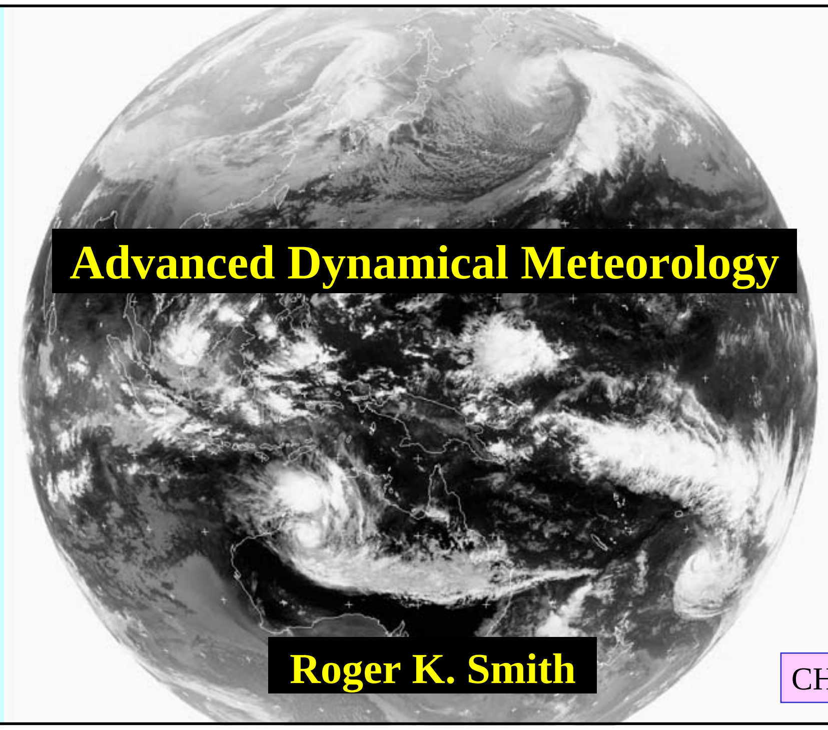
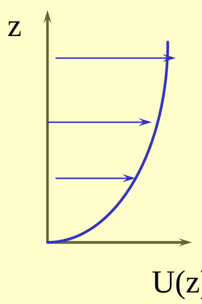


Advanced Dynamical Meteorology

Roger K. Smith

CH 05

Shear instability



Boussinesq fluid

In Chapter 3 we studied stable gravity waves, including those modified by shear.

The relative effects of stratification and shear are characterized by the Scorer parameter:

$$l^2(z) = \frac{N^2}{(U-c)^2} - \frac{U_{zz} + U_z / H_s}{U-c} - \frac{1}{4H_s^2}$$

We consider now the other extreme of sheared motion, possibly modified by stratification.

Cross-differentiate

$$u_t + Uu_x + wU_z = -P_x$$

and

$$w_t + Uw_x = -P_z + \sigma$$

Introduce a streamfunction such that the horizontal vorticity component

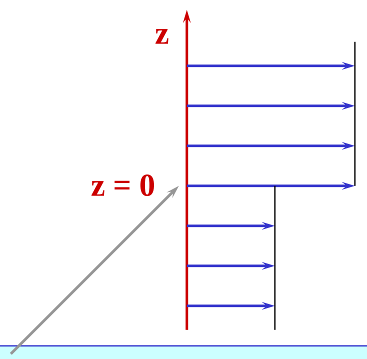
$$u_z - w_x = \nabla^2 \psi$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\nabla^2\psi - \psi_x U_{zz} = -\frac{\partial \sigma}{\partial x}$$

Helmholtz instability

Assume buoyancy effects are unimportant ($\sigma \approx 0$) and suppose that $U(z)$ changes from one relatively uniform value to another in a small height interval.

Idealization:



inviscid, homogeneous fluid

uniform flow, U_1

uniform flow, U_2

The interface $z = 0$ may be regarded as a vortex sheet.

Consider small-amplitude perturbations to this basic flow of the form

$$\psi(x,z,t) = \hat{\psi}(z)\exp[ik(x - ct)]$$

constants

An eigenvalue problem gives $c = c(k)$ which turns out to be complex.

→ The phase speed of the wave is $\mathrm{Re}[c]$.

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\nabla^2\psi - \psi_x U_{zz} = -\frac{\partial\sigma}{\partial x} \;\Rightarrow\; \hat{\psi}_{zz} - k^2\hat{\psi} = 0$$

$$\Rightarrow \begin{cases} \hat{\psi}_1 = A_1 e^{-kz} & \text{for } z > 0 \\ \hat{\psi}_2 = A_2 e^{+kz} & \text{for } z < 0 \end{cases}$$

$$\hat{\psi}_1 = A_1 e^{-kz} \quad \text{for } z > 0$$

$$\hat{\psi}_2 = A_2 e^{+kz} \quad \text{for } z < 0$$

Boundary conditions: pressure along the vortex sheet and vertical displacement of the sheet are continuous.

(as usual these condition are linearized to $z = 0$).

Exercise 3.6 → $\psi/(c - U)$ and $(c - U)\psi_z$ are continuous at $z = 0$.

$$\Rightarrow \quad c = \tfrac{1}{2}(U_1 + U_2) \pm \tfrac{1}{2}i(U_1 - U_2)$$

perturbation travels with the average flow speed

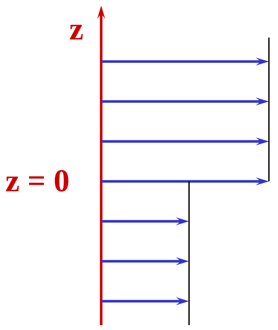
growth rate $kc_i = k(U_1 - U_2)$

Kelvin-Helmholtz instability

It happens frequently in the atmosphere that the region of strong shear coincides with a sharp, stable density gradient.

Idealization:

inviscid, homogeneous fluid layers



uniform flow, U_1
density ρ_1

$\rho_1 < \rho_2$

uniform flow, U_2
density ρ_2

Exercise 3.6

Show that continuity of interface displacement and total pressure at the mean position of the interface z = 0 requires that

$$\hat{\psi}\,/\,(c - U) \quad \text{and} \quad -\rho g \hat{\zeta} + \rho (c - U)\hat{\psi}_z$$

are both continuous at z = 0.

sinusoidal interface displacement $\zeta(x, t) = \mathrm{Re}[\hat{\zeta} e^{ik(x - ct)}]$

Exercise 5.1 Show that

$$\hat{\psi}/(c-U) \quad \text{and} \quad -\rho g\hat{\zeta} + \rho(c-U)\hat{\psi}_z \quad \text{at } z = 0$$

$$c = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[c_0^2 - \rho_1\rho_2 \left[\frac{U_1 - U_2}{\rho_1 + \rho_2} \right]^2 \right]^{\frac{1}{2}}$$

where $c_0^2 = \frac{g}{k}\left[\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right]$

c_0 is the speed of interfacial waves in the absence of mean currents (i.e. when U_1 and U_2 are both zero)

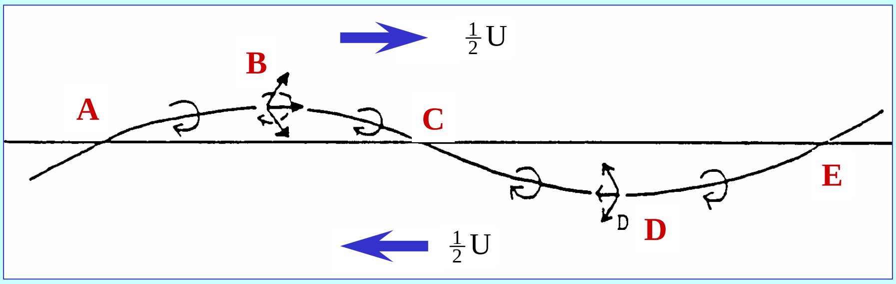
$$c = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[c_0^2 - \rho_1\rho_2 \left[\frac{U_1 - U_2}{\rho_1 + \rho_2} \right]^2 \right]^{\frac{1}{2}}$$

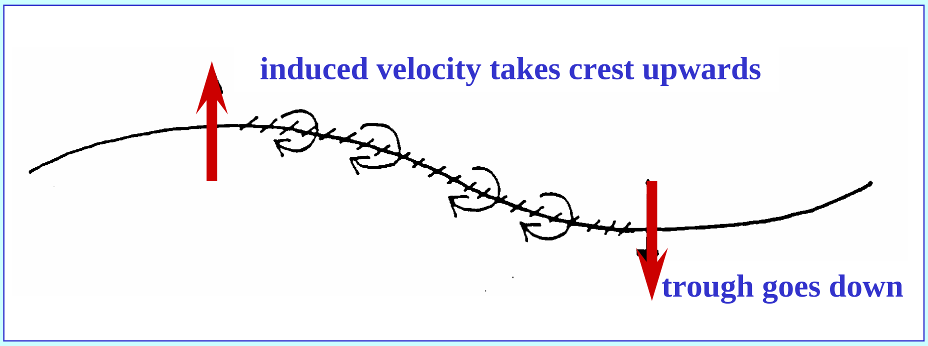
Kelvin-Helmholz flow is unstable to small-amplitude perturbations (with $\mathrm{Im}(c) > 0$) when

$$\rho_1\rho_2 \left[\frac{U_1 - U_2}{\rho_1 + \rho_2} \right]^2 > c_0^2$$

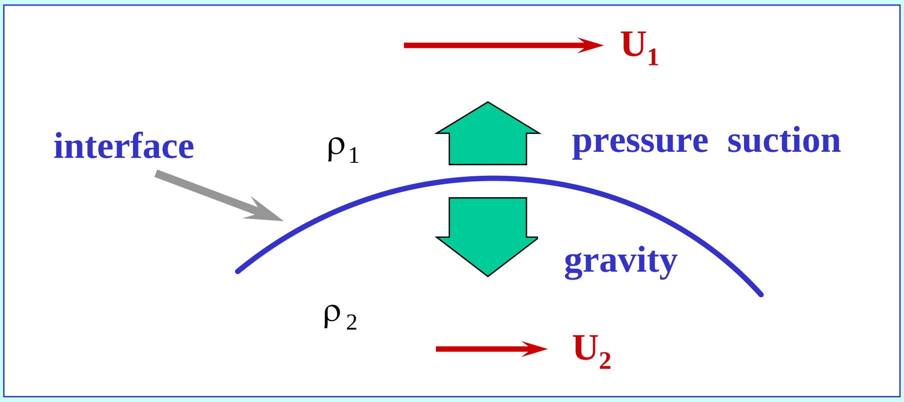
The Helmholtz and Kelvin-Helmholtz instabilities may be interpreted as being due to a redistribution of vorticity in the vortex sheet by the disturbance.

View the perturbation in the frame of reference in which it is stationary. Assume that $\rho_2 - \rho_1 << \rho_1 + \rho_2$

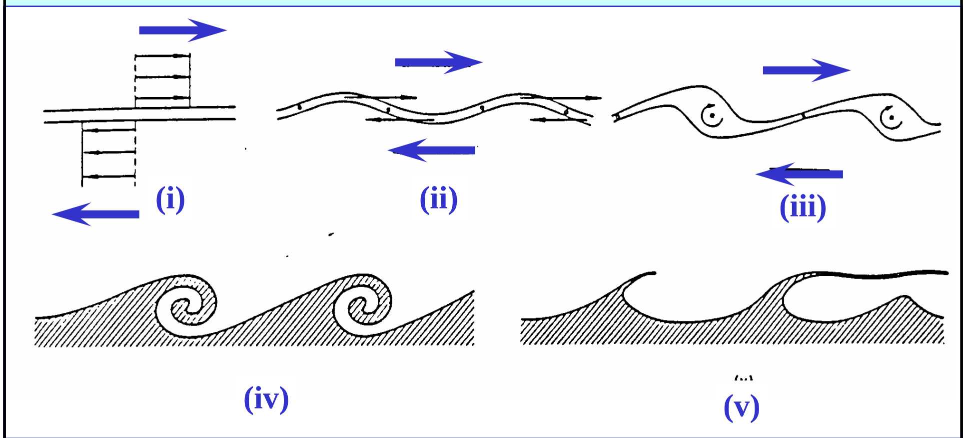


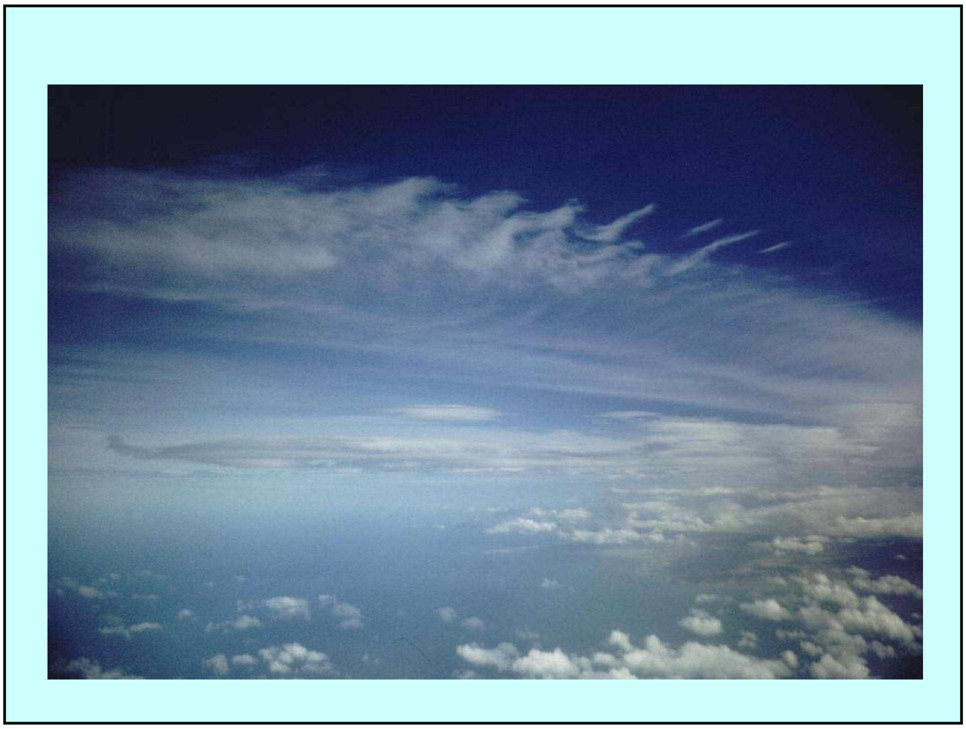


Force balance in Kelvin-Helmholtz instability



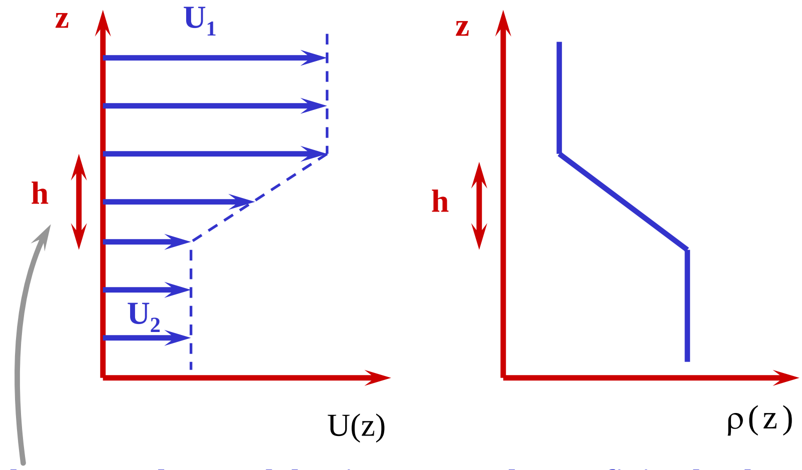
Sequence illustrating the growth of Kelvin-Helmholtz waves





Kelvin-Helmholtz waves displayed in a cloud over Stromness harbour, Orkney Islands, at about 2200 local time on 14th July 2006. The billows on the top of the cloud take the form of waves that curl and break. Kelvin-Helmholtz waves are formed between thermally stable layers of air that contain marked vertical windshear. © Keith Johnson.

Uniform shear model for Kelvin-Helmholtz instability



the vortex sheet and density contrast have a finite depth scale

the velocity shear $\frac{dU}{dz} = (U_1 - U_2)/h$

the density gradient $\frac{d\rho}{dz} = (\rho_2 - \rho_1)/h$

the static stability parameter analogous to N^2 is

$$\frac{g(\rho_2 - \rho_1)}{\overline{\rho} h}$$

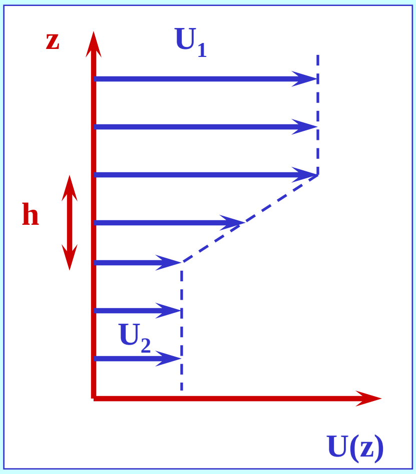
assume $(\rho_2 - \rho_1) << \overline{\rho}$

mean density $\overline{\rho} = \frac{1}{2}(\rho_2 + \rho_1)$

$$\rho_1\rho_2\left[\frac{U_1 - U_2}{\rho_1 + \rho_2}\right]^2 > c_0^2 \quad \Rightarrow \quad \tfrac{1}{4}(U_1 - U_2)^2 > \tfrac{1}{2} g(\rho_2 - \rho_1)/\overline{\rho}k$$

$$\Rightarrow \quad Ri = \frac{N^2}{\left(\frac{dU}{dz}\right)^2} < \tfrac{1}{2}kh$$

Exercise: Uniform shear model for Helmholz instability



$\rho(z)$ = constant

$$\left[c - \tfrac{1}{2}(U_1 + U_2)\right]^2 = \frac{(U_1 - U_2)^2}{4k^2h^2}\left[(1 - kh)^2 - e^{-2kh}\right]$$

Threshold for instability occurs when kh is about 1.3

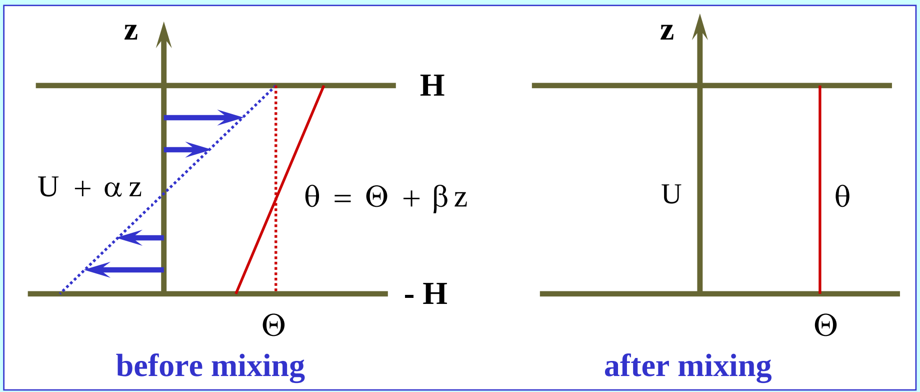
The Richardson number criterion for Kelvin-Helmholz instability

$$Ri = \frac{N^2}{\left(\frac{dU}{dz}\right)^2}$$

The nondimensional quantity Ri **is called the Richardson number.**

- **It is a measure of the stabilizing effect of the stratification compared with the destabilizing effect of the shear.**
- **With the threshold value of** 1.3 **for** kh **as obtained in the Helmholz problem for the case of no stratification, the Richardson number criterion gives** $Ri < 0.65$ **for instability.**

An energy criterion



Before and after states in the complete mixing of a uniform shear layer with a uniform potential-temperature gradient.

Does the total energy (kinetic + potential) increase or decrease?

Assume that no net work is done by the pressure field as a consequence of the mixing process.

$$\text{Change in KE} = \Delta KE = \int_{-H}^{H} \tfrac{1}{2}\bar{\rho}\left[U^2 - (U + \alpha z)^2\right]dz = -\tfrac{1}{3}\bar{\rho} H^3 \alpha^2$$

$$\text{Change in PE} = \Delta PE = \int_{-H}^{H} g(\rho_F - \rho_I) z\,dz$$

final and initial densities

zero pressure change

$$\frac{\Delta\rho}{\rho} = (1-\kappa)\frac{\Delta p}{p} - \frac{\Delta\theta}{\theta} \quad \textbf{or} \quad \frac{\Delta\rho}{\rho} = \frac{\rho_F - \rho_I}{\bar{\rho}} = \frac{\theta_I - \theta_F}{\Theta} = \frac{\beta z}{\Theta}$$

$$\Delta PE = (\bar{\rho} g \beta / \Theta)\int_{-H}^{H} z^2 dz = \tfrac{2}{3}\bar{\rho} g \beta H^3 / \Theta$$

$$\Delta KE = \int_{-H}^{H} \tfrac{1}{2}\bar{\rho}\left[U^2 - (U + \alpha z)^2\right]dz = -\tfrac{1}{3}\bar{\rho} H^3 \alpha^2$$

$$\Delta PE = (\bar{\rho} g \beta / \Theta)\int_{-H}^{H} z^2 dz = \tfrac{2}{3}\bar{\rho} g \beta H^3 / \Theta$$

Hence, the total energy is decreased if

$$\Delta(KE + PE) < 0, \text{ i.e. if } \tfrac{1}{3}\bar{\rho} H^3\left(2g\beta/\Theta - \alpha^2\right) < 0$$

$$Ri < \tfrac{1}{2}$$

Instability of stratified shear flows

Inviscid, Boussinesq fluid

$\eta(x,z,t)$ = **vertical displacement of a fluid parcel**

For linear disturbances $\eta_t + U\eta_x = w$

Assume $\eta(x,z,t) = \hat{\eta}(z)\exp[ik(x-ct)]$ etc.

$$\hat{w} = ikV\hat{\eta}$$

$$\Rightarrow \quad (V^2\hat{\eta}_z)_z + (N^2 - k^2V^2)\hat{\eta} = 0$$

Consider a flow between horizontal rigid boundaries at $z = 0$ **and** $z = d$. $\Rightarrow$ $\hat{\eta}(0) = \hat{\eta}(d) = 0$

Miles' theorem

If the Richardson number $Ri = \dfrac{N^2}{U_z^2} \geq \dfrac{1}{4}$ **everywhere, then the flow is stable.**

Proof **Put** $G(z) = V^{1/2}\hat{\eta}(z)$ $\Rightarrow$ $V^{1/2}\hat{\eta}_z = G_z - \tfrac{1}{2}(V_z/V)G$

$(V^2\hat{\eta}_z)_z + (N^2 - k^2V^2)\hat{\eta} = 0$ $\Rightarrow$

$$V^{1/2}(V^{3/2}\hat{\eta}_z)_z + \tfrac{1}{2}(V_z/V^{1/2})V^{3/2}\hat{\eta} + (N^2 - k^2V^2)\hat{\eta} = 0$$

Substitution for $\hat{\eta}$ **and** $\hat{\eta}_z$ **gives**

$$(VG_z)_z - [\tfrac{1}{2}U_{zz} + k^2V + V^{-1}(\tfrac{1}{4}U_z^2 - N^2)]G = 0$$

$$(VG_z)_z - [\tfrac{1}{2}U_{zz} + k^2 V + V^{-1}(\tfrac{1}{4}U_z^2 - N^2)]G = 0$$

Multiply by the complex conjugate G* **of** G **and integrate** $\int_0^d dz$

Note that

$$\int_0^d G^*(VG_z)_z dz = \int_0^d G^* d(VG_z) - \int_0^d VG_z G_z^* dz$$

$$\int_0^d [V(|G_z|^2 + k^2|G|^2) + \tfrac{1}{2}U_{zz}|G|^2 + (\tfrac{1}{4}U_z^2 - N^2)V^*|G/V|^2]dz = 0$$

If $c_i \neq 0$ **the imaginary part gives**

$$\int_0^d [V(|G_z|^2 + k^2|G|^2)dz + \int_0^d (N^2 - \tfrac{1}{4}U_z^2)|G/V|^2 dz = 0$$

$$\int_0^d [V(|G_z|^2 + k^2|G|^2)dz + \int_0^d (N^2 - \tfrac{1}{4}U_z^2)|G/V|^2 dz = 0$$

If $N^2 > \tfrac{1}{4}U_z^2$ **i.e. if** $Ri > \tfrac{1}{4}$**,** k **and** G_z **must be identically zero**

$G =$ **constant**

$$\hat{\eta}(z) = V^{-1/2}G(z) \propto [U(z) - c]^{-1/2}$$

Hence, if $Ri \geq \tfrac{1}{4}$ **everywhere, the flow is stable. q.e.d.**

Howard's semi-circle theorem

The complex wave speed c of any unstable mode must lie inside the semi–circle in the upper half of the c–plane with the range of U as its diameter.

Proof **Assume that** $c_i \neq 0$

Multiply $(V^2\hat{\eta}_z)_z + (N^2 - k^2V^2)\hat{\eta} = 0$ **by** $\hat{\eta}^*$ **and**

$\int_0^d dz$

$$\int_0^d (U-c)^2(|\hat{\eta}_z|^2 + k^2|\eta|^2)dz = \int_0^d N^2|\hat{\eta}|^2 dz$$

$$\int_0^d (U-c)^2(|\hat{\eta}_z|^2 + k^2|\eta|^2)dz = \int_0^d N^2|\hat{\eta}|^2 dz$$

Put $Q = |\hat{\eta}_z|^2 + k^2|\eta|^2$ **and take the real part**

$$\int_0^d [(U-c_r)^2 - c_i^2]Q\,dz = \int_0^d N^2|\hat{\eta}|^2 dz$$

and imaginary part

$$\int_0^d UQ\,dz = c_r\int_0^d Q\,dz$$

These equations can be combined to give

$$\int_0^d [U^2 - (c_r^2 + c_i^2)]Q\,dz - \int_0^d N^2|\hat{\eta}|^2\,dz = 0$$

Now

$$0 \geq \int_0^d (U - U_{min})(U - U_{max})Q\,dz$$

$$= \int_0^d [U^2 - U(U_{min} + U_{max}) + U_{min}U_{max}]Q\,dz$$

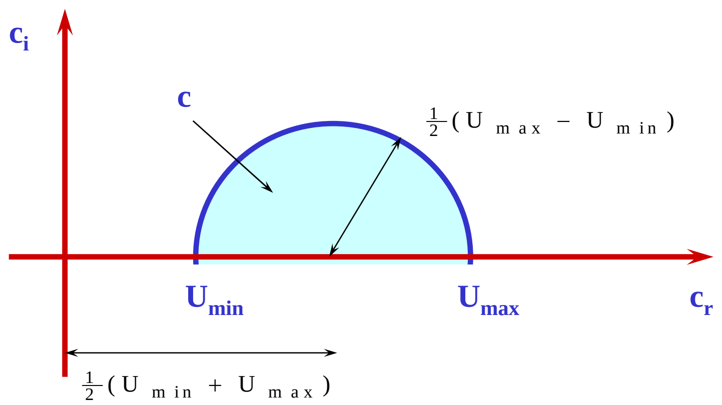
$$= \int_0^d [(c_r^2 + c_i^2) - (U_{min} + U_{max})c_r + U_{min}U_{max}]Q\,dz$$

$$+ \int_0^d N^2|\hat{\eta}|^2\,dz$$

➡ $$c_r^2 + c_i^2 - (U_{min} + U_{max})c_r + U_{min}U_{max} \leq 0$$

➡ $$[c_r - \tfrac{1}{2}(U_{min} + U_{max})]^2 + c_i^2 \leq \tfrac{1}{4}(U_{max} - U_{min})^2$$

Howard's semi-circle theorem



All eigenvalues of unstable modes lie in the semi-circle

Bound on the growth rates

$$\int_0^d [V(|G_z|^2 + k^2|G|^2)dz + \int_0^d (N^2 - \tfrac{1}{4}U_z^2)|G/V|^2 dz = 0$$

$$\Rightarrow \quad k^2\int_0^d |G|^2 dz = \int_0^d (\tfrac{1}{4}U_z^2 - N^2)|G/V|^2 dz - \int_0^d |G_z|^2 dz$$

Since $|U - c_r - ic_i| \geq c_i^2$ **and** $|V|^2 \leq c_i^{-2}$

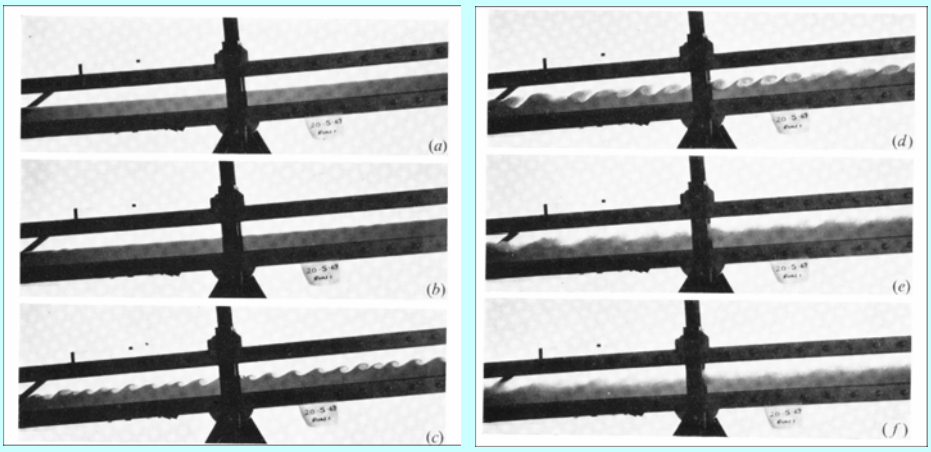
$$k^2\int_0^d |G|^2 dz \leq c_i^{-2}\max(\tfrac{1}{4}U_z^2 - N^2)\int_0^d |G|^2 dz$$

$$k^2 c_i^2 \leq \max(\tfrac{1}{4}U_z^2 - N^2)$$

This result contains Miles' theorem because

$(N^2 \geq \tfrac{1}{4}U_z^2) \Rightarrow k^2 c_i^2 \leq 0$ **and the flow is stable**

KH Instability in the laboratory



From S. A. Thorpe, *J. Fluid Mech.*, 46, 299-319