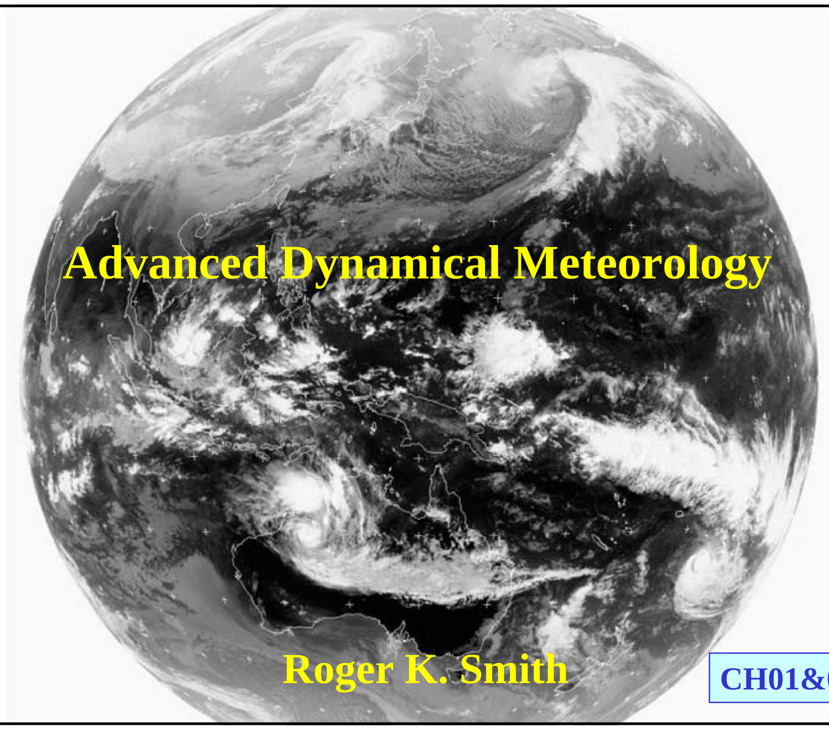

CH01&02

Skript - auf englisch!

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Contents

Dynamical Meteorology

- Dynamical meteorology concerns itself with the theoretical study of **atmospheric motion**.
- It aims to provide an understanding of such motion as well as a rational basis for the prediction of atmospheric events, including short and medium weather prediction and the forecasting of climate.

The Atmosphere

- The atmosphere is an extremely complex system involving motions on a very wide range of space and time scales.
- The dynamic and thermodynamic equations which describe the motions are too general to be easily solved.
- They have solutions representing phenomena that may not be of interest in the study of a particular problem.

Scaling

- We attempt to reduce the complexity of the equations by scaling.
- We try to retain a reasonably accurate description of motion on certain temporal and spatial scales.
- First, we need to identify the essential physical aspects of the motion we hope to study.

Atmospheric Waves

- Various types of wave motions occur in the atmosphere.
- Waves may propagate significant amounts of energy from one place to another.
- Waves will appear in solutions of the equations of motion when these integrated numerically.

Filtering Waves

- Some wave types cause difficulties in attempts to make numerical weather predictions.
- We shall explore ways to modify the equations in order to filter them out:
  - e.g. we may wish to 'sound-proof' the equations - see Chapter 2, or remove inertial gravity waves see Chapter 6.

Wave Instabilities

- Some waves may grow rapidly in amplitude, often as a result of instability. e.g.
  - Kelvin-Helmholtz instability (when a strong vertical shear occurs in the neighbourhood of a large stable density gradient such as an inversion layer) - a mechanism for clear air turbulence (CAT).
  - Baroclinic instability - a mechanism for cyclogenesis and relevant to atmospheric predictability.

General wave types in the atmosphere

Wave type	Speed controlled by:
acoustic waves	temperature
gravity waves	static stability
inertial waves	Coriolis forces
Rossby waves	latitudinal variation of the Coriolis parameter

The momentum equation for a rotating stratified in height coordinates

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p + \sigma\mathbf{k} - \mathbf{D}$$

$f = 2\Omega \sin\phi$	is the Coriolis parameter
ϕ	is the latitude
$p_T = p_o(z) + p$	is the perturbation pressure
p_T	is the total pressure
p_o	is the reference pressure
ρ_o	is the reference density
$\sigma = -g(\rho - \rho_o)/\rho$	is the buoyancy force per unit mass
$\mathbf{D}$	is the frictional force per unit mass

$$\frac{dp_o}{dz} = -g\rho_o$$

The Boussinesq approximation

- **density variations are considered only in as much as they give rise to buoyancy forces**
- $1/\rho$ **is set equal to** $1/\bar{\rho}$
  - where $\bar{\rho}$ is the average density over the whole flow domain

$$\sigma \cong -g(\rho - \rho_o)/\bar{\rho}$$

- **the continuity equation is** $\nabla \cdot \mathbf{u} = 0$

E. A. Spiegel

The anelastic approximation

- **The Boussinesq approximation is sometimes too restrictive and the anelastic approximation is more accurate.**
- $1/\rho$ **is set equal to** $1/\rho_o$

$$\sigma \cong -g(\rho - \rho_o) / \rho_o(z)$$

  - where $1/\rho_o$ is the horizontal average of ρ at height z, or the ambient density $\rho_o(z)$
- **the continuity equation is** $\nabla \cdot [\rho_o(z)\mathbf{u}] = 0$

Pressure coordinates

$$\frac{D_p}{Dt}\mathbf{u}_h + \omega\frac{\partial \mathbf{u}_h}{\partial p} + f\mathbf{k} \wedge \mathbf{u}_h = -\nabla_p \phi \qquad \omega = \frac{Dp}{Dt}$$

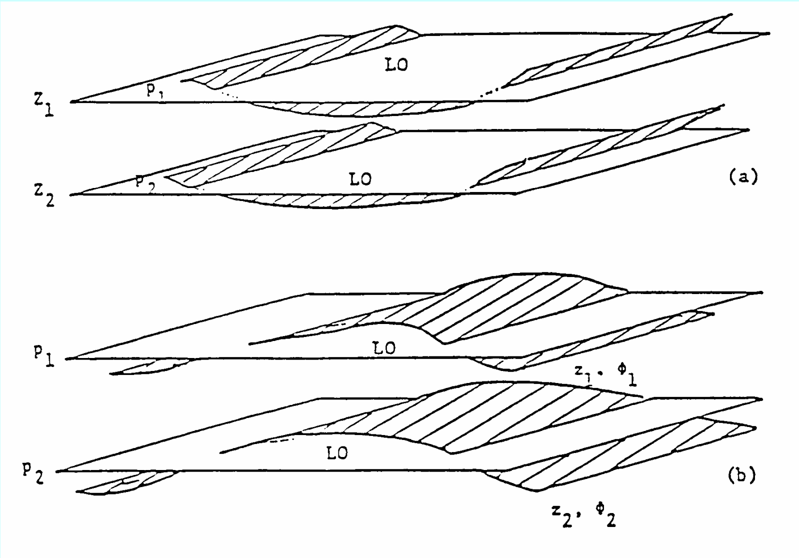
$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} \qquad \frac{D_p}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_p$$

$$\nabla_h \cdot \mathbf{u}_h + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{D_p}{Dt}\ln\theta + \omega\frac{\partial}{\partial p}\ln\theta = \frac{1}{c_p T}\frac{DQ}{Dt}$$

$$\theta = T\left(\frac{p_*}{p}\right)^{\kappa}$$

two isobaric surfaces in height coordinates



two geopotential height surfaces in pressure coordinates

Other coordinate systems

- $Z(p) = -H_s \ln(p^*/p)$, $H_s = RT_o/g$
  - introduced by Eliassen (1950)
  - equals exact height in an isothermal atmosphere
- $Z(p) = [1 - (p^*/p)^\kappa]H/\kappa$, $H = p_o/(\rho_o g)$
  - introduced by Hoskins and Bretherton (1972)
  - equals exact height in an adiabatic atmosphere
- **Isentropic coordinates** (x,y,θ)
- **Sigma coordinates** (x,y,σ) , $\sigma = p/p_s$ **or** $\sigma = \dfrac{p - p_{top}}{p_s - p_{top}}$

Chapter 2: Small-amplitude waves in a stably-stratified rotating atmosphere

Plan

- Start with the general equations
- Linearize
- Introduce tracers
- Look for travelling wave solutions
- Find the dispersion relation for such waves
- Filter out sound waves
- Filter out inertia-gravity waves

Small-amplitude waves in a stably-stratified rotating atmosphere at rest

In a stably-stratified atmosphere at rest $\frac{dp_o}{dz} = -g\rho_o$

Equations for inviscid, isentropic motion:

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k}\wedge\mathbf{u} = -\frac{1}{\rho}\nabla p_T - g\mathbf{k}$$ momentum

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{u} = 0$$ continuity

$$\frac{Ds}{Dt} = 0$$ specific entropy

$$p = \rho RT$$ state

Some basics

$$s = c_p \ln\theta + \text{constant} = c_p\phi + \text{constant}$$

$$\phi = \ln\theta$$

f = Coriolis parameter

$$N^2 = g\frac{d\phi_o}{dz} = \frac{g}{\theta_o}\frac{d\theta_o}{dz}$$

N = Brunt-Väisälä frequency, or buoyancy frequency

$$c_o^2 = \gamma R T_o = \gamma p_o / \rho_o$$

c_o = sound speed

$$\frac{1}{H_s} = -\frac{1}{\rho_o}\frac{d\rho_o}{dz} = \frac{g}{RT_o} + \frac{1}{T_o}\frac{dT_o}{dz}$$

H_s = density scale height

$$\gamma = c_p / c_v$$

ratio of specific heats

Note that $\frac{N^2}{g} = -\frac{g}{c_o^2} + \frac{1}{H_s}$

Assumptions

- two-dimensional perturbations

$$\frac{\partial}{\partial y} = 0 \text{, but} \quad v \neq 0$$

- small-amplitude perturbations

$$|\mathbf{u} \cdot \nabla| << \left|\frac{\partial}{\partial t}\right| \quad \text{in} \quad \frac{D}{Dt}$$

The linearized equations

$$u_t - fv + [p/\rho_o(z)]_x = 0$$

$$v_t + fu = 0$$

$$n_4 w_t + p_z/\rho_o + g\rho/\rho_o = 0$$

$$u_x + w_z + n_1 \frac{1}{\rho_o}\frac{d\rho_o}{dz} w + n_2 \frac{\rho_t}{\rho_o} = 0$$

$$\phi_t + N^2 \phi_{oz} = 0$$

tracers = 1 or 0

$$\phi = \frac{p}{\gamma p_o} - \frac{\rho}{\rho_o} = \frac{p}{\rho_o c_o^2} - \frac{\rho}{\rho_o}$$

$$n_4 w_t + p_z/\rho_o + g\rho/\rho_o = 0$$

substitute

$$\phi = \frac{p}{\rho_o c_o^2} - \frac{\rho}{\rho_o}$$

$$\rightarrow \quad n_4 w_t + (p/\rho_o)_z - \frac{p}{\rho_o}\frac{1}{H_s} + g\left(\frac{p}{\rho_o c_o^2} - \phi\right) = 0$$

use

$$\frac{N^2}{g} = -\frac{g}{c_o^2} + \frac{1}{H_s} \qquad -\frac{N^2}{g}\frac{p}{\rho_o} \; n_3$$

insert tracer

$$\sigma = g\phi = g\frac{\theta'}{\theta_o}$$

$$\rightarrow \quad n_4 w_t + P_z - n_3 \frac{N^2}{g} P - \sigma = 0$$

$$P = \frac{p}{\rho_o}$$

The linearized equations

$$u_t - fv + P_x = 0$$

$$v_t + fu = 0$$

$$n_4 w_t + P_z - n_3 \frac{N^2}{g} P - \sigma = 0$$

$$u_x + w_z + n_1 \frac{w}{H_s} + n_2 \frac{1}{c_o^2} P_t - n_2 \phi_t = 0$$

$$\sigma_t + N^2 w = 0$$

tracers = 1 or 0

five dependent variables: u, v, w, P, σ

Travelling wave solutions

Substitute $(u, v, w, P, \sigma) = (\hat{u}(z), \ldots) \exp[i(kx - \omega t)]$

→ **a set of ODEs or algebraic relationships between** $\hat{u}(z)$ **etc.**

$$-i\omega\hat{u} - f\hat{v} + ik\hat{P} = 0$$

$$-i\omega\hat{v} + f\hat{u} = 0$$

$$-i\omega n_4 \hat{w} + \hat{P}_z - n_3 \frac{N^2}{g} \hat{P} - \hat{\sigma} = 0$$

$$ik\hat{u} + \hat{w}_z + n_1 \frac{\hat{w}}{H_s} + -i\omega n_2 \left(\frac{1}{c_o^2} \hat{P} - \frac{\hat{\sigma}}{g} \right) = 0$$

$$-i\omega\hat{\sigma} + N^2 \hat{w} = 0$$

Three algebraic equations

$$\hat{u} = \frac{k\omega}{\omega^2 - f^2}\hat{P} \qquad \hat{v} = \frac{-ikf}{\omega^2 - f^2}\hat{P} \qquad \hat{\sigma} = \frac{N^2}{i\omega}\hat{w}$$

Two ODEs

$$L_1\hat{w} + A(z)\hat{P} = 0$$

$$\frac{d\hat{w}}{dz} + \left[\frac{N^2}{g}n_2 - \frac{n_1}{H_s}\right]\hat{w} + i\omega\left[\frac{k^2}{\omega^2 - f^2} - \frac{n_2}{c_o^2}\right]\hat{P} = 0$$

$$\left[\frac{d}{dz} - \frac{N^2}{g}n_3\right]\hat{P} - i\omega\left[n_4 - \frac{N^2}{\omega^2}\right]\hat{w} = 0$$

$$B(z)\hat{w} + L_2\hat{P} = 0$$

A single ODE

$$\left[\frac{d}{dz} - \frac{N^2}{g}n_3\right]\left[\left(\frac{k^2}{\omega^2 - f^2} - \frac{n_2}{c_o^2}\right)^{-1}\left(\frac{d}{dz} + \frac{N^2}{g}n_2 - \frac{n_1}{H_s}\right)\right]\hat{w} - \left[\omega^2 n_4 - N^2\right]\hat{w} = 0$$

If the boundary conditions are homogeneous, i.e. if $\hat{w} = 0$ at two horizontal boundaries, we have an eigenvalue problem for ω as a function of k.

In general N^2, c_o^2, H_s are functions of height, z.

In an isothermal atmosphere they are all constants.

Waves with all terms included

First we consider waves in an unbounded region of fluid assuming that all terms are important, i.e., $n_i = 1$ $(i = 1,5)$.

Put $\hat{w}(z) \propto \exp(imz + z/2H_s)$

We obtain the dispersion relation for small amplitude waves:

$$\frac{\omega^4}{c_0^2} - \omega^2\left(k^2 + m^2 + \frac{1}{4H_s^2} + \frac{f^2}{c_0^2}\right) + N^2k^2 + f^2\left(m^2 + \frac{1}{4H_s^2}\right) = 0$$

Each of the perturbation quantities u, v, w, σ, P, **varies in proportion to** $w \propto \exp[i(kx + mz - \omega t + z/2H_s)]$

or $w \propto \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t) + z/2H_s]$

Plane waves

Consider $w \propto \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t) + z/2H_s]$

Here $\mathbf{k} = (k, m)$. **It is not the unit vector in the vertical!**

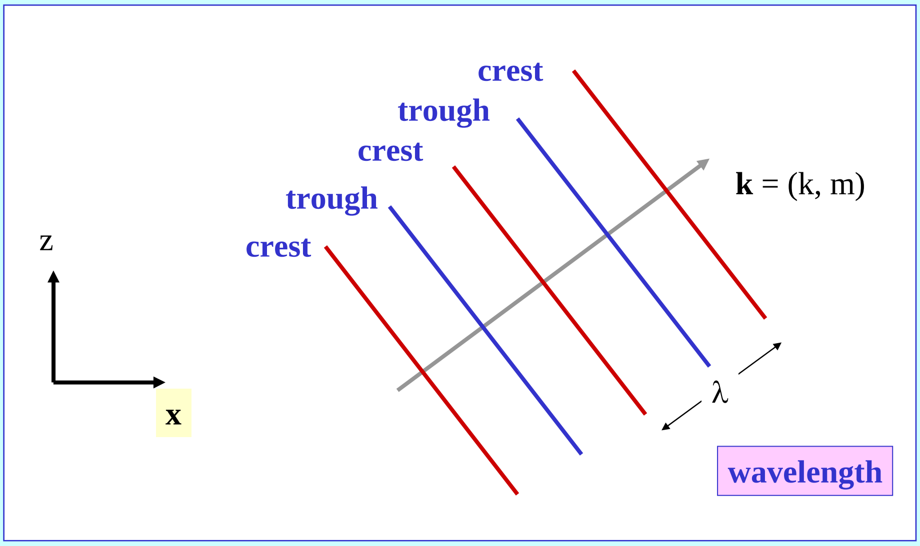
The planes $\mathbf{k}\cdot\mathbf{x} =$ constant **are surfaces of constant phase**

The wave 'crests' and 'troughs' are planes oriented normal to the vector k.

The wavelength is the distance in the direction of k over which $\mathbf{k}\cdot\mathbf{x}$ **increases (or decreases) by** 2π **, i.e.** $\lambda = 2\pi/|\mathbf{k}|$

Note that the wave amplitude is appreciably uniform over height ranges small compared with H_s**, but increases exponentially as** z**; this is related to the exponential decrease of the density with height.**

Wave crests and troughs in an unbounded plane wave

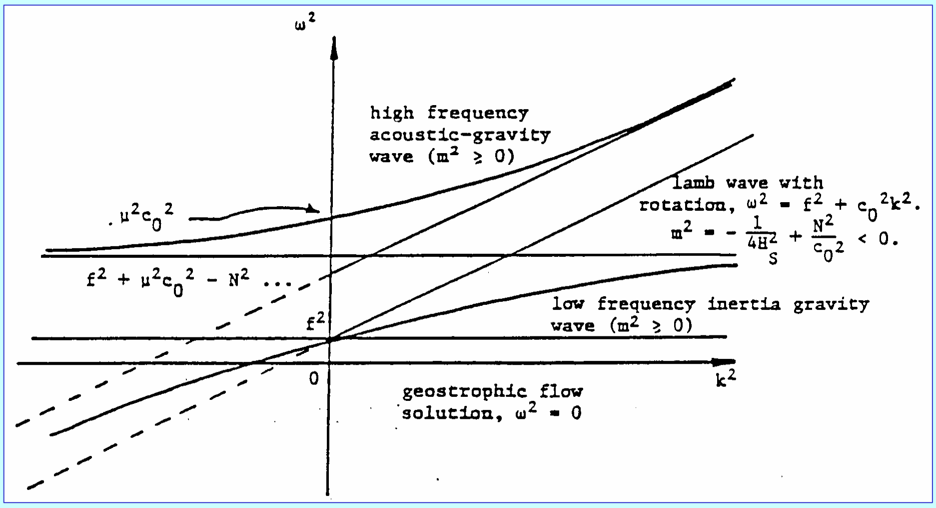


Exercises

1. Show that the phase speed of the plane wave is $\omega/|\mathbf{k}|$.

2. Show that the phase speed is ***not*** a vector; i.e., the components of the vector $\omega\,\mathbf{k}/|\mathbf{k}|^2$ are in general not equal to the components of the phase speed in the x and z directions.

Dispersion relation in k - ω space



Possible wave modes

1. **Geostrophic motion:** $\omega = 0$

$$-fv = -p_x / \rho_o , \quad u = 0 , \quad \text{and} \quad p_z / \rho_o = \sigma$$

Thus air parcels are displaced in the y-direction ($v \neq 0$).

The solution holds whether or not N^2, c_o^2, and H_s are constant and corresponds with a thermal wind in the y-direction

2. **Inertia-gravity waves:** $\omega^2 << (k^2 + m^2) c_o^2$

Then

$$\omega^2 \approx \frac{N^2 k^2 + f^2 (m^2 + 1/4H_s^2)}{k^2 + m^2 + 1/4H_s^2 + f^2/c_o^2}$$

This term is negligible

In the atmosphere at middle latitudes:

$f \sim 10^{-4}\ s^{-1}$, $N \sim 10^{-2}\ s^{-1}$, $H_s \sim 10^4 m$ and $c_0 \sim 10^3/3 ms^{-1}$.

Then $\varepsilon^2 = f^2 / N^2 = 10^{-4} << 1$

Also $1/4H_s^2 + f^2/c_0^2 = (1 + 4H_s^2 f^2/c_0^2)/4H_s^2$

and $4H_s^2 f^2 / c_0^2 \approx 4 \times 10^{-5}$ $\Rightarrow$ **it can be neglected.**

$$\omega^2 \approx \frac{N^2k^2 + f^2(m^2 + 1/4H_s^2)}{k^2 + m^2 + 1/4H_s^2}$$

$$\omega^2 \approx \frac{N^2k^2 + f^2(m^2 + 1/4H_s^2)}{k^2 + m^2 + 1/4H_s^2}$$

Stratification and rotation are equally important when

$$N^2k^2 \approx f^2(m^2 + 1/4H_s^2)$$

If $m^2 << 1/4H_s^2$ **, this** ⇒ $k^2 \approx f^2/4N^2H_s^2)$

⇒ **in the atmosphere.**

$$\lambda \approx \lambda_o = 4\pi N H_s / f \approx 12000 \text{ km}$$

Thus for $\lambda << \lambda_o$ **, rotation effects are negligible unless** |m| **is sufficiently large, implying a vertical wavelength much less than** H_s**.**

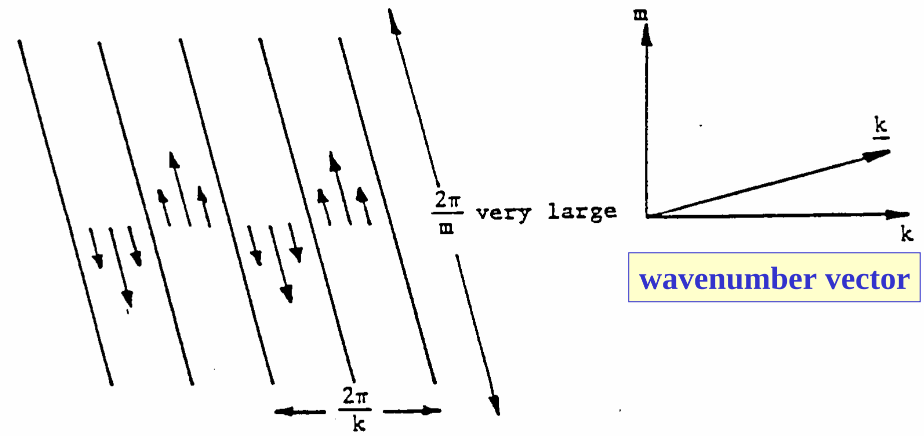
$$\frac{\hat{w}}{\hat{u}} \sim \frac{k}{(m^2 + 1/4H_s^2)^{1/2}} >> 1$$

$$\frac{\hat{P}_z - (N^2/g)\hat{P}}{\hat{\sigma}} = 1 - \frac{\omega^2}{N^2} = \frac{(m^2 + 1/4H_s^2)(1 - f^2/N^2)}{k^2 + m^2 + 1/4H_s^2} << 1$$

$$\frac{\hat{w}_t}{\hat{\sigma}} \sim \left| \frac{i\omega(k^2/\omega)(m^2 + 1/4H_s^2)^{-1/2}}{k^2(m^2 + 1/4H_s^2)^{-1/2}} \right| \sim 1$$

Particle motions in an internal gravity wave with

$k^2 >> m^2 + 1/4H_s^2$



wavenumber vector

3. Ultra-long waves: $k \to 0$

$$\omega^2 \approx \frac{N^2k^2 + f^2(m^2 + 1/4H_s^2)}{k^2 + m^2 + 1/4H_s^2}$$

- **In this limit,** $\omega \to f$.
- **Then only Coriolis forces are important and the motion is purely horizontal.**
- **Neither buoyancy nor pressure forces are significant.**
- **Such waves are called inertial waves.**
- **Pure inertial waves are regarded often as a meteorological curiosity (Holton, pp59-60, §3.2.3).**
- **They may be important in atmospheric tidal motions, they are certainly observed in the oceans.**
- **In addition, since their phase speed** $c_p = \omega/k = f/k$ **is large, they may be a computational nuisance.**

Inertial Effects

- **Nevertheless, inertial effects *are* observed in the atmosphere (see DM, Chapter 11).**
- **When both rotation and stratification are important, though not necessarily comparable in magnitude, the waves are called inertia-gravity waves.**
- **In the atmosphere, gravity waves usually have horizontal wavelengths** 10 **km and may be excited, for example, by airflow over orography, by convection penetrating a stably-stratified air layer, or through shearing instability.**

4. Acoustic waves: $\omega^2 >> N^2, f^2$

In this case

$$\omega^2 \approx (k^2 + m^2 + 1/4H_s^2)c_0^2$$

Consider the special case where $mH_s >> 1$

$$(\hat{u}, \hat{w}) \approx \omega^{-1}(k, m)\hat{P}$$

the particle motions are in the direction of the wave, **k**, and are associated with negligible entropy change, $\phi \approx 0$.

These are well known properties of acoustic waves

- the waves are longitudinal, and
- the phase speed $c_p = \omega/|\mathbf{k}| \approx c_o$ is large.

Mechanisms of excitation include lightening discharges and aeroplane noise (a rumble when clouds are around).

5. Lamb waves: $w = 0$

The equation

$$\left(\frac{d}{dz} - \frac{N^2}{g}n_3\right)\left[\left(\frac{k^2}{\omega^2 - f^2} - \frac{n_2}{c_o^2}\right)^{-1}\left(\frac{d}{dz} + \frac{N^2}{g}n_2 - \frac{n_1}{H_s}\right)\right]\hat{w} - \left(\omega^2 n_4 - N^2\right)\hat{w} = 0$$

has the trivial solution $\hat{w}(z) = 0$.

This does not lead to a trivial solution of the complete system of equations, a nontrivial solution of which is $\hat{w}(z) = 0$ provided that

$$\omega^2 = f^2 + k^2 c_0^2$$

When $\hat{w}(z) = 0$ **unchanged**

$$-i\omega\hat{u} - f\hat{v} + ik\hat{P} = 0 \qquad -i\omega\hat{v} + f\hat{u} = 0$$

$$-i\omega\hat{\sigma} + N^2\hat{w} = 0 \quad \Rightarrow \quad \hat{\sigma} = 0$$

$$-i\omega n_4\hat{w} + \hat{P}_z - n_3\frac{N^2}{g}\hat{P} - \hat{\sigma} = 0$$

$$\Rightarrow \quad \hat{P}_z - n_3\frac{N^2}{g}\hat{P} = 0 \quad \Rightarrow \quad \hat{P}(z) = \hat{P}(0)e^{N^2 z/g}$$

$$ik\hat{u} + \hat{w}_z + n_1\frac{\hat{w}}{H_s} - i\omega n_2\left(\frac{1}{c_o^2}\hat{P} - \frac{\hat{\sigma}}{g}\right) = 0$$

$$\Rightarrow \quad ik\hat{u} - i\omega n_2\hat{P} / c_o^2 = 0$$

These solutions correspond to

$$\hat{u} \approx (k/\omega)\hat{P}, \quad \hat{v} \approx 0 \quad \text{and} \quad \hat{\sigma} = 0$$

This solution is a so-called Lamb wave, modified slightly by rotation.

Its existence requires the ground ($z = 0$) **to be flat, so that**

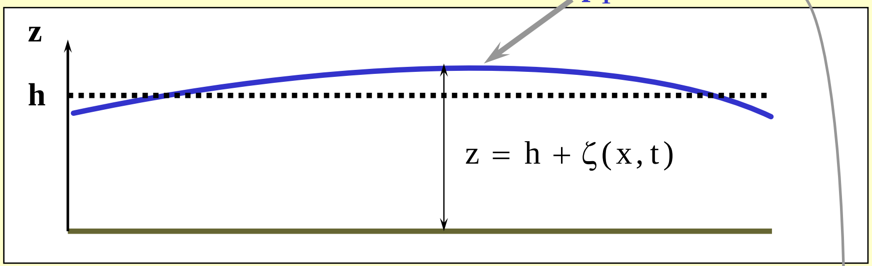
$$\hat{w}(0) = 0$$

The pressure perturbation in the Lamb wave is essentially supported by the ground.

6. Boundary waves

Example: water waves

p_T = constant



$$\frac{Dp_T}{Dt} = 0 \quad \textbf{at} \quad z = h + \zeta(x,t)$$

linearize

$$\frac{\partial p}{\partial t} + w\frac{dp_o}{dz} = \frac{\partial p}{\partial t} - \rho_o g w = 0 \quad \textbf{at } z = h$$

$$\frac{\partial p}{\partial t} + w\frac{dp_o}{dz} = \frac{\partial p}{\partial t} - \rho_o g w = 0 \quad \textbf{at } z = h$$

With $(u, v, w, P, \sigma) = (\hat{u}(z), \ldots)\exp[i(kx - \omega t)]$

$$\hat{P} = -(g / i\omega)\hat{w}$$

Using

$$\left[\frac{d}{dz} - \frac{N^2}{g}n_3\right]\hat{P} - i\omega\left[n_4 - \frac{N^2}{\omega^2}\right]\hat{w} = 0$$

set to 1

$$g\frac{d\hat{w}}{dz} - \omega^2\hat{w} = 0 \quad \textbf{at } z = h$$

The solution of

$$\left[\frac{d}{dz}-\frac{N^2}{g}n_3\right]\left[\left(\frac{k^2}{\omega^2-f^2}-\frac{n_2}{c_o^2}\right)^{-1}\left(\frac{d}{dz}+\frac{N^2}{g}n_2-\frac{n_1}{H_s}\right)\right]\hat{w}$$
$$-\left[\omega^2 n_4 - N^2\right]\hat{w} = 0$$

subject to

$$g\frac{d\hat{w}}{dz} - \omega^2\hat{w} = 0 \qquad \text{at } z = h$$

$$\hat{w} = 0 \qquad \text{at } z = 0$$

is straightforward

no essential features are lost if we

- neglect the stratification; i.e., set $N^2 = 0$ and omit the σ-equation, and
- assume the motion to be hydrostatic, $w_t \ll g$; i.e., set $n_4 = 0$ (this assumption is valid provided that the waves are long enough), and
- assume there is no coupling between pressure and density in the continuity equation ; i.e., set $n_2 = 0$.

This suppresses acoustic waves

Then **unchanged**

$$-i\omega\hat{u} - f\hat{v} + ik\hat{P} = 0 \qquad -i\omega\hat{v} + f\hat{u} = 0$$

$$-i\omega\hat{\sigma} + \cancel{N^2}\hat{w} = 0 \quad \Rightarrow \quad \hat{\sigma} = 0$$

$$-i\omega \cancel{n_4}\hat{w} + \hat{P}_z - n_3\frac{\cancel{N^2}}{g}\hat{P} - \cancel{\hat{\sigma}} = 0 \quad \Rightarrow \quad \hat{P}_z = 0$$

$$ik\hat{u} + \hat{w}_z + n_1\frac{\hat{w}}{H_s} - i\omega\cancel{n_2}\left(\frac{1}{c_o^2}\hat{P} - \frac{\hat{\sigma}}{g}\right) = 0$$

set $n_1 = 1$

$$\Rightarrow \quad ik\hat{u} + \hat{w}_z + \frac{\hat{w}}{H_s} = 0$$

Then

$$\left[\frac{d}{dz} - \frac{N^2}{g}n_3\right]\left[\left(\frac{k^2}{\omega^2 - f^2} - \frac{n_2}{c_o^2}\right)^{-1}\left(\frac{d}{dz} + \frac{N^2}{g}n_2 - \frac{n_1}{H_s}\right)\right]\hat{w}$$
$$-\left[\omega^2 n_4 - N^2\right]\hat{w} = 0$$

becomes

$$\frac{d}{dz}\left[\frac{d}{dz} - \frac{1}{H_s}\right]\hat{w} = 0$$

Assume

$$\hat{w} = 0 \qquad \text{at } z = 0$$

$$g\frac{d\hat{w}}{dz} - \omega^2\hat{w} = 0 \qquad \text{at } z = h$$

$$\Rightarrow \quad \hat{w}(z) = W[1 - \exp(z/H_s)]$$

where W **is a constant**

Notes:

1. $\hat{u}, \hat{v}, \hat{P}$ are related to W

2. $\hat{P}_z = 0$ → $\hat{P} = \text{constant}$

3. If $\hat{w}(z) = W[1 - \exp(z/H_s)]$ satisfies

$$g\frac{d\hat{w}}{dz} - \omega^2 \hat{w} = 0 \quad \text{at } z = h$$

then

$$\omega^2 = f^2 + gH_s k^2 [1 - \exp(-h/H_s)]$$

The solution corresponds to a free surface wave

The effects of shear

- **acoustic waves** are refracted by superimposed shear; enhanced downwind audibility results from the convergent refraction of sound waves, usually due to wind shear, but, occasionally, temperature variations may play a role also.
- **gravity waves** may be severely modified by shear. The refraction effect is considerable and leads to the total reflection of some (shorter) components which may be trapped in channels as well-marked trains of lee waves downstream of mountains. In certain situations waves may be absorbed also.
- where **vorticity gradients are large**, gravity waves may grow spontaneously by **Kelvin-Helmholtz instability**, giving rise to **billows or clear air turbulence** (CAT).

→ see later

The kinetic energy equation

$$u\times\frac{\partial u}{\partial t}+v\times\frac{\partial v}{\partial t}+w\times\frac{\partial}{\partial t}(n_4 w)\quad\Rightarrow$$

$$\tfrac{1}{2}(u^2+v^2+n_4w^2)_t+u\left(\frac{p}{\rho_o}\right)_x+w\left(\frac{p}{\rho_o}\right)_z-n_3\frac{pw}{\rho_o}\frac{N^2}{g}-gw\phi=0$$

The potential energy equation

$$\frac{p}{\rho_o}\times(\text{continuity})\quad\Rightarrow$$

$$\frac{p}{\rho_o}(u_x+w_z)-\frac{n_1}{H_s}\frac{pw}{\rho_o}+n_2\frac{p}{\rho_o}\left(\frac{p_t}{\rho_o c_o^2}-\phi_t\right)=0$$

The total energy equation

Add the K. E. and P. E. equations

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(u^2+v^2+n_4w^2+n_2\frac{p^2}{\rho_o^2c_o^2}+\frac{g^2\phi^2}{N^2}\right)\right]+\left(\frac{up}{\rho_o}\right)_x+\left(\frac{wp}{\rho_o}\right)_z-\frac{n_1}{H_s}\frac{pw}{\rho_o}+\frac{pw}{\rho_o}\frac{N^2}{g}(n_2-n_3)=0\,.$$

or

$$\rho_o\frac{\partial E}{\partial t}=-\nabla\cdot\mathbf{F}+(n_1-1)\frac{pw}{H_s}+pw\frac{N^2}{g}(n_3-n_2)$$

where

$$E=\frac{1}{2}\left(u^2+v^2+n_4w^2+n_2\frac{p^2}{\rho_o^2c_o^2}+\frac{g^2\phi^2}{N^2}\right)$$

E = **total wave energy per unit mass**

F = (pu, 0, pw) **can be interpreted as the wave energy flux per unit mass**

$$\rho_o \frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F} + (n_1 - 1)\frac{pw}{H_s} + pw\frac{N^2}{g}(n_3 - n_2)$$

Note 1: these terms are zero when $n_1 = n_2 = n_3 = 1$

Note 2: f **does not appear in the energy equation!**

$$\rho_o \frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F}$$

the local rate of change of total wave energy equals the convergence of wave energy flux

$$E = \frac{1}{2}\left(u^2 + v^2 + n_4 w^2 + n_2 \frac{p^2}{\rho_o^2 c_o^2} + \frac{g^2\phi^2}{N^2}\right)$$

In any particular type of wave motion, the energy will fluctuate between kinetic energy and some other energy form.

Pure wave types are:

(a) gravity waves, in which the energy is stored in potential energy form $\frac{1}{2}\sigma^2 / N^2$ **, when not in kinetic energy form, and**

(b) compressible, or acoustic waves, in which the energy is stored as internal energy $\frac{1}{2}p^2 / \rho_o^2 c_o^2$ **, when not in kinetic energy form.**

Inertia-gravity waves undergo energy conversions similar to pure gravity waves.

In general, waves of the **mixed gravity-acoustic** type are such that kinetic energy is converted partly into potential energy and partly into internal energy.

However, in this case, the interpretations of $\frac{1}{2}\sigma^2 / N^2$ as potential energy and $\frac{1}{2}p^2 / \rho_o^2 c_o^2$ as internal energy are not strictly correct.

Simplified solutions and filtered equations

If the tracers n_i are retained and the expression

$$\hat{w}(z) \propto \exp\left[\left(im + n_1 / 2H_s\right)z\right]$$

is substituted into

$$\left(\frac{d}{dz} - \frac{N^2}{g} n_3\right)\left[\left(\frac{k^2}{\omega^2 - f^2} - \frac{n_2}{c_0^2}\right)^{-1}\left(\frac{d}{dz} + \frac{N^2}{g} n_2 - \frac{n_1}{H_s}\right)\right]\hat{w} - (\omega^2 n_4 - N^2)\hat{w} = 0$$

we obtain

$$m^2 + \frac{n_1^2}{4H_s^2} + \frac{N^2}{g}\left[im(n_3 - n_2) + n_2(n_3 - 1)\frac{N^2}{g} + \frac{1}{H_s}\left\{ n_2 - \tfrac{1}{2} n_1 (n_2 + n_3) \right\} \right]$$
$$+ (n_4\omega^2 - N^2)\frac{k^2}{\omega^2 - f^2} - n_2 n_4 \frac{\omega^2}{c_o^2} = 0.$$

The consequences of omitting certain terms in the equations of motion may now be investigated.

It is desirable that any approximation yields a consistent energy equation.

'Sound-proofing' the equations

Examination of

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(u^2 + v^2 + n_4 w^2 + n_2 \frac{p^2}{\rho_o^2 c_o^2} + \frac{g^2\phi^2}{N^2} \right) \right] +$$
$$\left(\frac{up}{\rho_o} \right)_x + \left(\frac{wp}{\rho_o} \right)_z - \frac{n_1}{H_s}\frac{pw}{\rho_o} + \frac{pw}{\rho_o}\frac{N^2}{g}(n_2 - n_3) = 0\ .$$

suggests that setting $n_2 = 0$ **in**

$$u_x + w_z - n_1 \frac{w}{H_s} + \frac{n_2}{c_o^2}\left[\frac{p}{\rho_o} \right]_t - n_2 \phi_t = 0$$

will remove the acoustic mode from the equations.

The equation

$$m^2+\frac{n_1^2}{4H_s^2}+\frac{N^2}{g}\left[im(n_3-n_2)+n_2(n_3-1)\frac{N^2}{g}+\frac{1}{H_s}\{n_2-\tfrac{1}{2}n_1(n_2+n_3)\}\right]$$
$$+(n_4\omega^2-N^2)\frac{k^2}{\omega^2-f^2}-n_2n_4\frac{\omega^2}{c_o^2}=0.$$

then gives

$$\omega^2=\frac{N^2k^2+f^2\left(m^2+\frac{n_1^2}{4H_s^2}+n_3\frac{N^2}{g}\left(im-\frac{n_1}{2H_s}\right)\right)}{n_4k^2+m^2+\frac{n_1^2}{4H_s^2}+n_3\frac{N^2}{g}\left(im-\frac{n_1}{2H_s}\right)}$$

When $n_1 = n_4 = 1$, this differs from the dispersion relation for inertia-gravity waves only in respect of the terms involving n_3.

$$\omega^2=\frac{N^2k^2+f^2\left(m^2+\frac{n_1^2}{4H_s^2}+n_3\frac{N^2}{g}\left(im-\frac{n_1}{2H_s}\right)\right)}{n_4k^2+m^2+\frac{n_1^2}{4H_s^2}+n_3\frac{N^2}{g}\left(im-\frac{n_1}{2H_s}\right)}$$

note that ω^2 is complex → **ω is complex**

→ **there exist exponentially growing wave solutions**

Such solutions must have an energy source, although none is available in the unapproximated system!

Filtering the sound waves

The equation

$$\rho_o \frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F} + (n_1 - 1)\frac{pw}{H_s} + pw\frac{N^2}{g}(n_3 - n_2)$$

shows that with $n_2 = 0$, there is an energy source unless $n_3 = 0$.

Hence, to filter sound waves from the system of equations, we must take both n_2 and n_3 to be zero to preserve energetic consistency.

Note that sound waves are filtered out also by letting $c_o^2 \to \infty$

The hydrostatic approximation

In many atmospheric situations, the pressure is very close to its hydrostatic value.

In the foregoing analysis, pressure is hydrostatic if $n_4 = 0$.

This eliminates Dw/Dt, or in linearized form $\partial w/\partial t$, from the vertical momentum equation.

With $n_1 = n_2 = n_3 = 1$, the dispersion relation gives

$$-n_4\frac{\omega^4}{c_o^2} + \omega^2\left(m^2 + \frac{1}{4H_s^2} + n_4\frac{f^2}{c_o^2} + n_4 k^2\right) - N^2k^2 - f^2\left(m^2 + \frac{1}{4H_s^2}\right) = 0$$

$$-n_4\frac{\omega^4}{c_o^2}+\omega^2\left(m^2+\frac{1}{4H_s^2}+n_4\frac{f^2}{c_o^2}+n_4k^2\right)-N^2k^2-f^2\left(m^2+\frac{1}{4H_s^2}\right)=0$$

The terms involving n_4 are negligible if

(a) $\frac{\omega^2}{c_o^2} << m^2+\frac{1}{4H_s^2}$ **typically well satisfied for inertia-gravity waves**

(b) $\omega^2 << N^2$ **More discriminating**

If a layer is close to adiabatic, then $N^2 \approx 0$, and the dispersion relation becomes

$$\omega^2\left(m^2+\frac{1}{4H_s^2}+n_4k^2\right)=f^2\left(m^2+\frac{1}{4H_s^2}\right) \Rightarrow k^2 << m^2$$

Since for finite N^2

$$\omega^2=\left[N^2k^2+f^2\left(m^2+\frac{1}{4H_s^2}\right)\right]/\left(n_4k^2+m^2+\frac{1}{4H_s^2}\right)$$

The condition $k^2 << m^2$ implies that $\omega^2 << N^2$

The hydrostatic approximation may be considered appropriate whenever $k^2 << m^2$ is satisfied.

The hydrostatic approximation is valid for waves whose horizontal wavelength is much larger than the vertical wavelength.

Note that the magnitude of $g^{-1}Dw/Dt$ is irrelevant to the usefulness of the hydrostatic approximation for finding the acceleration.

Even if this quantity is small, the dynamics will be incorrectly modelled by its neglect if the motion is in tall narrow columns, i.e. if $m^2 << k^2$.

The condition $k^2 << m^2$ is usually well satisfied for large-scale atmospheric motions and the hydrostatic approximation is used exclusively in 'primitive equation' (PE) numerical weather prediction (NWP) models.

Sound-proofed hydrostatic approximation

If we formally set $n_1 = 1$ and $n_2 = n_3 = n_4 = 0$, valid under the same conditions (a) and (b) for the hydrostatic approximation alone, the dispersion relation gives

$$\omega^2 = f^2 + N^2k^2 / (m^2 + 1/4H_s^2)$$

The Lamb wave now disappears as it does when the other acoustic waves are eliminated. However, the free surface wave still exists and leads to a spurious fast moving wave. It is filtered out by using a rigid-lid condition. A primitive equation, numerical weather prediction model can be devised for the set of equations with $n_1 = 1$ and $n_2 = n_3 = n_4 = 0$ together with a rigid upper boundary condition.

Variation of mean density with height; the equivalent incompressible atmosphere

The upward decrease of mean density that distinguishes between continuity of volume and mass is represented by the term in n_1 and appears in the multiplier $\exp(z/2H_s) \propto \sqrt{\rho_o}$ in $\hat{w}$ etc.

Over tropospheric depths this factor is significant, but elsewhere n_1 appears only in the combination $m^2 + n_1/4H_s^2$ and then it is often negligible, e.g. if $m = \pi/H_s$, $1/4H_s^2 = \frac{1}{4}(m/\pi)^2 \approx m^2/40$

When the vertical length scale of the motion is not large, the density variation can be neglected by setting $n_1 = 0$ as long as the factor $\sqrt{\rho_o}$ is included implicitly.

A prediction of u obtained from the incompressible (Boussinesq) model should then be compared with $\sqrt{(\rho_o/\rho_s)}u$ observed in the (compressible) atmosphere, where $\rho_s = \rho_o(0)$.

Geostrophic motion

The solution of

$$\frac{\omega^4}{c_0^2} - \omega^2\left(k^2 + m^2 + \frac{1}{4H_s^2} + \frac{f^2}{c_0^2}\right) + N^2k^2 + f^2\left(m^2 + \frac{1}{4H_s^2}\right) = 0$$

corresponding with $\omega = 0$ is independent of the values of $n_1 - n_4$, but the more general "quasi-geostrophic" solutions are not.

If we are interested only in the slowly-moving, nearly geostrophic waves, we can omit sound waves and suppose that the pressure is hydrostatic, i.e., put $n_2 = n_3 = n_4 = 0$.

Exercise (2.10)

The Boussinesq equations for a non-rotating stratified liquid are:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_*}\nabla p + \sigma\mathbf{k} \qquad \nabla\cdot\mathbf{u} = 0 \qquad \frac{D\rho}{Dt} = 0$$

Show that the dispersion relation for small-amplitude waves in the x-z plane is

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2}$$

The End