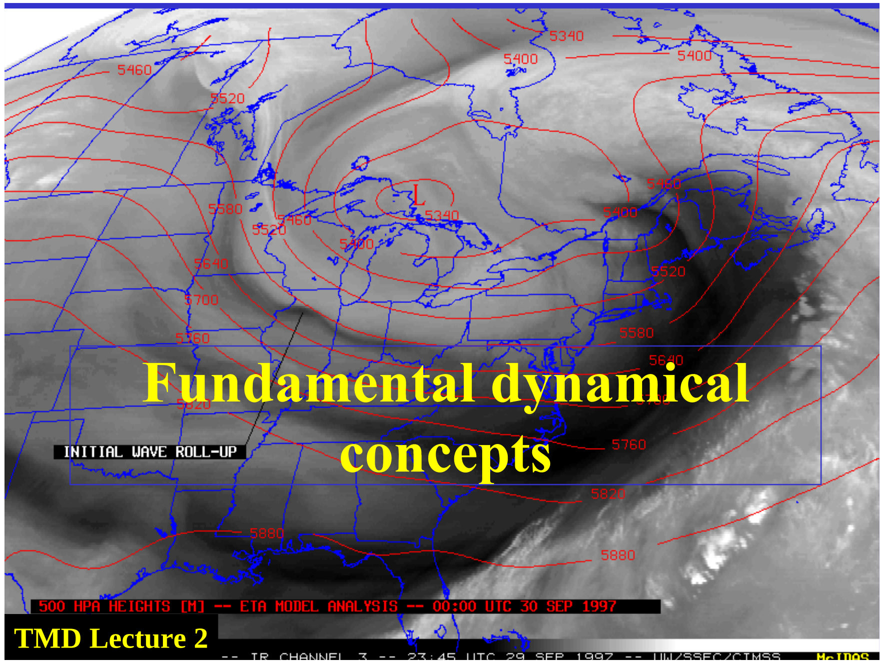
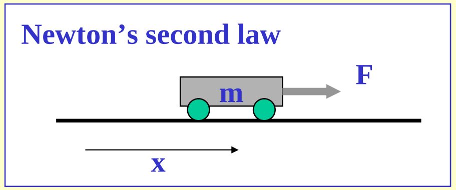


Atmospheric Motion

Dynamics



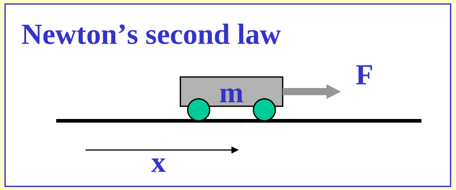
Mass × Acceleration = Force

$$m\frac{d^2x}{dt^2} = F$$

Thermodynamics

Concerned with changes in the internal energy and state of moist air.

Newton's second law



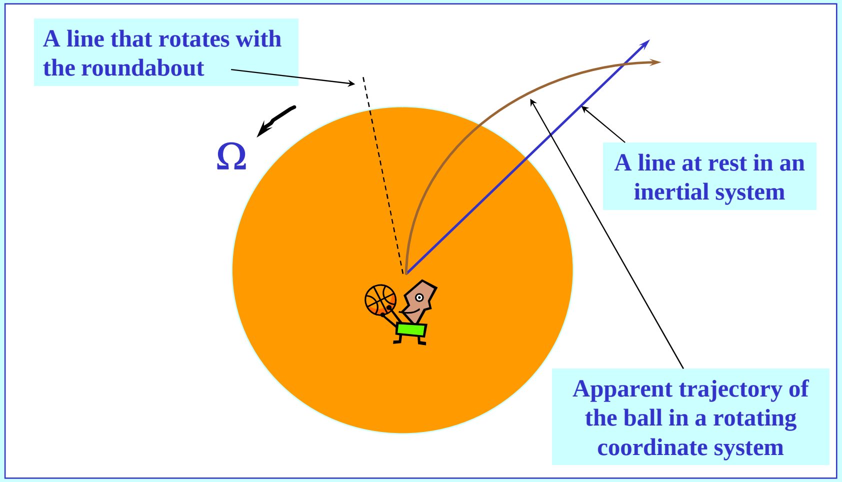
$$m\frac{d^2x}{dt^2} = F$$

Mass × Acceleration = Force applies in an inertial frame of reference.

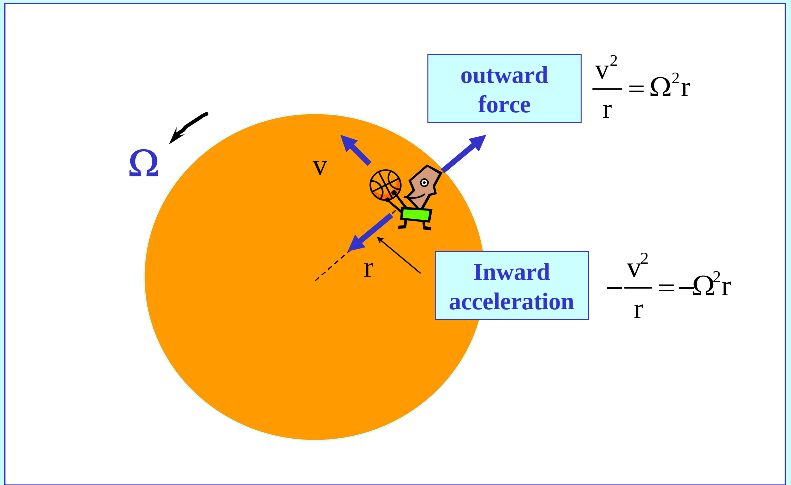
But we like to make measurements relative to the Earth, which is rotating!

To do this we must add correction terms in the equation, the centrifugal and Coriolis accelerations.

The Coriolis force

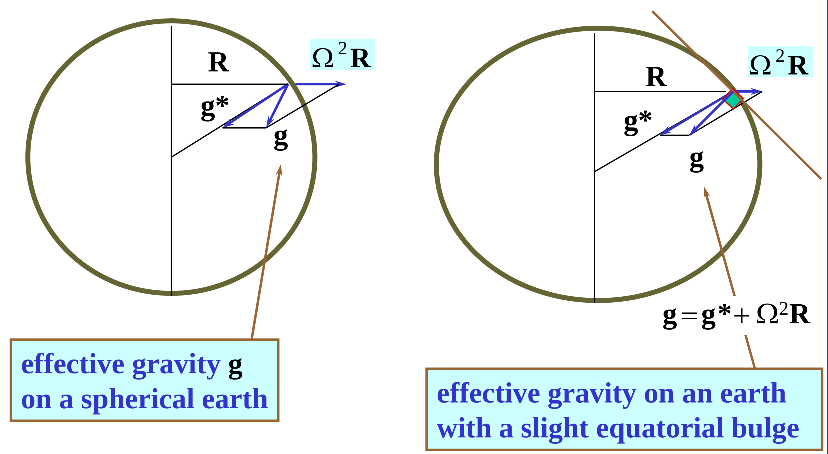


The centripetal acceleration/centrifugal force



Effective Gravity

g is everywhere normal to the earth's surface



Effective Gravity

If the earth were a perfect sphere and not rotating, the only gravitational component **g*** would be radial.

Because the earth has a bulge and is rotating, the effective gravitational force **g** is the vector sum of the normal gravity to the mass distribution **g***, together with a centrifugal force $\Omega^2 R$, and this has no tangential component at the earth's surface.

$$\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$$

When frictional forces can be neglected, **F** is the pressure gradient force

total pressure

$$\mathbf{F} = -\nabla p_T \quad \text{force per unit volume}$$

$$\rho\left(\frac{d\mathbf{u}}{dt} - 2\Omega \wedge \mathbf{u}\right) = \mathbf{F} + \rho\mathbf{g}$$

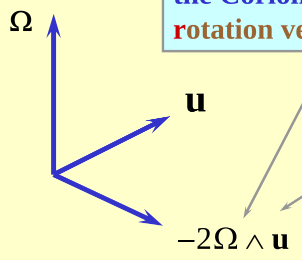
$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p_T + \mathbf{g} - 2\Omega \wedge \mathbf{u} \qquad \text{per unit mass}$$

This is Euler's equation of motion in a rotating reference frame.

The Coriolis force does no work

the Coriolis force acts normal to the rotation vector and normal to the velocity.

is directly proportional to the magnitude of **u** and Ω.



Note: the Coriolis force does no work because $\mathbf{u}\cdot(2\Omega\wedge\mathbf{u})\equiv 0$

Perturbation pressure, buoyancy force

Define $p_T = p_0(z) + p$ where $\dfrac{dp_0}{dz} = -g\rho_0$

$p_0(z)$ and $\rho_0(z)$ are reference pressure and density fields

p is the perturbation pressure

Euler's equation becomes

$\mathbf{g} = (0, 0, -g)$

$$\frac{D\mathbf{u}}{Dt} + 2\Omega\wedge\mathbf{u} = -\frac{1}{\rho}\nabla p + \underbrace{\mathbf{g}\left[\frac{\rho-\rho_0}{\rho}\right]}_{\text{the buoyancy force}}$$

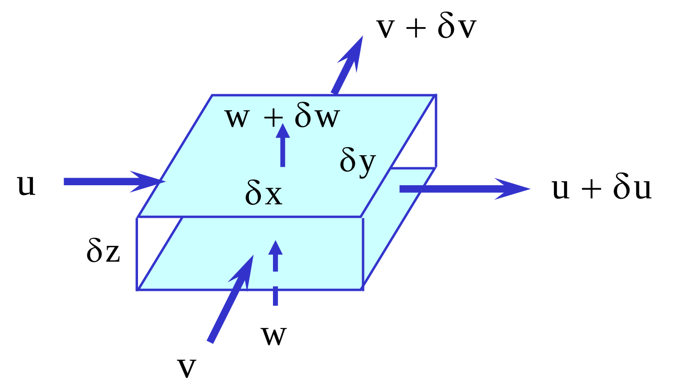
Important: the perturbation pressure gradient $-\dfrac{1}{\rho}\nabla p$

and buoyancy force $\mathbf{g}\left(\dfrac{\rho-\rho_0}{\rho}\right)$ are not uniquely defined.

But the total force $-\dfrac{1}{\rho}\nabla p + \mathbf{g}\left(\dfrac{\rho-\rho_0}{\rho}\right)$ is uniquely defined.

Indeed $$-\frac{1}{\rho}\nabla p + \mathbf{g}\left(\frac{\rho-\rho_0}{\rho}\right) = -\frac{1}{\rho}\nabla p_T + \mathbf{g}$$

Mathematical formulation of the continuity equation for an incompressible fluid



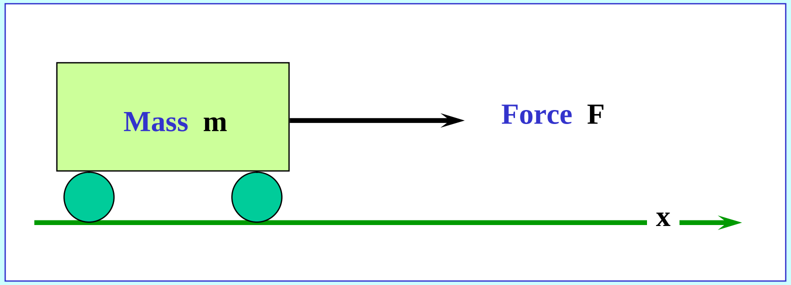
The mass continuity equation

Incompressible fluid $\nabla \cdot \mathbf{u} = 0$

Compressible fluid $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$

Anelastic approximation $\nabla \cdot (\rho_o(z)\mathbf{u}) = 0$

Rigid body dynamics



Newton's equation of motion is: $m\frac{d^2x}{dt^2} = F$

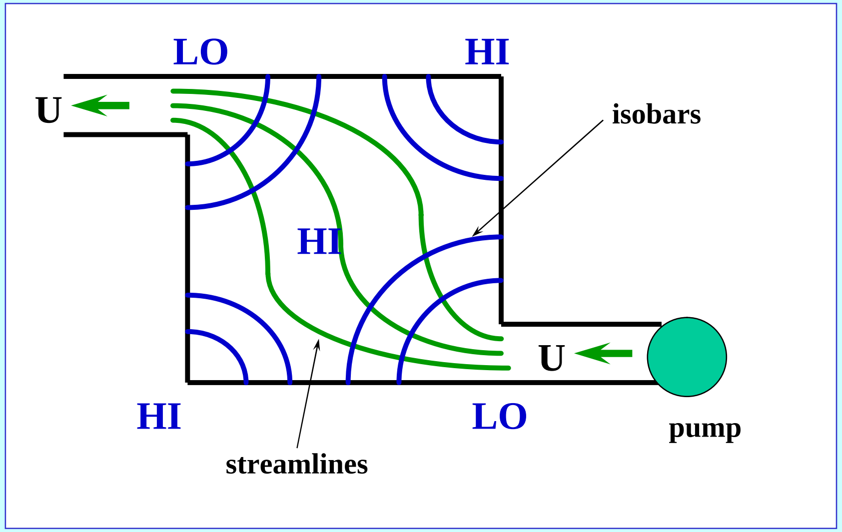
Problem is to calculate $\mathbf{x}(t)$ given the force $\mathbf{F}$

Fluid dynamics problems

- The force field is determined by the overall constraints provided by
  - the requirement of continuity
  - the boundary conditions
- In particular, the pressure field at any instant is determined by the flow configuration
  - I will now illustrate this with an example!
  - Let us forget about density differences and rotation for this example

Fluid dynamics problems

- The aim of any fluid dynamics calculation is to calculate the flow field $\mathbf{U}(x,y,z,t)$ in a given region subject to appropriate boundary conditions and the constraint of continuity.
- The calculation of the force field (i.e. the pressure field) may not be necessary, depending on the solution method.



Assumptions: inviscid, irrotational, incompressible flow

A mathematical demonstration

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p' \qquad \nabla\cdot\mathbf{u} = 0$$

Momentum equation **Continuity equation**

The divergence of the momentum equation gives:

$$\nabla^2 p' = -\nabla\cdot\left(\rho\mathbf{u}\cdot\nabla\mathbf{u}\right)$$

This is a diagnostic equation!

But what about the effects of rotation?

Newton's 2nd law vertical component

mass × acceleration = force

$$\rho\frac{Dw}{Dt} = -\frac{\partial p_T}{\partial z} - g\rho$$

buoyancy form

Put $p_T = p_o(z) + p'$, $\rho = \rho_o(z) + \rho'$ **where** $\frac{dp_o}{dz} = -g\rho_o$

Then $\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b$ **where** $b = -g\left(\frac{\rho-\rho_o}{\rho}\right)$

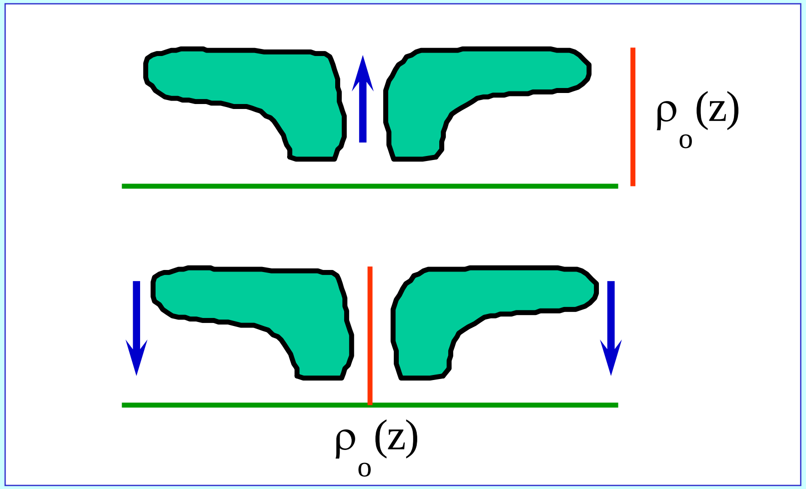
buoyancy force is NOT unique

$$b = -g\left(\frac{\rho - \rho_o}{\rho}\right)$$

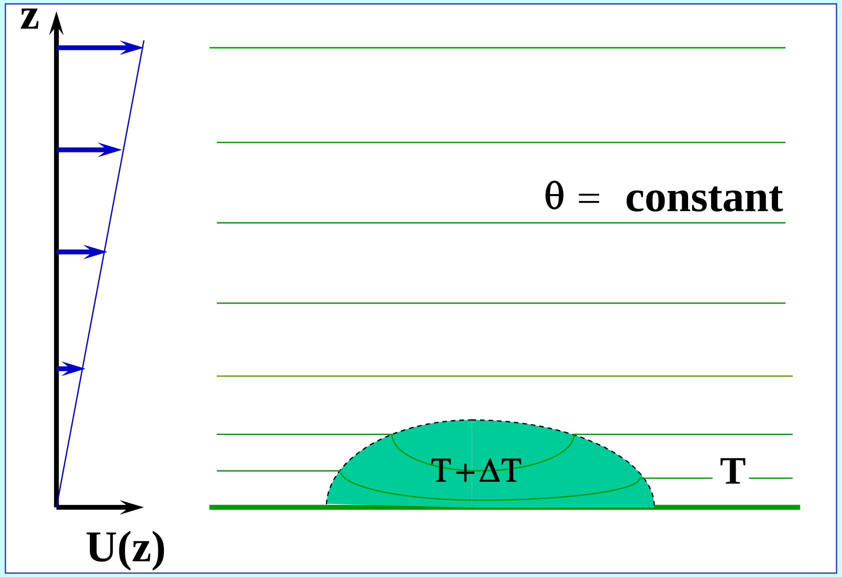
it depends on choice of reference density $\rho_o(z)$

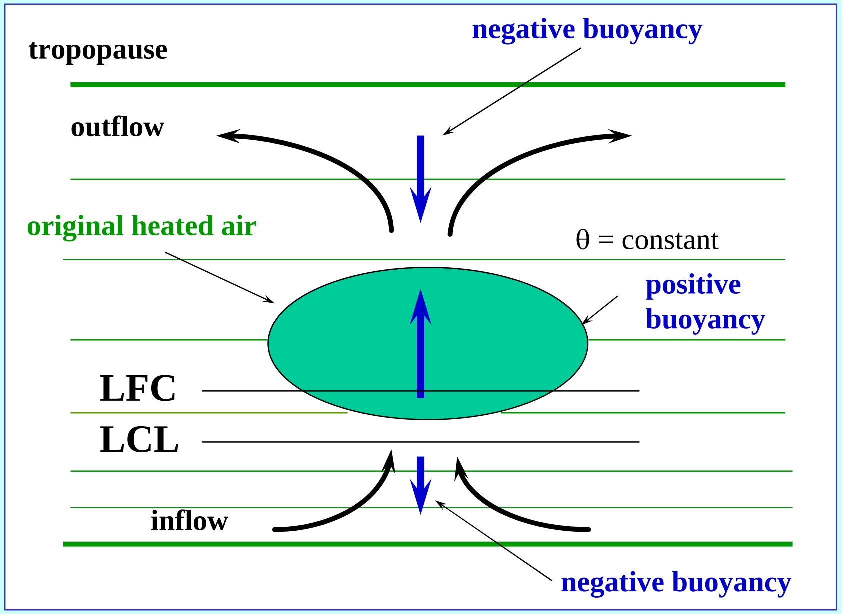
but $$-\frac{1}{\rho}\frac{\partial p_T}{\partial z} - g = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b$$ **is unique**

Buoyancy force in a hurricane



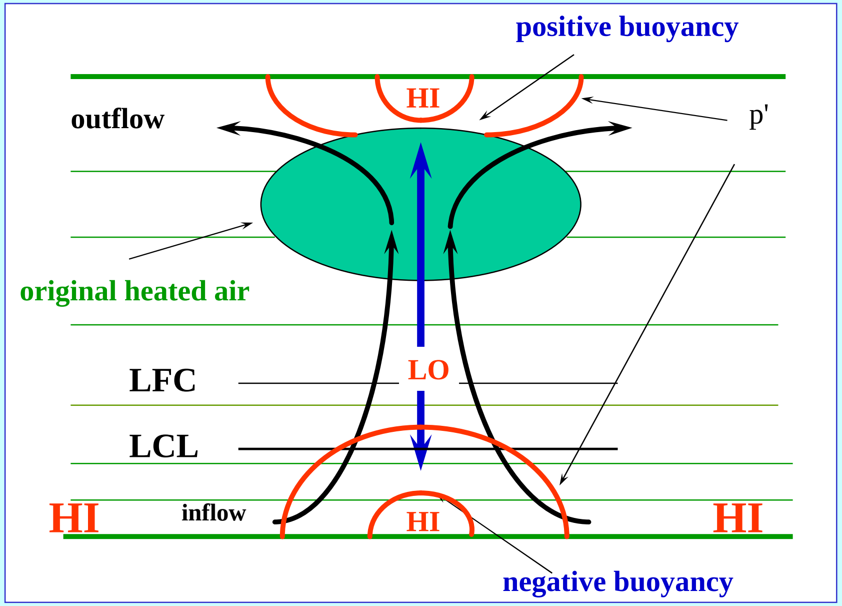
Initiation of a thunderstorm





Some questions

- How does the flow evolve after the original thermal has reached the upper troposphere?
- What drives the updraught at low levels?
  - Observation in severe thunderstorms: the updraught at cloud base is negatively buoyant!
  - Answer: - the perturbation pressure gradient



The geostrophic approximation

For frictionless motion ($\mathbf{D} = \mathbf{0}$) the momentum equation is

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

perturbation pressure

Let $Ro \to 0$

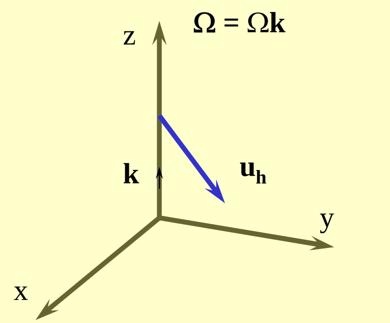
$$\Rightarrow \quad 2\boldsymbol{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$$

This is called the geostrophic equation

We expect this equation to hold approximately in synoptic scale motions in the atmosphere and oceans, except possibly near the equator.

Choose rectangular coordinates:

$\mathbf{k} = (0,0,1)$



velocity components $\mathbf{u} = (u,v,w)$, $\quad \mathbf{u} = \mathbf{u}_h + w\mathbf{k}$

$\mathbf{u}_h = (u,v,0)$ is the horizontal flow velocity

Take $\mathbf{k} \wedge$ $\quad 2\Omega \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p$

$(\mathbf{k} \cdot \mathbf{u})\mathbf{k} = (0, 0\ w)$

$$2\Omega\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{u}) = 2\Omega[\underbrace{(\mathbf{k} \cdot \mathbf{u})\mathbf{k} - \mathbf{u}}_{-\mathbf{u}_h}] = -\frac{1}{\rho}\mathbf{k} \wedge \nabla p$$

$\nabla_h p = (\partial p/\partial x, \partial p/\partial y, 0)$

$$\boxed{\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k} \wedge \nabla_h p \quad \text{and} \quad 0 = \frac{\partial p}{\partial z}}$$

This is the solution for geostrophic flow.

The geostrophic wind

$$\mathbf{u}_h = \frac{1}{2\Omega\rho}\mathbf{k} \wedge \nabla_h p$$

- The geostrophic wind blows parallel to the lines (or more strictly surfaces) of constant pressure - the isobars, with low pressure to the left.
- Well known to the layman who tries to interpret the newspaper "weather map", which is a chart showing isobaric lines at mean sea level.
- In the southern hemisphere, low pressure is to the right.

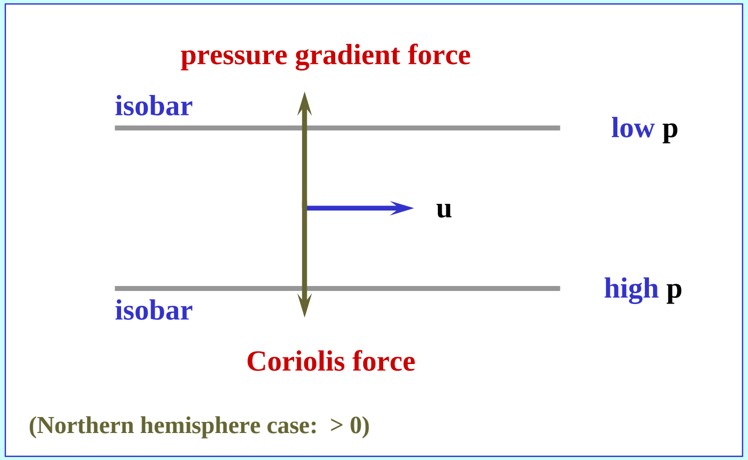
Choice of coordinates

- For simplicity, let us orientate the coordinates so that x points in the direction of the geostrophic wind.
- Then $v = 0$, implying that $\partial p/\partial x = 0$.

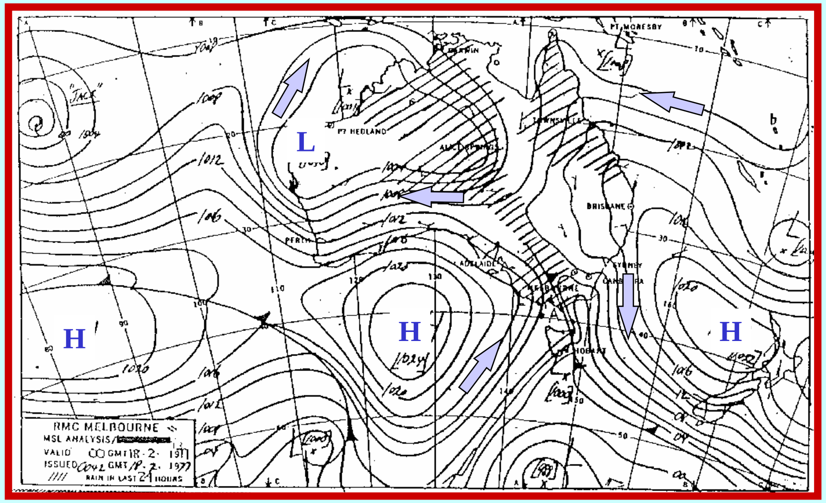
$$\boxed{u = -\frac{1}{2\Omega\rho}\frac{\partial p}{\partial y}}$$

- Note that for fixed $\mathbf{\Omega}$, the winds are stronger when the isobars are closer together and, for a given isobar separation, they are stronger for smaller $|\mathbf{\Omega}|$.

Geostrophic flow



A mean sea level isobaric chart over Australia



Note also that the solution $\mathbf{u}_h = \dfrac{1}{2\Omega\rho}\mathbf{k} \wedge \nabla_h p$

and $0 = \dfrac{\partial p}{\partial z}$

tells us nothing about the vertical velocity w.

- For an incompressible fluid, $\nabla \cdot \mathbf{u} = 0$.
- Also, for geostrophic flow, $\nabla_h \cdot \mathbf{u_h} = 0$.
- then $\partial w/\partial z = 0$ implying that w is independent of z.

If $w = 0$ at some particular z, say $z = 0$, which might be the ground, then $w \equiv 0$.

The geostrophic equation is degenerate!

- The geostrophic equation is **degenerate**, i.e. time derivatives have been eliminated in the approximation.
- We cannot use the equation to **predict** how the flow will evolve.
- Such equations are called **diagnostic** equations.
- In the case of the geostrophic equation, for example, a knowledge of the isobar spacing at a given time allows us to calculate, or '**diagnose**', the geostrophic wind.
- We cannot use the equation to **forecast** how the wind velocity will change with time.

Vortex flows: the gradient wind equation

- **Strict geostrophic motion** requires that the isobars be straight, or, equivalently, that **the flow be uni-directional.**
- To investigate **balanced flows with curved isobars**, including vortical flows, it is convenient to express Euler's equation in cylindrical coordinates.
- To do this we need an expression for the total horizontal acceleration $D\mathbf{u_h}/Dt$ in cylindrical coordinates.

The **radial** and **tangential** components of Euler's equation may be written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

The **axial** component is

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

The case of pure circular motion with $u = 0$ and $\partial/\partial\theta \equiv 0$.

$$\Rightarrow \quad \frac{v^2}{r} + fv = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

- This is called the **gradient wind equation**.
- It is a generalization of the geostrophic equation which takes into account centrifugal as well as Coriolis forces.
- This is necessary when the curvature of the isobars is large, as in an extra-tropical depression or in a tropical cyclone.

The gradient wind equation

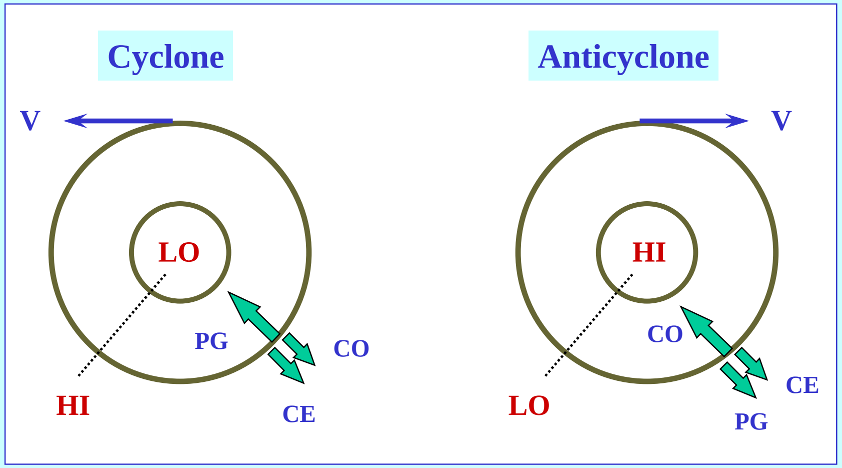
Write $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{v^2}{r} + fv$$

$$\underline{\hphantom{-\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{v^2}{r} + fv}}$$

terms interpreted as forces

- The equation expresses a balance of the centrifugal force (v^2/r) and Coriolis force (fv) with the radial pressure gradient.
- This interpretation is appropriate in the coordinate system defined by $\hat{\mathbf{r}}$ and $\hat{\theta}$, which rotates with angular velocity v/r.

Force balances in low and high pressure systems



The equation $0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{v^2}{r} + fv$

is a diagnostic equation for the tangential velocity v **in terms of the pressure gradient:**

$$\Rightarrow \quad v = -\frac{1}{2}fr + \left[\frac{1}{4}f^2r^2 + \frac{r}{\rho}\frac{\partial p}{\partial r}\right]^{\frac{1}{2}}$$

Choose the positive sign so that geostrophic balance is recovered as $r \to \infty$ **(for finite** v**, the centrifugal force tends to zero as** $r \to \infty$ **).**

$$v = -\frac{1}{2}fr + \left[\frac{1}{4}f^2r^2 + \frac{r}{\rho}\frac{\partial p}{\partial r}\right]^{\frac{1}{2}}$$

- **In a low pressure system,** $\partial p/\partial r > 0$ **and there is no theoretical limit to the tangential velocity** v**.**
- **In a high pressure system,** $\partial p/\partial r < 0$ **and the local value of the pressure gradient cannot be less than** $-\rho r f^2/4$ **in a balanced state.**
- **Therefore the tangential wind speed cannot locally exceed** $rf/2$ **in magnitude.**
- **This accords with observations in that wind speeds in anticyclones are generally light, whereas wind speeds in cyclones may be quite high.**

Limited wind speed in anticyclones

In the anticyclone, the Coriolis force increases only in proportion to v**: => this explains the upper limit on** v **predicted by the gradient wind equation.**