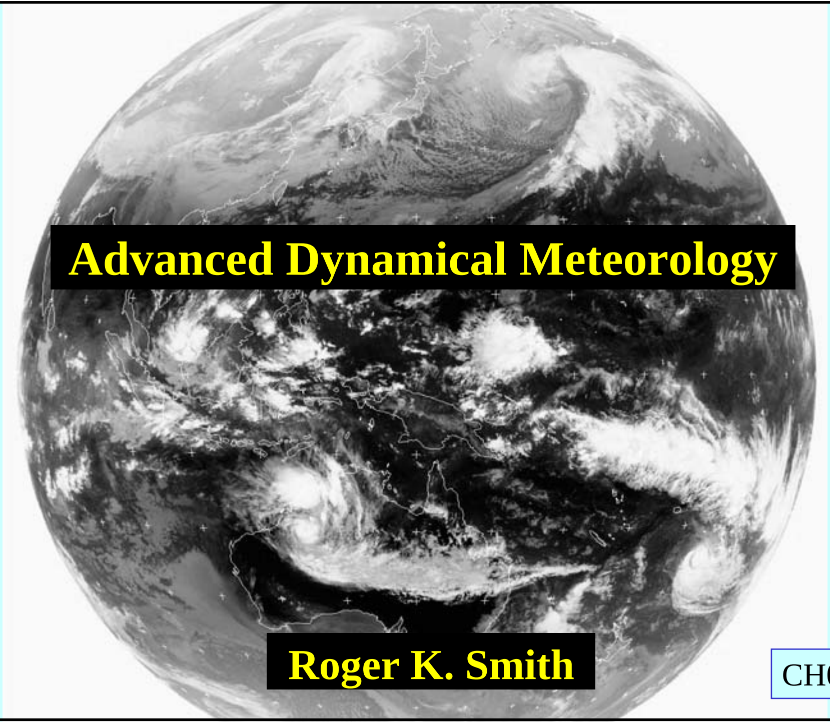


Advanced Dynamical Meteorology

Roger K. Smith

CH03

Waves on moving stratified flows

Small-amplitude internal gravity waves in a stratified shear flow U = (U(z),0,0)**, including the special case of uniform flow** U(z) = constant**.**

Linearized anelastic equations:

$$u_t + Uu_x + wU_z = -P_x$$

$$w_t + Uw_x = -P_z + b$$

$$b_t + Ub_x + N^2 w = 0$$

$$u_x + w_z - w / H_s = 0$$

Travelling wave solutions

Substitute

$$(u, w, P, b) = (\hat{u}(z), \hat{w}(z), P(z), b(z)) \exp[i(kx - \omega t)]$$

→ **a set of ODEs or algebraic relationships between** $\hat{u}(z)$ **etc.**

$$-i(\omega - Uk)\hat{u} + \hat{w}U_z = -ik\hat{P}$$

$$-i(\omega - Uk)\hat{w} = -\hat{P}_z + \hat{b}$$

$$-i(\omega - Uk)\hat{b} + N^2\hat{w} = 0$$

$$iku + \hat{w}_z - \frac{\hat{w}}{H_s} = 0$$

Relate quantities $\hat{u}, \hat{P}, \hat{b}$ **to** $\hat{w}$**:**

$$(\hat{u}, \hat{P}, \hat{b}) = \left[\frac{i}{k}\left(\hat{w}_z - \frac{\hat{w}}{H_s}\right), \frac{i}{k^2}\left\{\omega^*\hat{w}_z + \left(kU_z - \frac{\omega^*}{H_s}\right)\hat{w}\right\}, \frac{N^2\hat{w}}{i\omega^*}\right]$$

$$\omega^* = \omega - Uk$$

is the intrinsic frequency of the wave.

The frequency measured by an observer moving with the local flow speed U(z).

Then the second equation gives an ODE for $\hat{w}$

$$\hat{w}_{zz} - \frac{1}{H_s}\hat{w}_z + \left[\frac{k}{\omega^*}U_{zz} + \frac{k}{\omega^*}\frac{U_z}{H_s} - \left(\frac{1}{H_s}\right)_z + \left(\frac{N^2}{\omega^{*2}} - 1\right)k^2\right]\hat{w} = 0$$

Put $\hat{w}(z) = \exp(z/2H_s)\tilde{w}(z)$

$c = \omega / k$

$\omega^* = k(c - U)$

Then $\tilde{w}_{zz} + (l^2(z) - k^2)\tilde{w} = 0$

Scorer's equation

Scorer parameter

$$l^2(z) = \frac{N^2}{(U-c)^2} - \frac{U_{zz} + U_z / H_s}{U - c} - \frac{1}{4H_s^2}$$

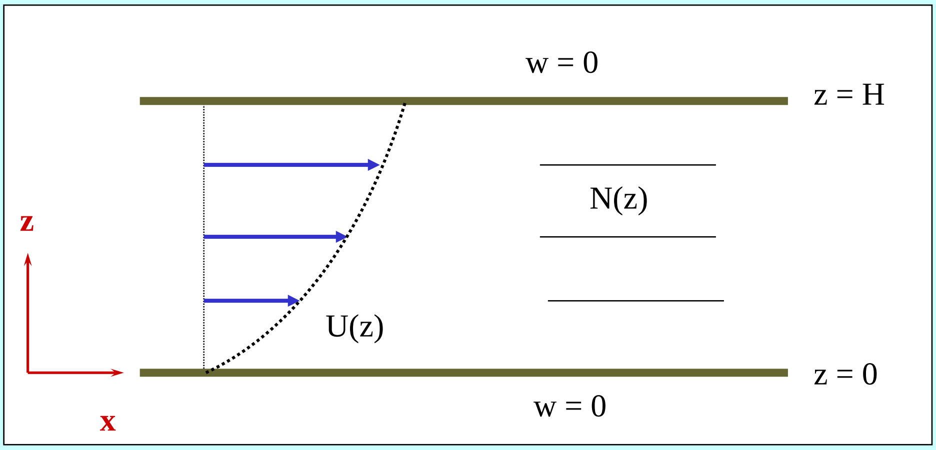
Free waves

- When w satisfies homogeneous boundary conditions (e.g. w = 0) at two particular levels, we have an eigenvalue problem.

$w = 0$ at $z = 0, H$ $\longrightarrow$ $\tilde{w} = 0$ at $z = 0, H$

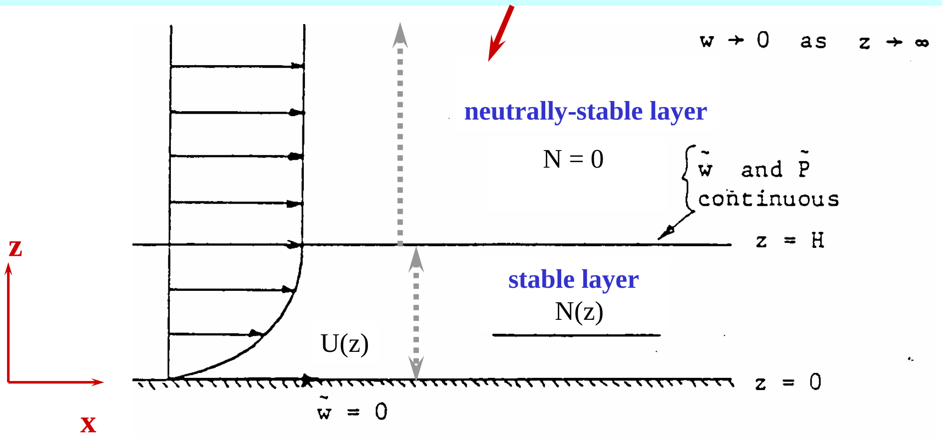
- For a given horizontal wavenumber k, free waves are possible only for certain phase speeds c.
- Alternatively, when c is fixed, there is a constraint on the possible wavenumbers.
- The possible values of c(k), or given c, the possible values of k, and the corresponding vertical wave structure are obtained as solutions of the eigenvalue problem.
- Often the eigenvalue problem is analytically intractable.

Example 1: Stratified shear flow between rigid horizontal boundaries



Example 2: Wave-guide for waves on a surface-based stable layer underlying a deep neutrally-stratified layer

no vertical wave propagation!



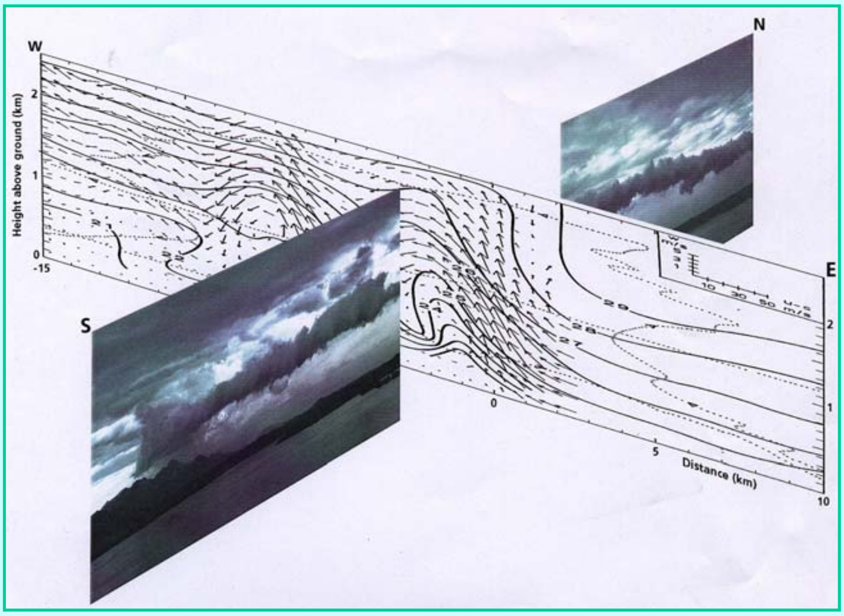
A model for the "Morning-Glory" wave-guide

The Morning-Glory

The Gulf of Carpentaria Region

Morning-Glory
cloud lines

A Morning Glory over Bavaria



Forced waves

- If the waves are being generated at a particular source level, the boundary conditions on Scorer’s equation are inhomogeneous and a different type of mathematical problem arises: -

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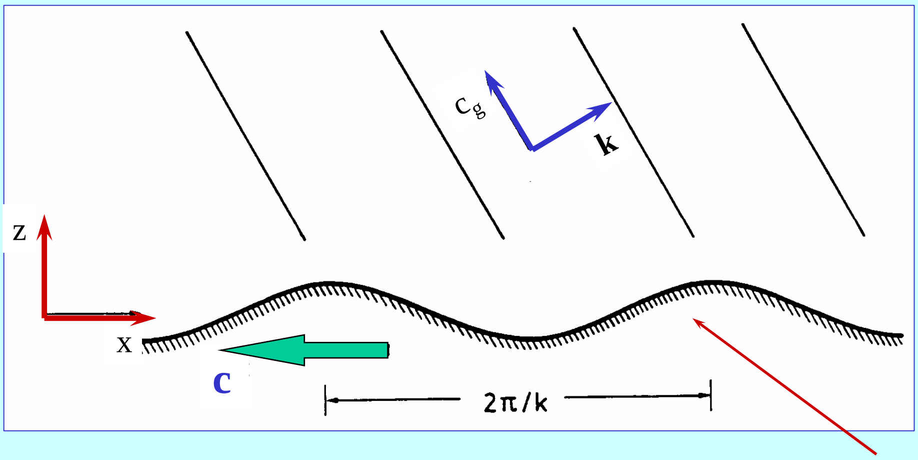
Example 1: Waves produced by the motion of a (sinusoidal) corrugated boundary underlying a stably-stratified fluid at rest.



corrugated boundary

Assume:

1. no basic flow ($U = 0$)
2. the Boussinesq approximation is valid ($1/H_s = 0$)
3. N and c are given constants

➡ the vertical wavenumber m is determined by the dispersion relation

$$m^2 = N^2 / c^2 - k^2$$

c is the speed of the boundary

k is the **horizontal wavenumber**

⬇

the wavenumber of the corrugations

The group velocity of the waves is

$$\mathbf{c_g} = \frac{c^3}{N^2}(m^2, -mk)$$

The vertical flux of mean wave energy is

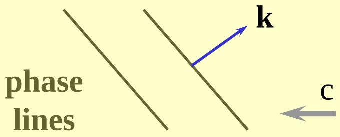
$$F = \overline{pw} = Ew_g$$

the mean wave energy density

the vertical component of the group velocity

Here $c < 0$ ➡ $\operatorname{sgn}(w_g) = \operatorname{sgn}(mk)$

➡ these waves propagate energy vertically



The solution of $\tilde{w}_{zz} + (l^2(z) - k^2)\tilde{w} = 0$ **is determined by**

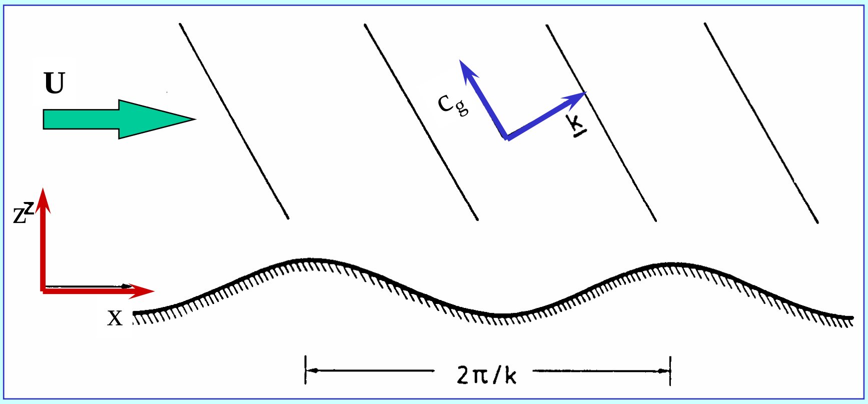
1. **A condition at** $z = 0$ **relating** $\tilde{w}$ **to the amplitude of the corrugations, and the speed of boundary motion,** c.
2. **A radiation condition as** $z \to \infty$

- **the condition at** $z = 0$ **is an inhomogeneous condition**
- **the condition as** $z \to \infty$ **ensures that the direction of energy flux at large heights is away from the wave source, i.e. vertically upwards.**

This condition is applied by ensuring that only wave components (in this case there is only one) with a positive group velocity are contained in the solution.

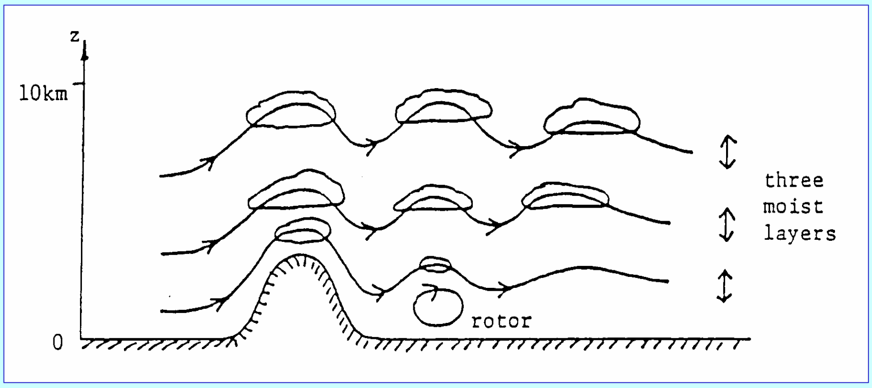
Conditions 1 and 2 determine the forced wave solution uniquely

Example 2: Waves produced by the flow of a stably-stratified fluid over a corrugated boundary.



This flow configuration is a Galilean transformation of the first: the boundary is at rest and a uniform flow U (> 0) **moves over it.**

Mountain waves



When a stably-stratified airstream crosses an isolated ridge or, more generally, a range of mountains, stationary wave oscillations are frequently observed over and to the lee of the ridge or range. These so-called mountain waves, or **lee waves**, may be marked by smooth lens-shaped clouds, known as **lenticular** clouds.

Linear Theory

Assume: **1. Uniform flow** U

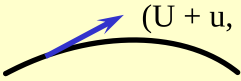
2. Boussinesq fluid ($1/H_s = 0$)

3. Small amplitude sinusoidal topography:

$$z = h(x) = h_m \cos kx$$

Surface boundary condition: streamline slope equals the terrain slope

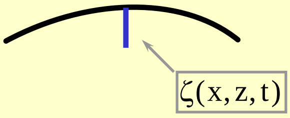
(U + u, w) = (dx, dh)



$$\Rightarrow \quad \frac{w}{U+u} = \frac{dh}{dx} \quad \text{on } z = h(x)$$

linearize $\Rightarrow \quad w = U\frac{dh}{dx} \quad \text{at } z = 0$

Let $\zeta(x,z,t)$ **be the vertical displacement of a fluid parcel.**



→ $w = \dfrac{D\zeta}{Dt}$

linearize → $w = U\dfrac{\partial\zeta}{\partial x}$ → $\hat{w} = ikU\hat{\zeta}$

U = constant → $\hat{\zeta}$ **satisfies the same equation as** $\hat{w}$

$$\boxed{\frac{d^2\hat{\zeta}}{dz^2} + \left(\frac{N^2}{U^2} - k^2\right)\hat{\zeta} = 0}$$

$\boxed{m^2 = N^2 / U^2 - k^2}$

The general solution is $\hat{\zeta}(z) = Ae^{imz} + Be^{-imz}$

constants

$$\hat{\zeta}(z) = Ae^{imz} + Be^{-imz}$$

The boundary condition at the ground is $\zeta(x,0) = h_m e^{ikx}$

→ $\hat{\zeta}(0) = h_m$

The upper boundary condition and hence the solution depends on the sign of $m^2 = \underline{N^2 / U^2} - k^2$

call l^2 → $l = \dfrac{N}{U} > 0$

There are two cases: $0 < |k| < l$ → m **real**

$l < |k|$ → m **imaginary**

→

Two cases:

I. $0 < |k| < l$ m **real**

if m, k > 0, **we must choose** B = 0 **to satisfy the radiation condition.**

⇓

The phase lines slope upstream with height ⇒ sgn(mk) > 0

Then the complete wave solution is

$$\hat{\zeta}(x,z) = h_m e^{i(kx+mz)}$$

The real part is implied

II. $l < |k|$ m **imaginary** **(**= im_0 **say) where** $m_0 = +\sqrt{k^2 - l^2}$

Then $\hat{\zeta}(z) = Ae^{-m_0 z} + Be^{m_0 z}$

The upper boundary condition requires that $\hat{\zeta}(z)$ **remains bounded as** $z \to \infty$

⇒ B = 0

Then the complete wave solution is

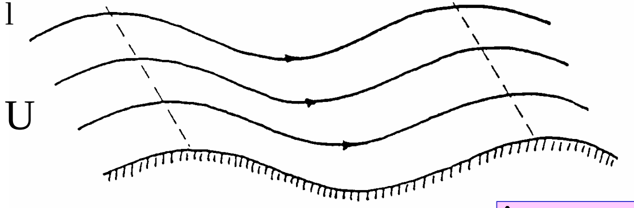
$$\hat{\zeta}(x,z) = h_m e^{-(\sqrt{k^2 - l^2})z + ikx}$$

Here, the phase lines are vertical

Streamline patterns for uniform flow over sinusoidal topography

vertical propagation: $U|k| < N$

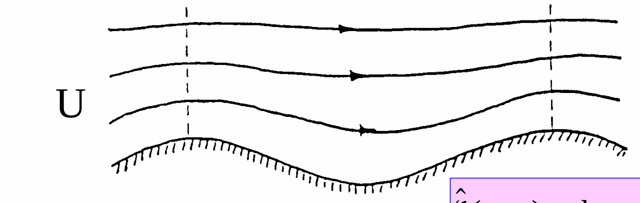
$0 < |k| < l$



vertical decay: $N < U|k|$

$$\hat{\zeta}(x,z) = h_m e^{i(kx+mz)}$$

$l < |k|$



$$\hat{\zeta}(x,z) = h_m e^{-(\sqrt{k^2-l^2})z+ikx}$$

Exercise

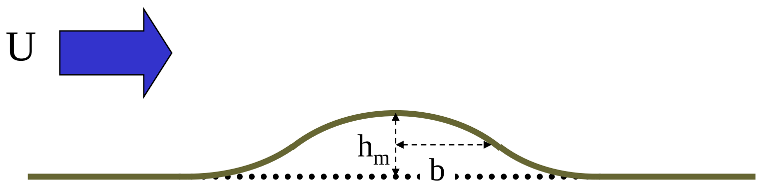
Calculate the mean rate of working of pressure forces at the boundary z = 0 for the two solutions

Show that for $\hat{\zeta}(x,z) = h_m e^{i(kx+mz)}$ **the boundary exerts a drag on the airstream**

whereas for $\hat{\zeta}(x,z) = h_m e^{-(\sqrt{k^2-l^2})z+ikx}$ **it does not.**

Show also that, in the former case, there exists a mean downward flux of horizontal momentum and that this is independent of height and equal to the drag exerted at the boundary.

Flow over isolated topography



Consider flow over an isolated ridge with
$h(x) = h_m/[1 + (x/b)^2]$

maximum height of the ridge = h_m
characteristic half width = b

The ridge is symmetrical about $x = 0$

➡ **it may be expressed as a Fourier cosine integral**

$$h(x) = \int_0^\infty \overline{h}(k)\cos(kx)dk$$
$$= \mathrm{Re}\int_0^\infty \overline{h}(k)e^{ikx}dk$$

where

$$\overline{h}(k) = \frac{2}{\pi}\int_0^\infty h(x)\cos kx\,dx = \frac{2h_m}{\pi}\mathrm{Re}\int_0^\infty \frac{e^{ikx}dx}{1+(x/b)^2}$$
$$= \frac{2h_m b}{\pi}\mathrm{Re}\int_0^\infty \frac{e^{ikbu}du}{1+u^2}$$

The solution for flow over the ridge is just the Fourier synthesis of the two solutions for $0 < |k| < l$ **and** $l < |k|$.

$$\zeta(x,z) = \mathrm{Re}\left[\int_0^l \overline{h}(k)e^{i[kx+(l^2-k^2)^{1/2}z]}\,dk + \int_l^\infty \overline{h}(k)\,e^{ikx-(k^2-l^2)^{1/2}z}dk\right]$$

Put $kb = u$ →

$$\zeta(x,z) = h_m\,\mathrm{Re}\left[\int_0^{lb} \exp\left[-u\left(1-\frac{ix}{b}\right)+i\left(l^2b^2-u^2\right)^{1/2}\frac{z}{b}\right]du\right.$$
$$\left.+\int_{lb}^{\infty} \exp\left[-u\left(1-\frac{ix}{b}\right)-\left(u^2-l^2b^2\right)^{1/2}\frac{z}{b}\right]du\right]$$

$= I_1 + I_2$ **, say**

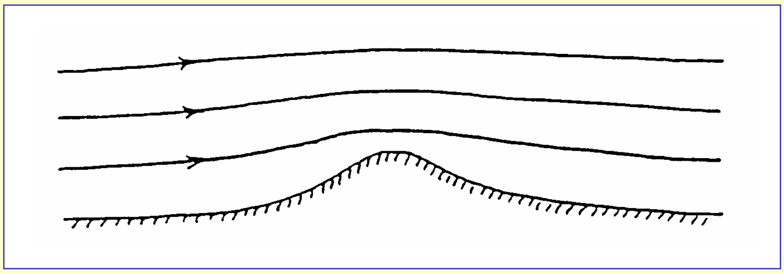
Two limiting cases:

narrow ridge, $lb << 1$ → $I_1 << I_2$ **and**

$$\zeta(x,z) \approx h_m\,\mathrm{Re}\int_0^\infty \exp\left[-u\left(1-\frac{ix}{b}+\frac{z}{b}\right)\right]du$$
$$= \left(\frac{b}{b+z}\right)\frac{h_m}{[1+x^2/(b+z)^2]}\,.$$

- **each streamline has the same general shape as the ridge**
- **the width of the disturbed portion of a streamline increases linearly with** z
- **its maximum displacement decreases in proportion to** $1/(1 + z/b)$ **, becoming relatively small for heights a few times greater than the barrier width.**

narrow ridge, lb << 1 ➔ Nb/U << 1



Steady flow of a homogeneous fluid over an isolated two-dimensional ridge, given by

$$\zeta(x,z) = \left(\frac{b}{b+z}\right)\frac{h_m}{[1+x^2/(b+z)^2]}\,.$$

The highest wind speed and the lowest pressure occurs at the top of the ridge.

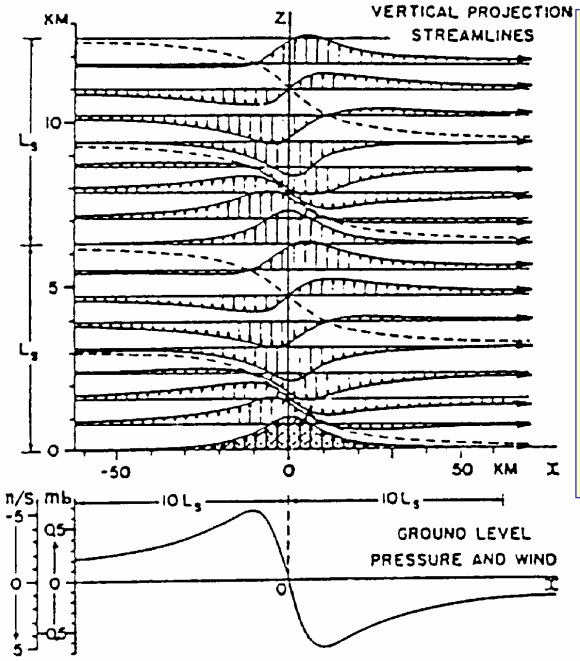
---

broad ridge, lb >> 1 ➔ Nb/U >> 1

➡ $I_1 >> I_2$ **and**

$$\zeta(x,z) \approx h_m \operatorname{Re}\{e^{ilz}\int_0^\infty \exp[-u(1-ix/b)]du\} = h_m \operatorname{Re}\left[\frac{be^{ilz}}{b-ix}\right]$$
$$= h_m\left[\frac{\cos lz-(x/b)\sin lz}{1+(x/b)^2}\right].$$

- **In this limit, essentially all Fourier components propagate vertically.**



Buoyancy-dominated hydrostatic flow over an isolated two-dimensional ridge, given by

$$\zeta(x,z) = h_m \times \left(\frac{\cos lz - (x/b)\sin lz}{1+(x/b)^2} \right)$$

- **The pressure difference across the mountain results in a net drag on it.**
- **The drag can be computed either as the horizontal pressure force on the mountain**

$$D = \int_{-\infty}^{\infty} p(x,0)\frac{dh}{dx}dx$$

or as the vertical flux of horizontal momentum in the wave motion

$$D = \rho_0(z)\int_{-\infty}^{\infty} uwdx$$

Boussinesq approximation ➡ $\rho_0(z) = \rho^*$

moderate ridge, $lb \approx 1$ → $Nb / U \approx 1$

The integrals I_1 and I_2 are too difficult to evaluate analytically, but their asymptotic expansions at large distances from the mountain (compared with 1/l) are revealing.

Let $u = lb\cos\alpha$ **where** $0 \le \alpha \le \pi / 2$

$x = r\cos\theta\,,\quad z = r\sin\theta$ **where** $0 \le \theta \le \pi$

Then

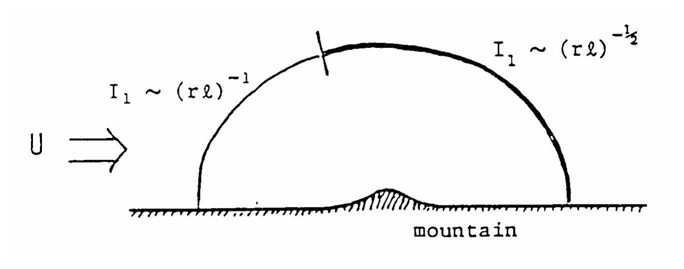
$$I_1 = h_m lb\,\mathrm{Re}\int_0^{\frac{\pi}{2}} (\sin\alpha\, e^{-lb\cos\alpha})\, e^{irl\cos(\theta-\alpha)}\, d\alpha$$

→ $$\boxed{I_1 = \mathrm{Re}\int_0^{\frac{\pi}{2}} h(\alpha)\, e^{irl\, g(\alpha)}\, d\alpha}$$

Asymptotic expansion for large r by stationary phase method

$$I_1 = h_m lb\,\mathrm{Re}\int_0^{\frac{\pi}{2}} (\sin\alpha\, e^{-lb\cos\alpha})\, e^{irl\cos(\theta-\alpha)}\, d\alpha$$

Far field behaviour of I_1



$$\left|I_2\right| \le h_m \max_{lb \le u \le \infty} \left| \exp\left[-u\left(1-\frac{ix}{b}\right)\right]\right| \left| \int_{lb}^{\infty} e^{-(u-lb)z/b} du \right.$$

$$= h_m b \exp(-lb)/z.$$

I_2 **is at most** O(b/r)

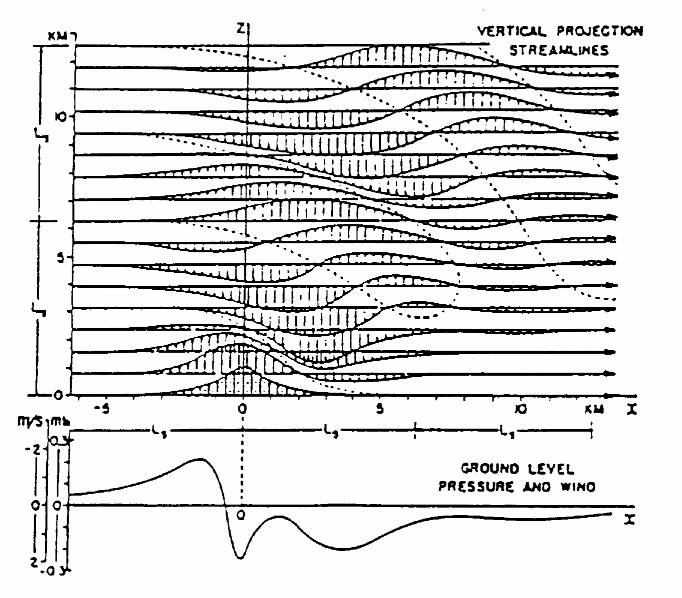
Since lb ≈ O(1) **,** $I_1 + I_2$ **is always the same order as** I_1

not surprising since I_1 **contains all the vertically propagating wave components.**

Then for x > 0 **and** θ **not to close to** 0 **or** π

$$\zeta(x,z) \approx h_m lb \sin\theta e^{-lb\cos\theta} \left(\frac{2\pi}{lr}\right)^{1/2} \cos\left(lr - \frac{\pi}{4}\right)$$

Lee waves in the case where lb ≈ 1



Scorer's Equation

$$\tilde{w}_{zz} + (l^2(z) - k^2)\tilde{w} = 0$$

$$l^2(z) = \frac{N^2}{(U-c)^2} - \frac{U_{zz} + U_z / H_s}{U-c} - \frac{1}{4H_s^2}$$

Trapped lee waves

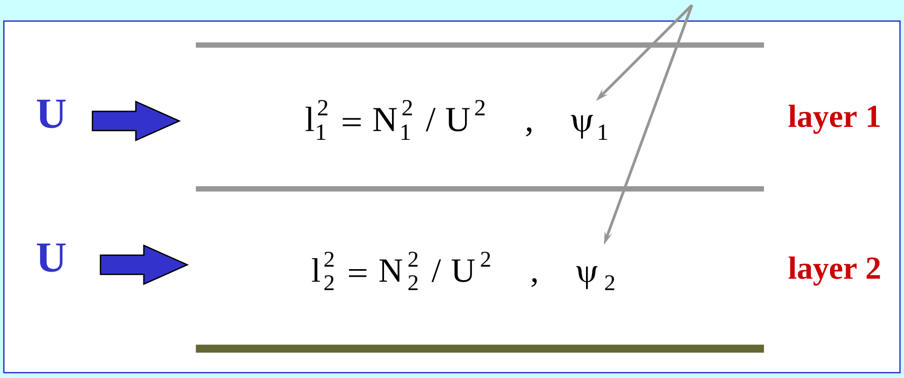
- It was pointed out by **Scorer** (1949) that, on occasion, a long train of lee waves may exist downstream of a mountain range.
- Scorer showed that a long wave train is theoretically possible if the parameter $l^2(z)$ decreases sufficiently rapidly with height.
- Eigensolutions of

$$\tilde{w}_{zz} + (l^2(z) - k^2)\tilde{w} = 0$$

are difficult to obtain when l^2 varies with z.

A two-layer model

perturbation streamfunction



In the Boussinesq approximation $\tilde{\psi}$ **satisfies the same ODE as** $\tilde{w}$

$$\frac{d^2\tilde{\psi}_n}{dz^2} + (l_n^2 - k^2)\tilde{\psi}_n = 0\,, \quad (n = 1{,}2)$$

$$u_x + w_z = 0 \quad \Rightarrow \quad w = -\psi_x \quad \Rightarrow \quad \tilde{w} = -ik\tilde{\psi}$$

$$\frac{d^2\tilde{\psi}_n}{dz^2} + (l_n^2 - k^2)\tilde{\psi}_n = 0\,, \quad (n = 1{,}2)$$

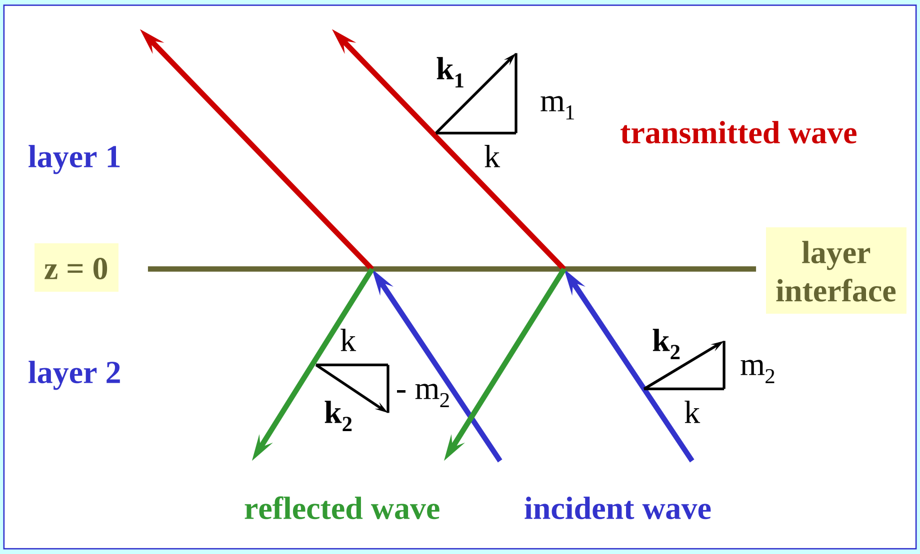
Assume stationary wave solutions so that $c = 0$

then $l^2 = N^2/U^2$ **as before**

Solutions: $\tilde{\psi} = \exp(imz)$ **where** $m^2 = l^2 - k^2$

Assume that $m_2^2 > 0$**, so that vertical wave propagation is possible in the lower layer.**

Case (a): upper layer more stable ($l_1 > l_2$)



Streamfunctions:

transmitted wave

$$\psi_1 = \alpha\, e^{i(kx+m_1 z)}$$

incident plus reflected wave

$$\psi_2 = a\, e^{i(kx+m_2 z)} + \beta e^{i(kx-m_2 z)}$$

$m_1 , m_2 > 0$

a **given**

Wave solutions are coupled by conditions expressing the continuity of interface displacement and pressure at z = 0.

Continuity of interface displacement at z = 0

$$\hat{\zeta}_1 = \hat{\zeta}_2 \quad \text{at } z = 0$$

$$\hat{w}_1 = \hat{w}_2 \quad \text{at } z = 0$$

provided, as here, that $U_1 = U_2$

$$\hat{\psi}_1 = \hat{\psi}_2 \quad \text{at } z = 0$$

When the density is continuous across the interface, continuity of pressure implies that

$$\hat{\psi}_{1z} = \hat{\psi}_{2z} \quad \text{at } z = 0$$

see Ex. 3.6

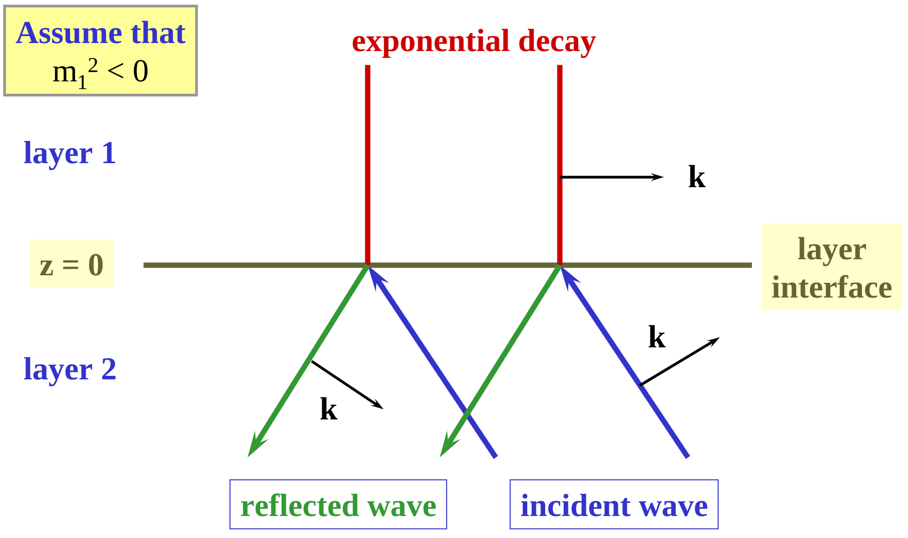
$$\alpha = \frac{2m_2 a}{m_1 + m_2} \text{ and } \beta = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) a$$

always a transmitted wave

no trapped "resonant" solutions in the lower layer

Case (a): upper layer less stable ($l_1 < l_2$)

Assume that $m_1^2 < 0$



The appropriate solution in the upper layer is the one which decays exponentially with height:

$$\psi_1 = \alpha e^{ikx-|m_1|z}$$

$$\psi_2 = a\,e^{i(kx+m_2z)} + \beta e^{i(kx-m_2z)}$$

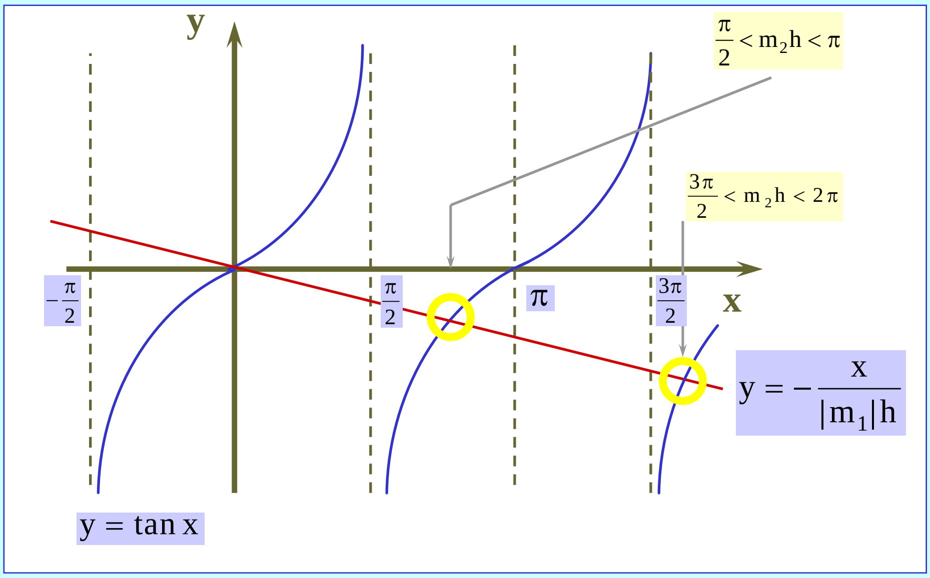
$$\Rightarrow \qquad \beta = -a$$

$$\Rightarrow \qquad \psi_2 = Ae^{ikx}\sin m_2 z$$

$$w_2 = -ik\psi_2 = -ikAe^{ikx}\sin m_2 z$$

$$w_2 = 0 \quad \text{at} \quad z = -h \quad \text{if} \quad \sin m_2 h = 0$$

$\tan m_2 h = -\dfrac{m_2 h}{|m_1|h}$ ➡ **put** $x = m_2h$



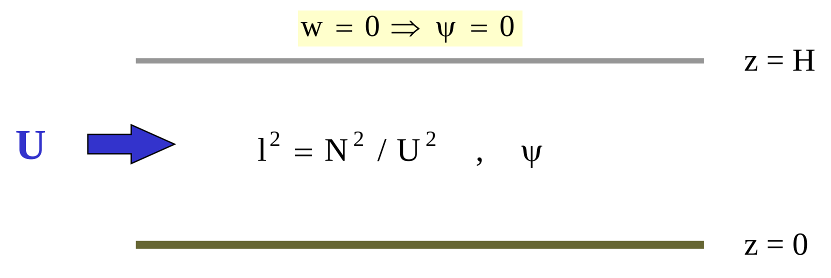
$$\frac{\pi}{2} < m_2 h < \pi \Rightarrow \frac{\pi^2}{4h^2} < m_2^2 = l_2^2 - k^2$$

Recall that
$m_1^2 < 0$

$$\Rightarrow l_1^2 < k^2$$

$$\Rightarrow l_1^2 < k^2 < l_2^2 - \frac{\pi^2}{4h^2}$$

To illustrate the principles involved in obtaining solutions for trapped lee waves, we assume that the upper layer behaves as a rigid lid for some appropriate range of wavelengths



Solution

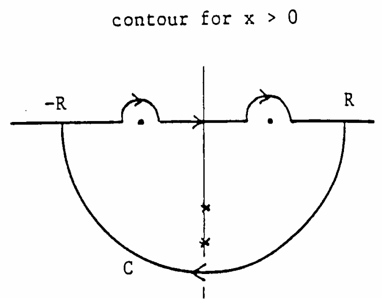
$$\psi(x,z) = \underset{\substack{\uparrow \\ \text{constant}}}{A}\, e^{-ikx} \sin \underset{\substack{\uparrow \\ m^2 = l^2 - k^2}}{m}(H - z)$$

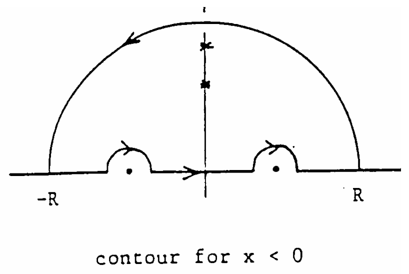
The general solution may be expressed as a Fourier integral of

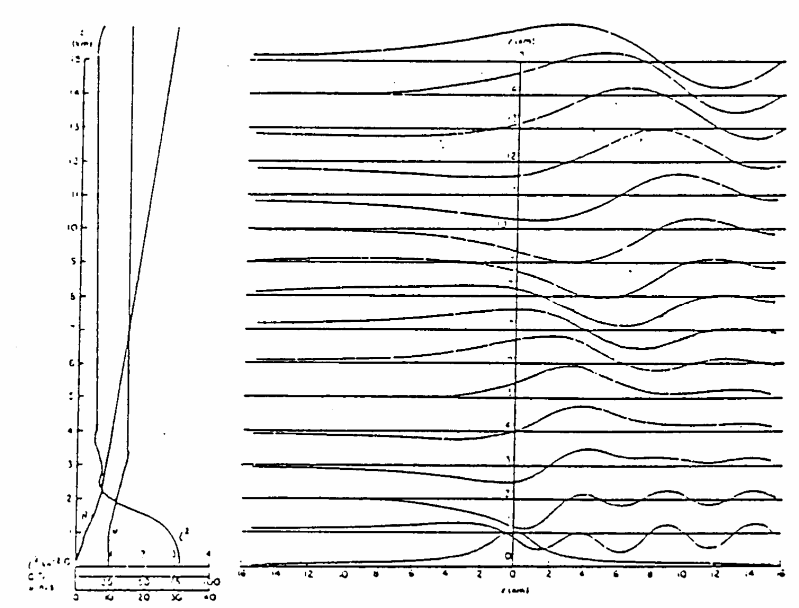
$$\psi(x,z) = Ae^{-ikx}\sin m(H-z)$$

$$\psi(x,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{-ikx}\sin m(H-z)dk$$

$$-\frac{Uh_m}{1+(x/b)^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{-ikx}\sin mH\,dk$$



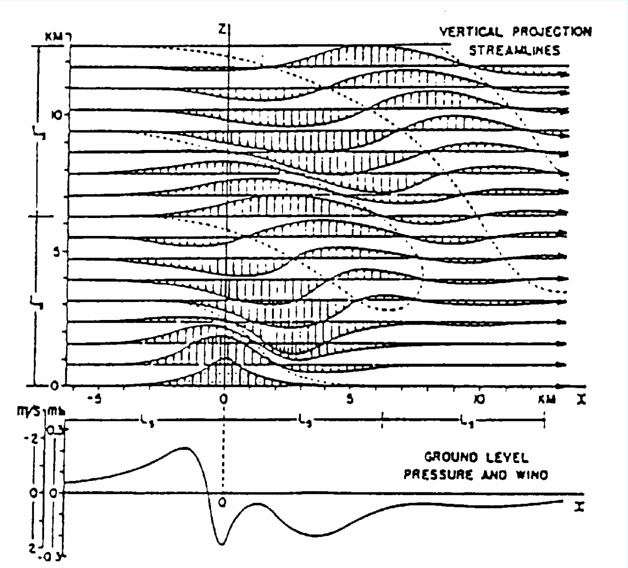




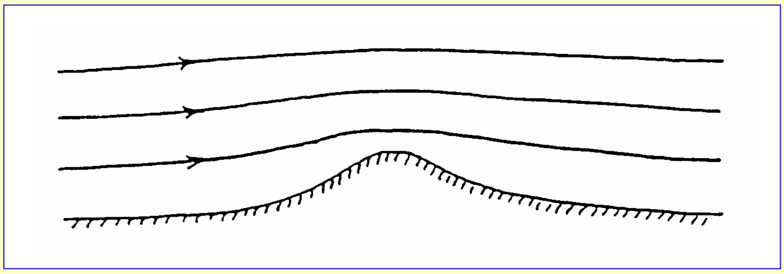
A solution showing trapped lee waves.

Summary

Lee waves for lb ≈ 1 → Nb/U ≈ 1



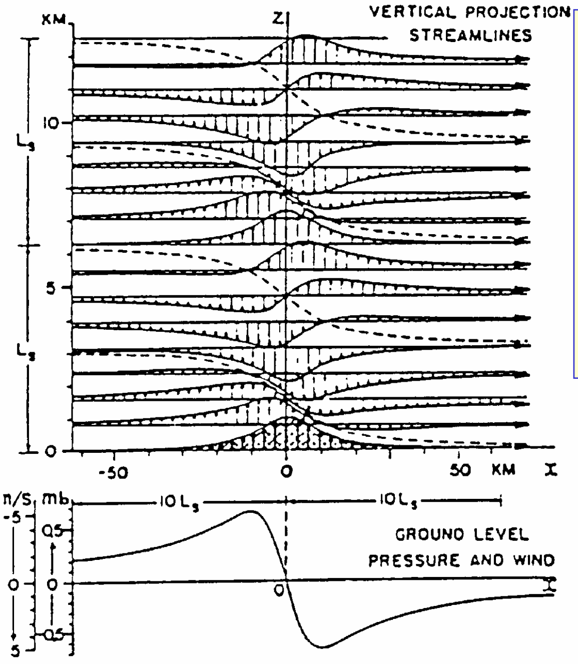
narrow ridge, lb << 1 ➔ Nb/U << 1



Steady flow of a homogeneous fluid over an isolated two-dimensional ridge, given by

$$\zeta(x,z)=\left(\frac{b}{b+z}\right)\frac{h_m}{[1+x^2/(b+z)^2]}\,.$$

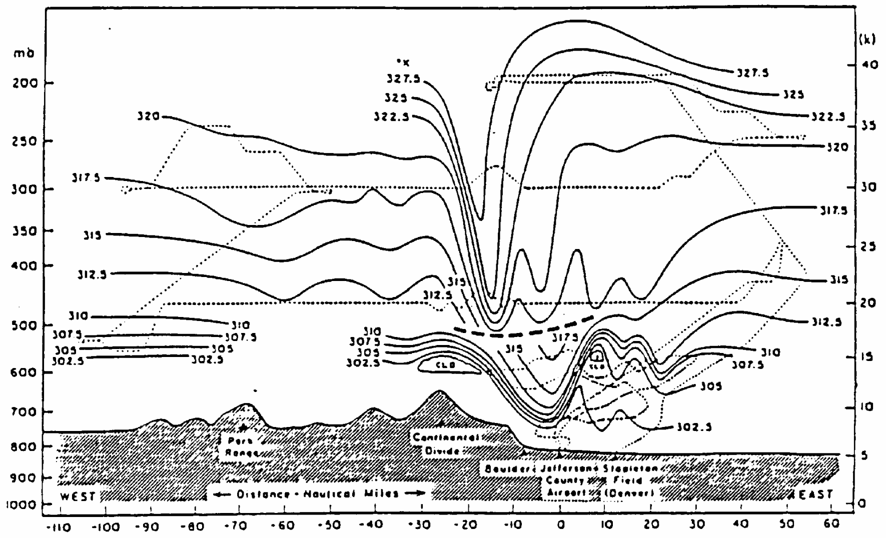
The highest wind speed and the lowest pressure occurs at the top of the ridge.



Broad ridge 1 << lb

➔ Nb/U << 1

$$\zeta(x,z)=h_m\times\left(\frac{\cos lz-(x/b)\sin lz}{1+(x/b)^2}\right)$$



Cross-section of potential temperature field (K) during a severe downslope wind situation in the Colorado Rockies on 11 January 1972. From Klemp and Lilly, 1975.

The End