

The Ertel potential vorticity

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On the use and significance of isentropic potential vorticity maps

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