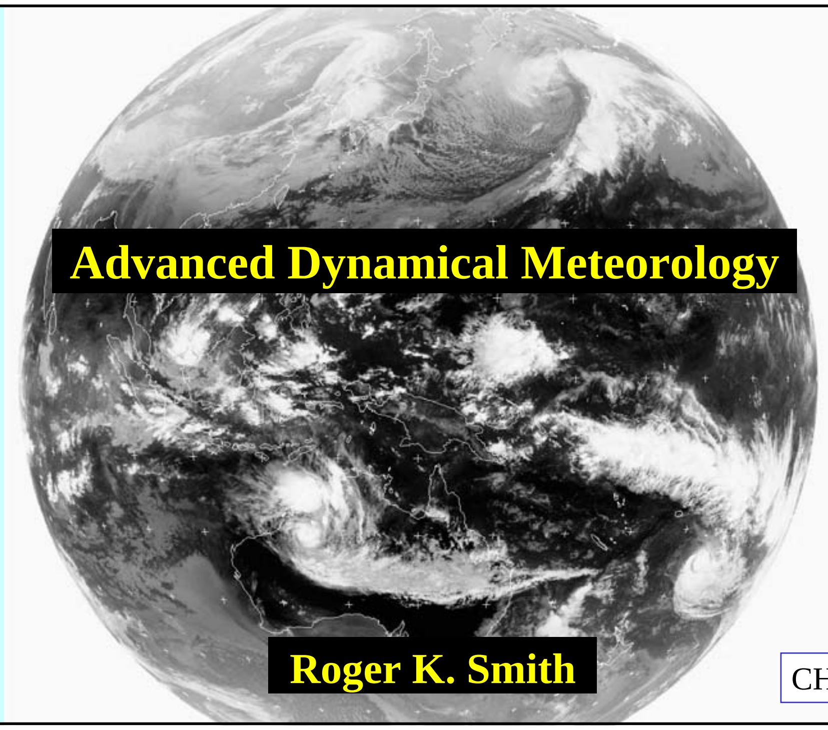
Advanced Dynamical Meteorology



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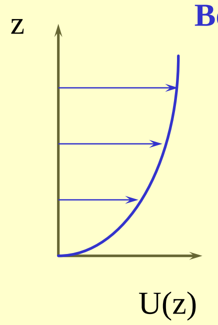
CH 04

Energetics of waves on stratified shear flows



Boussinesq fluid

Wave energy equation (see Ex. 3.4)

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z$$

E = **mean wave energy density**

F = **mean rate of working of the disturbance pressure force in the vertical** = $\overline{pw}$

$$F = \overline{pw}$$

In the case of a non-moving medium (U = 0), F is interpreted as a mean flux of wave energy and equals Ew_g (Ex. 2.11).

It is tempting to retain this interpretation of F in a moving fluid and to regard the term U_z in

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z$$

as a 'source' of mean wave energy associated with the interaction of the wave with the basic shear, U_z.

This interpretation can be misleading!

Question: What is meant by 'energy flux' in a moving medium?

To see why $-\rho\overline{uw}U_z$ **in** $\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z$ **ought not to be interpreted as an energy source, consider the equation for the mean flow to second-order in wave amplitude.**

Full horizontal momentum equation in flux form

$$\rho u^*_t + (\rho u^{*2})_x + (\rho u^* w)_z = -p_x$$

u^* **is the total horizontal wind speed**

Put $u^* = \overline{u}(z,t) + u(x,z,t)$

u = **deviation from the mean wind** $\overline{u}$

mean refers to an average over a wavelength: $\overline{(\ )} = \frac{1}{\lambda}\int_0^{\lambda} (\ )dx$

or for an non-periodic disturbance which vanishes as $x \to \pm\infty$:

$$\overline{(\ )} = \int_{-\infty}^{\infty} (\ )dx$$

Substitute for u* in $\rho u_t^* + (\rho u^{*2})_x + (\rho u^* w)_z = -p_x$ and average

→ **the mean flow momentum equation**

$$\rho\bar{u}_t + \left(\rho\overline{uw}\right)_z = 0$$

Assume that the wave amplitude is sufficiently small

→ $\bar{u} = U(z)$ and $\bar{w} = 0$

→ the interaction between the perturbation and the basic flow can be ignored.

However, the waves have a second-order effect in amplitude on the mean flow governed by

$$\rho\bar{u}_t + \left(\rho\overline{uw}\right)_z = 0$$

(see ADM)

$\times\ \bar{u} \Rightarrow$ **the mean flow kinetic energy equation**

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\bar{u}^2\right) = -\bar{u}\left(\rho\overline{uw}\right)_z$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho \bar{u}^2\right) = -\bar{u}\left(\rho\overline{uw}\right)_z$$

local second-order changes in the mean flow are associated with nonzero values of $\bar{u}\left(\rho\overline{uw}\right)_z$

this term should appears as the "source term" in

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z$$

Rewrite as
$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{pw} + U\rho\overline{uw}\right) + U\left(\rho\overline{uw}\right)_z$$

Interpret $F = \overline{pw} + U\rho\overline{uw}$ as the total or net energy flux.

Not Galilean invariant

Perturbation analysis

Put $u = \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$ and $\bar{u} = U(z) + \varepsilon^2 \bar{u}_2(z,t) + \ldots$

where $\varepsilon << 1$ and the u_i etc. are O(1), in $\rho\bar{u}_t + \left(\rho\overline{uw}\right)_z = 0$

$$\rho\bar{u}_{2t} + \left(\rho\overline{u_1 w_1}\right)_z = 0$$

Assumes that the wave-induced vertical motion is zero at $O(\varepsilon^2)$

Wave energy equation

$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{pw} + U\rho\overline{uw}\right) + U\left(\rho\overline{uw}\right)_z$$

Mean flow kinetic energy

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho \bar{u}^2\right) = -\bar{u}\left(\rho\overline{uw}\right)_z$$

At $O(\varepsilon^2)$
$$\frac{\partial}{\partial t}(E + \rho U\bar{u}_2) = -\frac{\partial}{\partial z}\left(\overline{p_1 w_1} + U\rho\overline{u_1 w_1}\right)$$

$$\frac{\partial}{\partial t}(E + \rho U \bar{u}_2) = -\frac{\partial}{\partial z}\left(\overline{p_1 w_1} + U\rho\overline{u_1 w_1}\right)$$

wave energy plus mean flow energy

$$\rho U \bar{u}_2 = \tfrac{1}{2}\rho(U + \varepsilon^2 \bar{u}_2 + \ldots)^2 - \tfrac{1}{2}\rho U^2$$

to $O(\varepsilon^2)$

the divergence of the vertical advective flux of total kinetic energy

$$\tfrac{1}{2}\rho(U + \varepsilon u_1 + \ldots)^2 \quad \text{at} \quad O(\varepsilon^2)$$

= the divergence of the first nonzero term in the expression

$$\tfrac{1}{2}\rho(U + \varepsilon u_1 + \ldots)^2 \;\; \varepsilon w_1$$

Exercise

Show that the perturbation and mean flow equations:

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z \qquad \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\bar{u}^2\right) = -\bar{u}\left(\rho\overline{uw}\right)_z$$

form an energetically closed system in the sense that, for a free wave with

$$F(0) = 0 \quad \textbf{and} \quad F \to 0 \;\; \text{as} \;\; z \to \infty$$

recall that $F = \overline{pw}$

$$\int_0^\infty \left(E + \tfrac{1}{2}\rho\bar{u}^2\right)dz \quad \textbf{is a constant}$$

The nonacceleration theorem

The perturbation equations for waves in a Boussinesq fluid are:

$$u_t + Uu_x + wU_z = -P_x$$
$$w_t + Uw_x = -P_x + \sigma$$
$$\sigma_t + U\sigma_x + N^2 w = 0$$
$$u_x + w_z = 0$$

Look for steady travelling wave solutions of the form

$$(u, w, P, \sigma) = \mathrm{Re}\left[(\hat{u}(z), \ldots.)e^{ik(x-ct)}\right]$$

$$ik(U-c)\hat{u} + U_z\hat{w} = -ik\hat{P}$$
$$ik(U-c)\hat{w} = -\hat{P}_z + \hat{\sigma}$$
$$ik(U-c)\hat{\sigma} + N^2\hat{w} = 0$$
$$ik\hat{u} + \hat{w}_z = 0$$

Put V = U - c

$$\hat{\sigma} = \frac{iN^2}{kV}\hat{w}$$

$$\hat{P} = -\frac{i}{k}(V\hat{w}_z - V_z\hat{w})$$

$$\hat{P}_z = \left[-ikV + \frac{iN^2}{kV}\right]\hat{w}$$

$$\hat{w}_{zz} + \left[\frac{N^2}{V^2} - \frac{V_{zz}}{V} - k^2\right]\hat{w} = 0$$

$$\hat{w}_{zz} + \left[\frac{N^2}{V^2} - \frac{V_{zz}}{V} - k^2\right]\hat{w} = 0$$

This is the Boussinesq form of Scorer's equation

We can write the components of

$$(u, w, P, \sigma) = \mathrm{Re}\left[(\hat{u}(z), \ldots .)e^{ik(x-ct)}\right]$$

as $u = \frac{1}{2}\hat{u}(z)e^{ik(x-ct)} + (\quad)^*$ **etc.**

complex conjugate

Then, for any two dependent variables a **and** b

$$\overline{ab} = \frac{1}{4}\left(\hat{a}\hat{b}^* + \hat{a}^*\hat{b}\right)$$

Multiply $ik(U-c)\hat{u} + U_z\hat{w} = -ik\hat{P}$ **by** $\rho\hat{w}^*/ik$

and use $ik\hat{u} + \hat{w}_z = 0$ →

$$(U-c)\rho\hat{u}\hat{w}^* - \frac{i}{k}\rho U_z|\hat{w}|^2 = -\hat{p}\hat{w}^*$$

Add this equation to its complex conjugate and use

$$\overline{ab} = \frac{1}{4}\left(\hat{a}\hat{b}^* + \hat{a}^*\hat{b}\right)$$

→ $$(U-c)\rho\overline{uw} = -\overline{pw}$$

Also, for wave perturbations such as

$$(u, w, P, \sigma) = \mathrm{Re}\left[(\hat{u}(z), \ldots.)e^{ik(x-ct)}\right]$$

$$F = \overline{pw} + U\rho\overline{uw} = \frac{cF}{c-U} = c\rho\overline{uw}$$

For a steady (in amplitude) sinusoidal wave $\partial E/\partial t = 0$ **and from**

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial z} - \rho\overline{uw}U_z \quad \textbf{and} \quad (U-c)\rho\overline{uw} = -\overline{pw}$$

$$\frac{\partial F}{\partial z} = -\rho\overline{uw}U_z \qquad (U-c)\rho\overline{uw} = -F$$

$$(U-c)\frac{d}{dz}\left(\rho\overline{uw}\right) = 0$$

$$(U-c)\frac{d}{dz}\left(\rho\overline{uw}\right) = 0 \quad \Rightarrow \quad U = c \quad \textbf{or} \quad \frac{d}{dz}\left(\rho\overline{uw}\right) = 0$$

$$\frac{d}{dz}\left(\rho\overline{uw}\right) = 0 \quad \textbf{and} \quad F = c\rho\overline{uw} \quad \Rightarrow \quad \frac{dF}{dz} = 0$$

$$\rho\bar{u}_t = -\left(\rho\overline{uw}\right)_z$$

⇒ **the waves do not force any second-order acceleration of the mean flow**

The nonacceleration theorem

For a steady (in amplitude) sinusoidal wave

$$\frac{d}{dz}\left(\rho\overline{uw}\right) = 0 \quad \textbf{and} \quad \rho\bar{u}_t = -\left(\rho\overline{uw}\right)_z \quad \Rightarrow \quad \bar{u}_t$$

This result is now known as the nonacceleration theorem.

It was first obtained in a slightly less general form by Eliassen and Palm (1960) and has been shown to be a quite general result by Andrews and McIntyre (1978).

The quantity $F = c\rho\overline{uw}$ **is the total vertical flux of wave energy.**

Flow over sinusoidal orography

For waves which radiate vertically ($0 < |k| < l$)**, there exists a downward flux of mean horizontal momentum (** $\rho\overline{uw} < 0$ **).**

$\rho\overline{uw} < 0$ **is independent of height and equal to the drag per unit wavelength exerted by the boundary on the airstream (see Ex. 3.5).**

$$\rho\overline{uw} = -\tfrac{1}{2}\rho U^2 mkh_m^2$$

where sgn(mk) > 0 **for upward propagation.**

Evidently, the momentum flux originates at infinity, where, presumably, the drag exerted by the boundary on the air stream acts. This is at first sight puzzling !

When a general airstream U(z) flows over mountain ridge and produces upward radiating waves, a forward wave drag is exerted on the mountain.

↓

The mountain exerts a drag on the airstream

Question: How is the stress on the airstream distributed? i.e. at what level(s) does the drag act on the airstream?

For steady waves, the nonacceleration theorem rules out the possibility of interaction except at a critical level where $U = c$, a level where the intrinsic frequency of the waves vanishes (see Eq. 3.4).

In the case of stationary mountain waves, $c = 0$.

- Linear theory suggests that at a critical level, the wave is almost completely absorbed, leading to a deceleration of the mean flow at that level.
- However, nonlinear and viscous effects may be important near the critical level.
- I shall not address the critical-layer problem in this course - for further details, see the important paper by Booker and Bretherton (1967).
- For a propagating wave packet with a spectrum of horizontal phase speeds, there may be a range of critical levels.
- Then absorption by the mean flow takes place in a finite layer.
- For stationary mountain waves, $c = 0$ for all Fourier components.

The analysis of unsteady wave development and of wave propagation in arbitrary shear flows is mathematically difficult.

For unsteady wave development Laplace transforms methods for initial-value problems usually lead to uninvertible transforms.

For wave propagation in arbitrary shear flows, typical eigenvalue problems are analytically intractable, or at best, very complicated.

Some analytical progress and further physical insight may be obtained by studying slowly-modulated wave trains or wave packets, in which the waves are locally plane, with wavelength and amplitude varying significantly only over a space scale of many wavelengths

Slowly varying wave trains or wave packets

Theory of Acheson (1976, pp 452 – 455):

Assume the waves have constant frequency ω and horizontal wavenumber k, but their amplitude varies with height and time on scales very long compared with one wavelength and one period, respectively.

Define 'slow' variables: $Z = \alpha z$ and $T = \alpha t$, where $\alpha \ll 1$ is a dimensionless measure of how slowly the wave train is modulated.

Slowly varying means that at any given height/time the wave amplitude varies by a factor $O(1)$ over a height/time scale of $O(\alpha^{-1})$ wavelengths/periods.

Assume a Boussinesq fluid with N constant, but with a basic shear (U(Z),0,0).

Consider linear wave perturbations with streamfunction

$$\psi = \psi_1 + \alpha\psi_2 + \ldots$$

where $$\psi_n = \mathrm{Re}\left[\hat{\psi}_n(Z,T)e^{i(kx+\theta(z)-\omega t)}\right]$$

Similar expansions are taken for other flow quantities.

A local vertical wavenumber defined in terms of the phase function θ(z) is

$$m(Z) = \frac{d\theta}{dz}$$

The multiple-scaling technique

Replace time and height derivatives by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial T} \quad \textbf{and} \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}$$

$$u_t + Uu_x + P_x = -(\alpha u_T + wU_Z)$$

$$w_t + Uw_x + P_z - \sigma = -(\alpha w_T + wP_Z)$$

$$\sigma_t + U\sigma_x + N^2 w = -\alpha\sigma_T$$

$$u_x + w_z = -\alpha w_Z$$

Term describing the effect of vertical shear appears only at O(α).

Zero-order solution

Substitution of the expansions for u, w, P, **gives to** $O(\alpha^0)$ **a locally-plane wave solution, identical with that which would be obtained if** U **were a constant.**

For this solution

$$\omega^{*2} = \frac{N^2 k^2}{k^2 + m^2}$$

$\omega^*(Z) = \omega - kU(Z)$ **is the intrinsic frequency.**

ω^* **is a function of** Z **through** U(Z) **and** m(Z)**.**

First-order solution

At $O(\alpha)$ **in the expansion, the equations for subscript ´2´ quantities become:**

$$i(\omega^* \hat{u}_2 - k\hat{P}_2) = \hat{u}_{1T} + U_Z \hat{w}_1$$

$$i(\omega^* \hat{w}_2 - m\hat{P}_2) + \hat{\sigma}_2 = \hat{w}_{1T} + \hat{P}_{1Z}$$

$$i\omega^* \hat{\sigma}_2 - N^2 \hat{w}_2) = \hat{\sigma}_{1T}$$

$$i(k\hat{u}_2 + m\hat{w}_2) = -\hat{w}_{1Z}$$

Eliminating $\hat{u}_2$**, … etc. we obtain an expression of the form**

$(\dots)\hat{w}_2$ = **expression involving** T **and** Z **derivatives of subscript '1' quantities**

where the coefficient of $\hat{w}_2$ **is zero.**

$(\ldots)\hat{w}_2$ = [expression involving T and Z derivatives of subscript '1' quantities]

This expression, when set equal to zero, gives a solvability condition for the $O(\alpha)$ problem.

This 4 * 4 set of linear equations has the general form $Ax = b$, where A is a 4 * 4 matrix with $\det A = 0$ and x and b are column vectors.

It has a non-unique solution, but only if b is orthogonal to the solution y of the adjoint homogenous problem $A'y = 0$.

Wave action

After some algebraic manipulation (see Appendix to ADM), the solvability condition for $(\ldots)\hat{w}_2 = \ldots$ may be written in the form

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}(w_g A) = 0$$

where

$A = E / \omega^*$ called the wave action

$$E = \tfrac{1}{2}\rho_o(1 + m^2 / k^2)|\hat{w}_1|^2$$

$w_g = \dfrac{\partial \omega}{\partial m}$ is the local vertical component of the group velocity

$\dfrac{\partial A}{\partial T} + \dfrac{\partial}{\partial Z}(w_g A) = 0$ expresses the conservation of wave action

Exercise

Assuming that the mean second-order perturbation to the basic flow forced by the foregoing slowly-varying wave varies only with Z and T, the mean horizontal momentum equation may be written

$$\alpha\left(\partial \bar{u}_2 / \partial t + U_Z \bar{w}\right) = -\left(\overline{u_1 w_1}\right)_z$$

where, from continuity $\bar{w}_z = 0$

Show that
$$\rho \frac{\partial \bar{u}_2}{\partial T} = -k \frac{\partial}{\partial Z}(w_g A)$$

Moreover, show that if $\bar{u}_2 = 0$ in the absence of waves (i.e., when $A = 0$), then

$$\rho \bar{u}_2 = \frac{E}{c - U}$$

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}(w_g A) = 0 \quad \Rightarrow$$

when the amplitude of the wave is independent of time, Aw_g is independent of height.

$\Rightarrow$ $\overline{p_1 w_1} / (c - U)$ is independent of height.

$$\frac{d}{dz}\left(\rho \overline{uw}\right) = 0 \quad \Rightarrow$$

when the wave amplitude is steady, this is true even when no restriction is placed on how fast U varies over a vertical wavelength.

Consider now a wave train set up by the horizontal translation with speed c of a corrugated boundary at z = 0.

The corrugations are assumed to increase gradually in amplitude (from zero at time t) on the slow time scale T as governed by the formula

$$\zeta(x,t;T) = A(T) \cos k(x - ct)$$

$$\Longrightarrow \quad \overline{w} = 0 \quad \text{at} \quad z = 0$$

Thus $\overline{w}_z = 0 \quad \Longrightarrow \quad \overline{w} \equiv 0$

As the slowly-modulated wave train propagates upwards past any given level, the local wave energy density E will slowly increase to a maximum and (in the absence of dissipation) then diminish again to zero as the wave train passes.

According to $$\rho \overline{u}_2 = \frac{E}{c - U}$$

the local modification $\overline{u}_2$ to the mean flow varies similarly, the mean flow being accelerated (decelerated) if c > U (c < U).

If the forcing $\zeta(x,t;T) = A(T) \cos k(x - ct)$ slowly attains a constant amplitude A_0 on the time scale T and persists at that amplitude thereafter, the wave train will consist of a precursor (which contains $O(\alpha^{-1})$ wavelengths and whose amplitude increases with depth from effectively zero to that amplitude A_0 which the source ultimately attains) and a lower part of constant amplitude A_0 extending all the way down to the source.

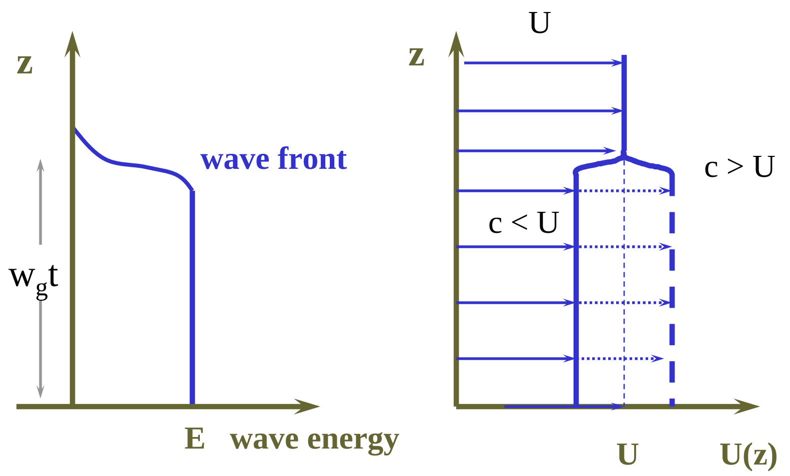
When U is a constant, so are m and w_g and

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}(w_g A) = 0$$

reduces to the statement that amplitude modulations propagate upwards at the group velocity.

In particular, what we call for convenience the '**front**' of the wave train moves upwards at this speed.

the tolerably well-defined highest point at which the amplitude is A_0



Wave energy density E as a function of height for a wave source switched on at $z = 0$ at $t = 0$ (**left**) and corresponding mean flow changes (**right**).

We may now understand the result that for steady flow over sinusoidal topography there is a downward flux of mean horizontal momentum from infinity.

If we imagine such a flow to be established by the gradual evolution of the topography as described by

$$\zeta(x,t;T) = A(T) \cos k(x - ct)$$

with $c = 0$**, it is clear that the source of the downward momentum flux in the steady wave regime (i.e. at heights below** $z = w_g t$**) is the deceleration of the mean flow in the region constituting the front of the wave train. At no finite time is there a momentum flux at infinity.**

The End