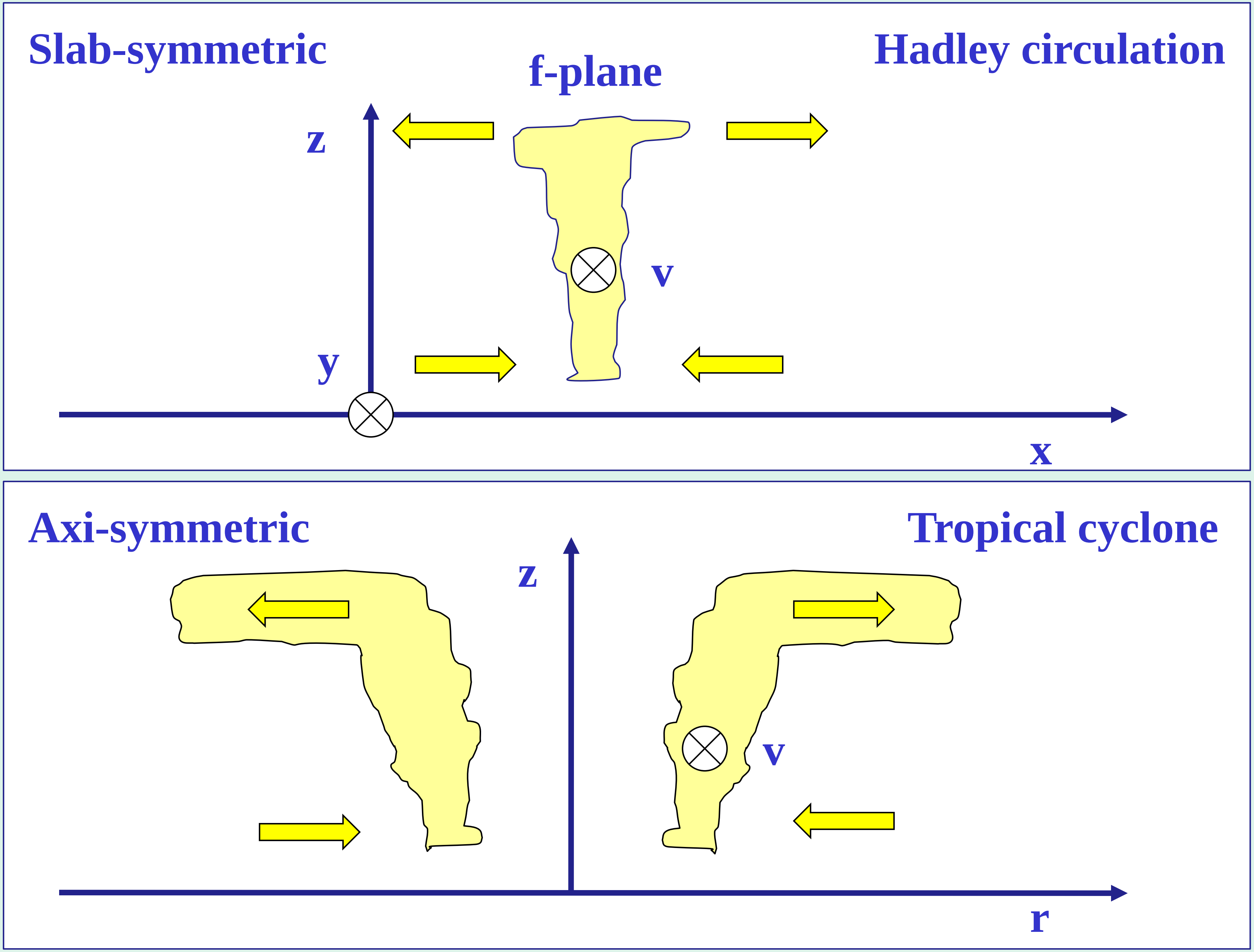
Slab-symmetric and axi-symmetric models


Slab-symmetric
f-plane
Hadley circulation
z
v
y
x
Axi-symmetric
Tropical cyclone
z
v
r

Slab-symmetric

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + fu = \dot{V}$$

$$\zeta = \frac{\partial v}{\partial x}$$

$$\boxed{\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V}}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + w\left(N^2 + \frac{\partial b}{\partial z}\right) = \dot{B}$$

$$fS = \boxed{f\frac{\partial v}{\partial z} = \frac{\partial b}{\partial x}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = \frac{\partial \psi}{\partial z} \qquad w = -\frac{\partial \psi}{\partial x}$$

Thermal wind

Axi-symmetric

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{v^2}{r} + fu = \dot{V}$$

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r}$$

$$\chi = 1/\theta$$

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta}$$

$$\boxed{g\frac{\partial(\ln \chi)}{\partial r} + C\frac{\partial(\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}}$$

$$\frac{\partial}{\partial r}\rho r u + \frac{\partial}{\partial r}\rho r z = 0$$

$$u = -\frac{1}{r\rho}\frac{\partial \psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial \psi}{\partial r}$$

Potential vorticity

Ertel PV

$$P = \frac{(\omega + \mathbf{f}) \cdot \nabla\theta}{\rho}$$

Slab-symmetric form

$$q = \omega_{\mathbf{a}} \cdot \nabla b = \left(N^2 + \frac{\partial b}{\partial z} \right) \zeta_a - \frac{\partial v}{\partial z}\frac{\partial b}{\partial x} = \left(N^2 + \frac{\partial b}{\partial z} \right) \zeta_a - f S^2$$

Sawyer-Eliassen Equation

Slab-symmetric

$$\left(N^2 + \frac{\partial b}{\partial z}\right)\frac{\partial^2 \psi}{\partial x^2} - 2fS\frac{\partial^2 \psi}{\partial x \partial z} + f\zeta_a \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial \dot{V}}{\partial z} - \frac{\partial \dot{B}}{\partial x}$$

Transform $X = x + v/f, \qquad Z = z$

$$\frac{\partial}{\partial X}\left(q\frac{\partial \psi}{\partial X}\right) + f^3\frac{\partial^2 \psi}{\partial Z^2} = \frac{f^2}{\zeta_a}\left(\frac{\partial \dot{V}}{\partial Z} + \frac{S}{f}\frac{\partial \dot{V}}{\partial X}\right) - \frac{\partial \dot{B}}{\partial X}$$

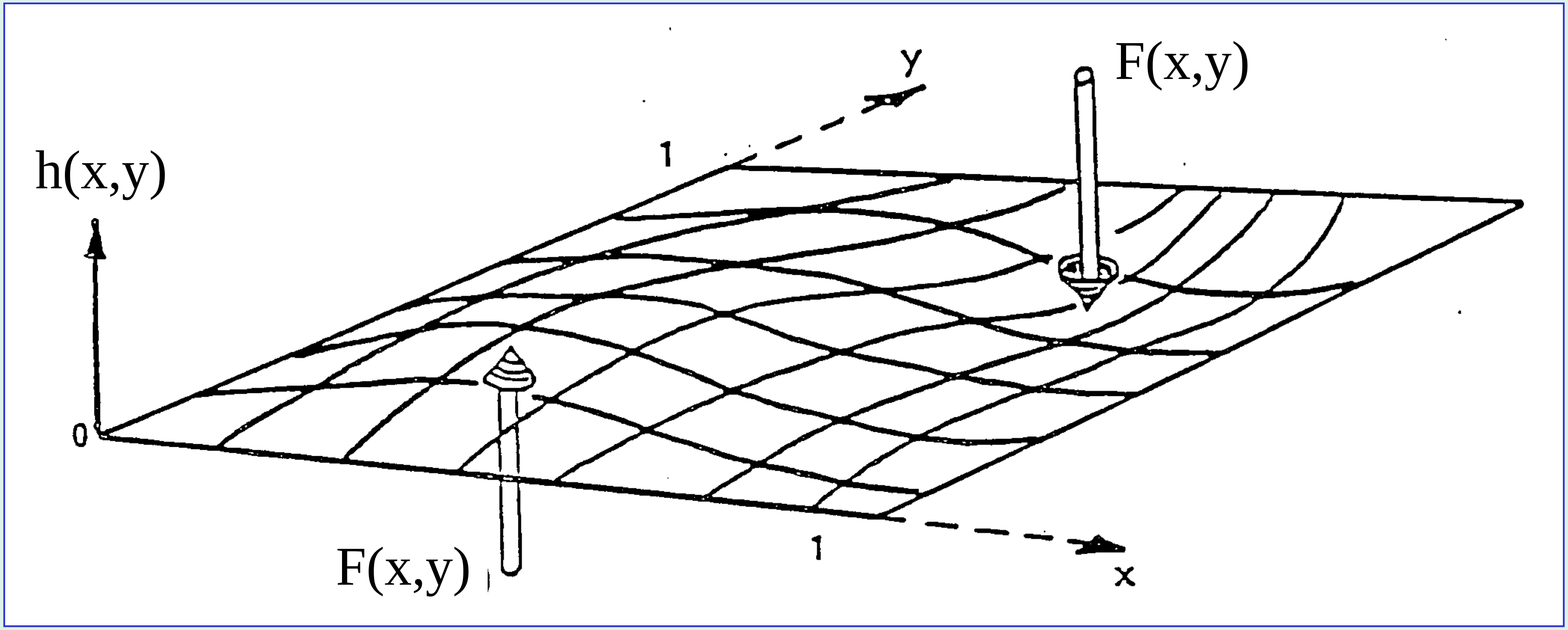
Special case q = constant

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial \bar{Z}^2} = \frac{f^2}{q\zeta_a}\left(\frac{\partial \dot{V}}{\partial Z} + \frac{S}{f}\frac{\partial \dot{V}}{\partial X}\right) - \frac{1}{q}\frac{\partial \dot{B}}{\partial X}$$

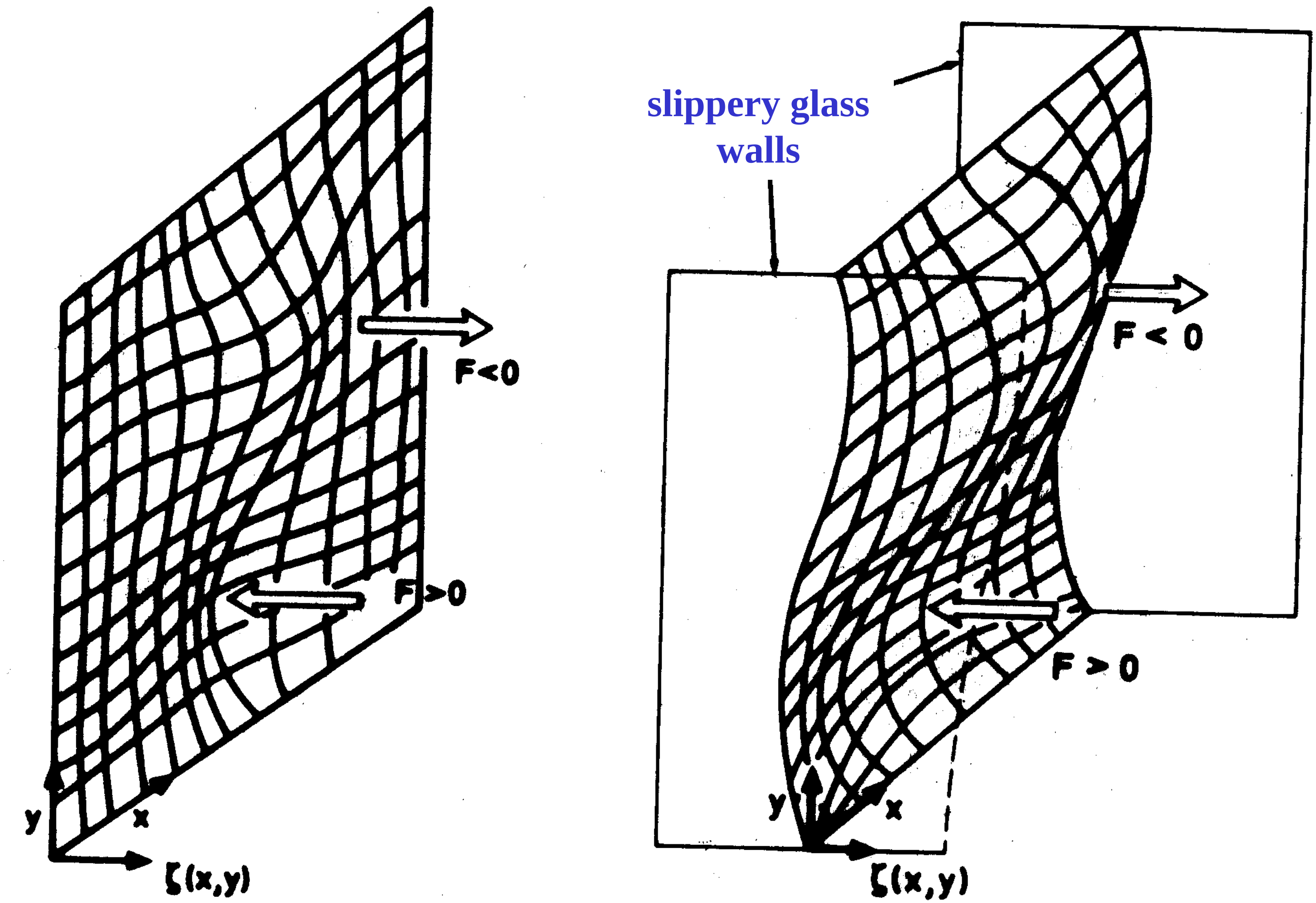
where $\bar{Z} = \sqrt{q/f}\,Z/f$ is a vertical length scale

The membrane analogy

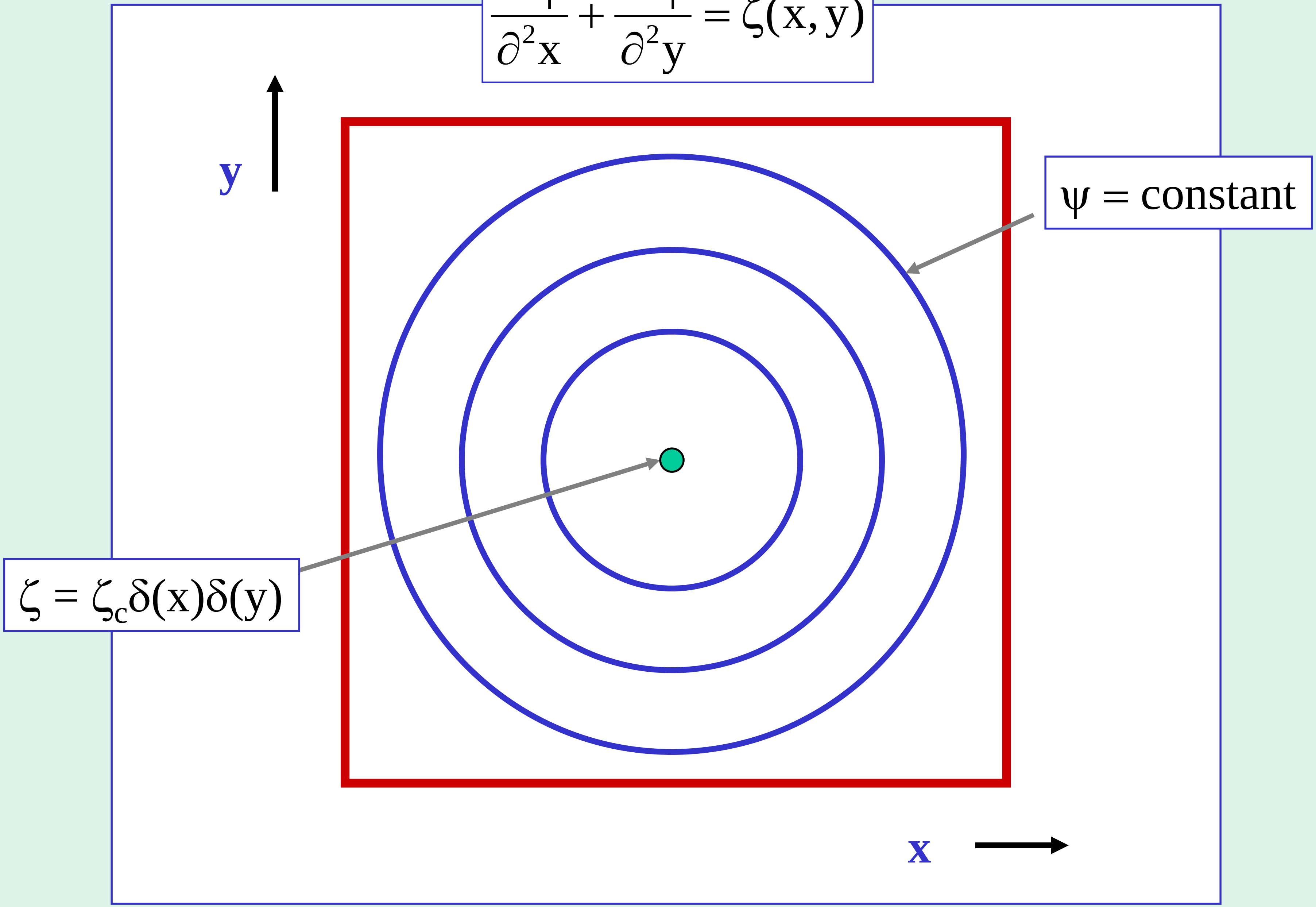
$$\frac{\partial^2 h}{\partial^2 x} + \frac{\partial^2 h}{\partial^2 y} = -F(x, y)$$



Equilibrium displacement of a stretched membrane over a square under the force distribution F(x,y).


F<0
F>0
y
x
ζ(x,y)
slippery glass walls
F < 0
F > 0
y
x
ζ(x,y)

$$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = \zeta(x, y)$$

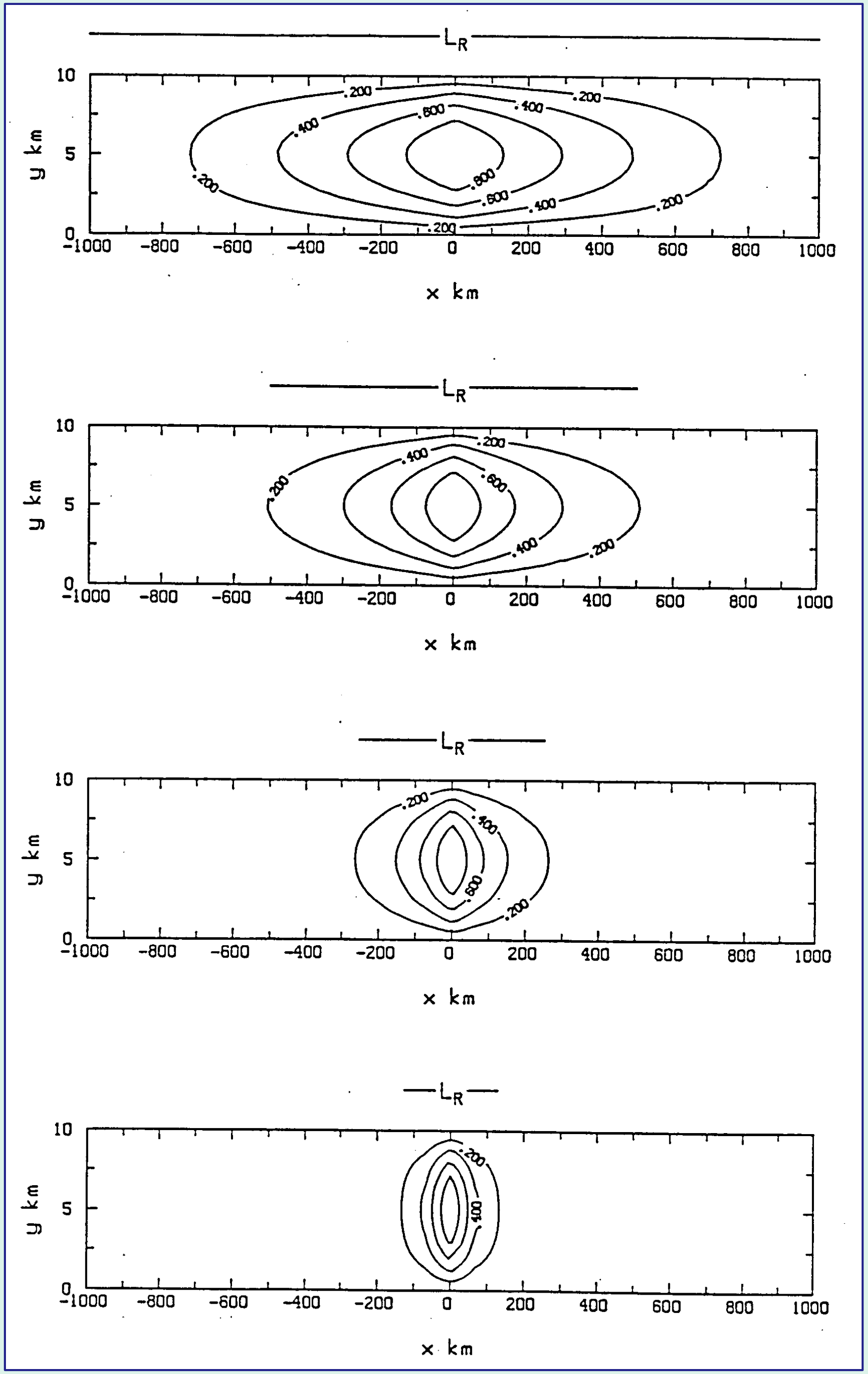


$$N^2 \frac{\partial^2 \psi}{\partial^2 x} + f^2 \frac{\partial^2 \psi}{\partial^2 z} = F(x,z)$$

$$\text{Put} \quad \bar{z} = \frac{N}{f} z \quad \Rightarrow$$

$$\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial^2 \bar{z}} = \frac{1}{N^2} \bar{F}(x,\bar{z})$$

$$L_R = \frac{NH}{f}$$



The Sawyer-Eliassen equation

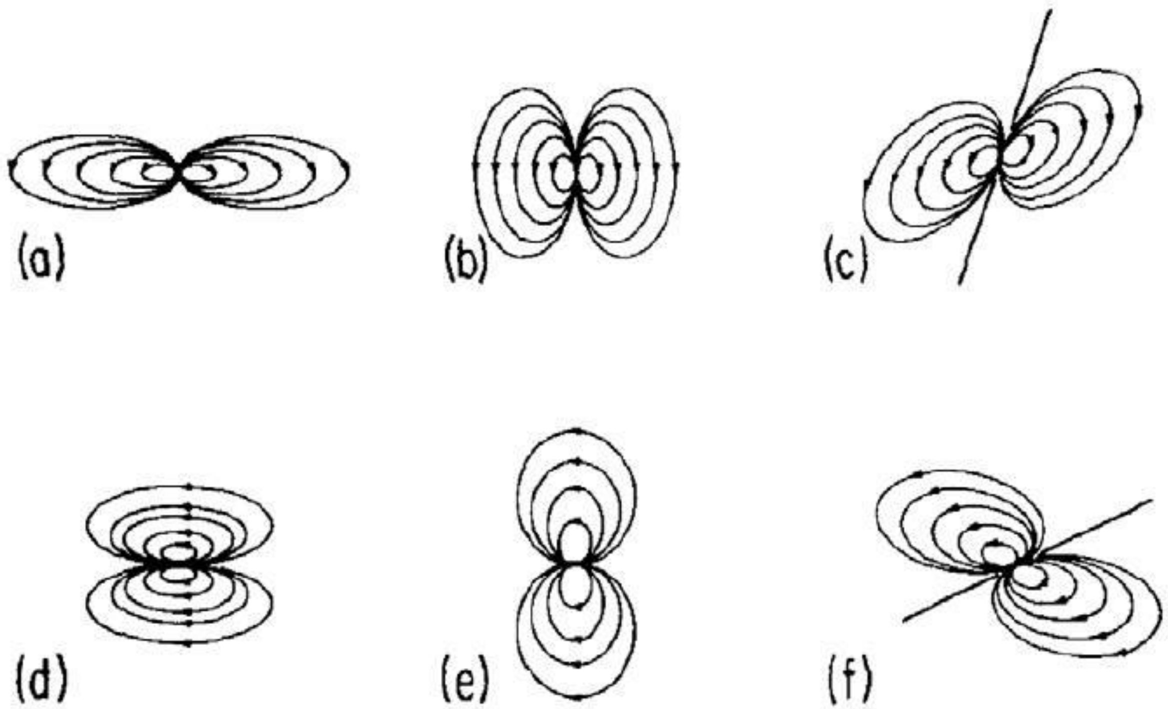


Figure 3.11: Streamfunction responses to point sources of: (a) Heat in a barotropic vortex with weak inertial stability, (b) heat in a barotropic vortex with strong inertial stability, (c) heat in a baroclinic vortex, (d) momentum in a barotropic vortex with weak inertial stability, (e) momentum in a barotropic vortex with strong inertial stability, and (f) momentum in a baroclinic vortex. (Based on Figs. 8, 9, 11, and 12

Sawyer-Eliassen Equation

Axi-symmetric

$$\frac{\partial}{\partial r}\left[g\frac{\partial \chi}{\partial z}\frac{1}{\rho r}\frac{\partial \psi}{\partial r}+\frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial z}\right]-$$

$$\frac{\partial}{\partial z}\left[\left(\xi\chi(\zeta+f)+C\frac{\partial \chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial \psi}{\partial z}-\frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right]=g\frac{\partial Q}{\partial r}+\frac{\partial}{\partial z}(CQ)+\frac{\partial}{\partial z}(\chi\xi\dot{V})$$

Discriminant

$$D=-g\frac{\partial \chi}{\partial z}\left(\xi\chi(\zeta+f)+C\frac{\partial \chi}{\partial r}\right)-\left[\frac{\partial}{\partial z}(\chi C)\right]^2$$

SE equation is elliptic if D > 0

$$u=-\frac{1}{r\rho}\frac{\partial \psi}{\partial z}\qquad w=\frac{1}{r\rho}\frac{\partial \psi}{\partial r}$$

Parameters in SE Equation

- the *static stability*

$$N^2 = -g\frac{\partial ln\chi}{\partial z};$$

- the *inertial stability*

$$I^2 = \frac{1}{r^3}\frac{\partial M^2}{\partial r} = \xi(\zeta + f)$$

- the *baroclinicity*

$$B^2 = \frac{1}{r^3}\frac{\partial M^2}{\partial z} = \xi S.$$

$$\xi = \frac{2v}{r} + f$$

Potential vorticity

Ertel PV

$$P = \frac{(\omega + \mathbf{f}) \cdot \nabla\theta}{\rho}$$

Slab-symmetric

$$q = \omega_{\mathbf{a}} \cdot \nabla b = \left(N^2 + \frac{\partial b}{\partial z}\right)\zeta_a - \frac{\partial v}{\partial z}\frac{\partial b}{\partial x} = \left(N^2 + \frac{\partial b}{\partial z}\right)\zeta_a - f S^2$$

Axi-symmetric

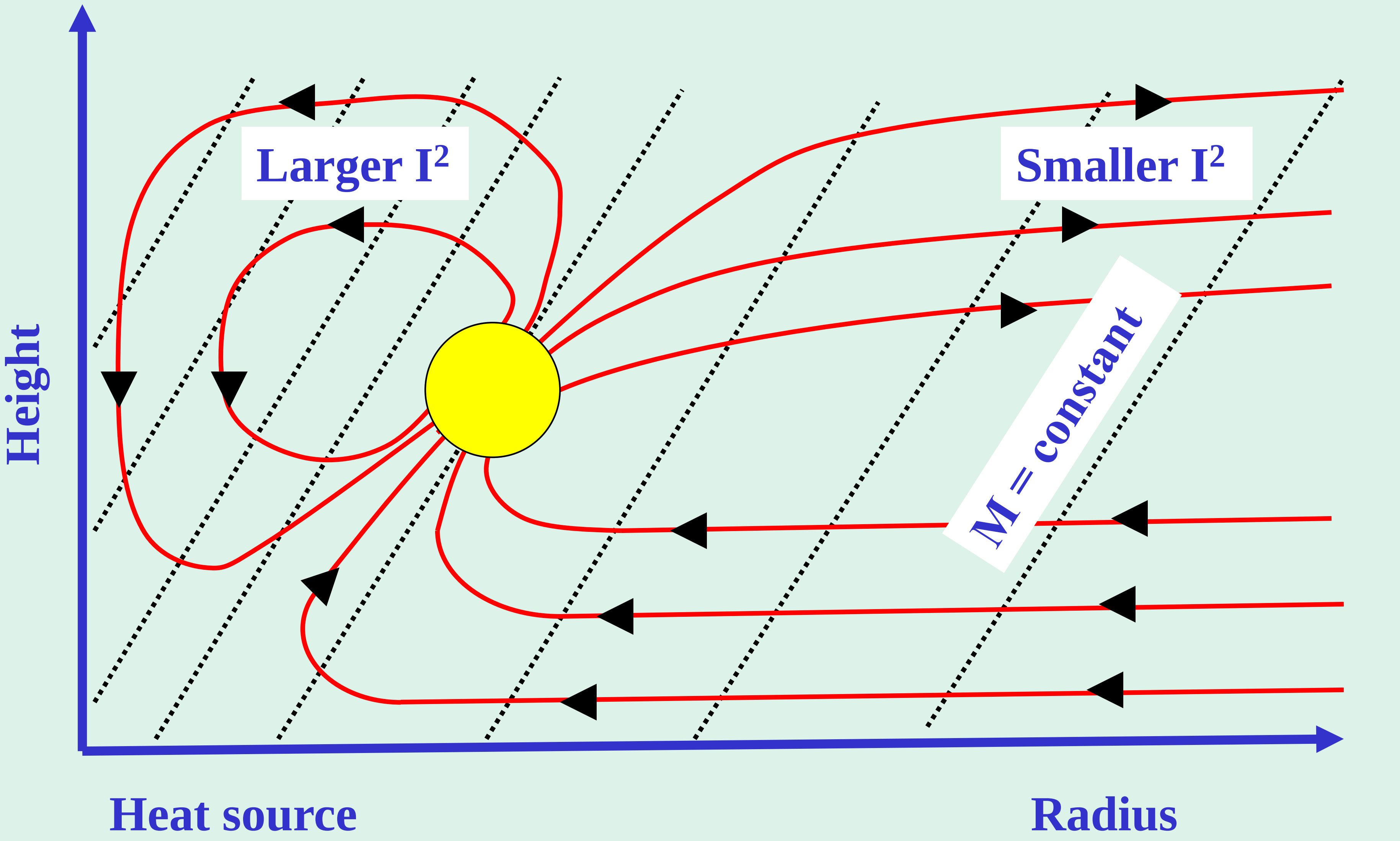
$$P = \frac{1}{\rho\chi^2}\left[\frac{\partial v}{\partial z}\frac{\partial \chi}{\partial r} - (\zeta + f)\frac{\partial \chi}{\partial z}\right]$$

Discriminant

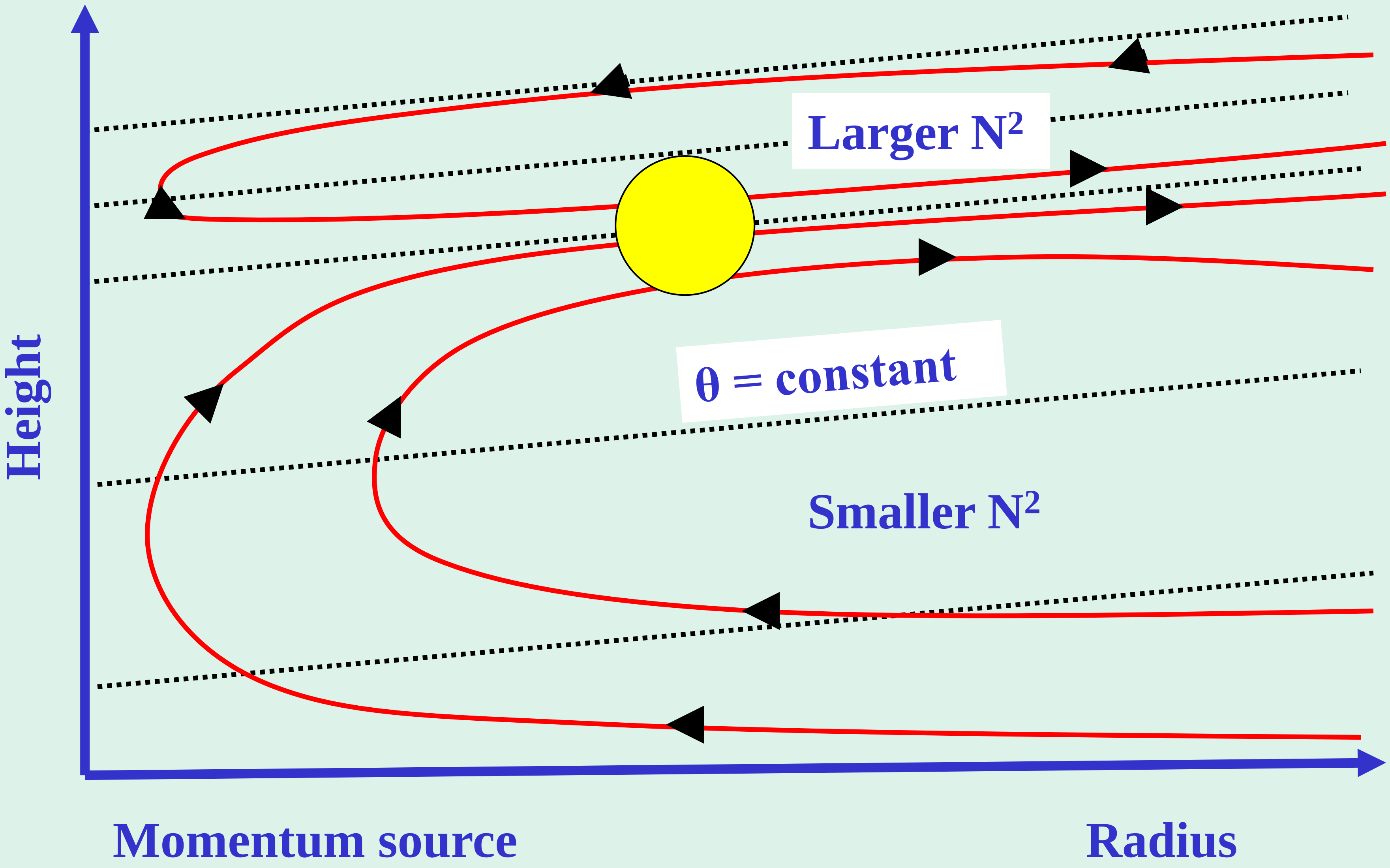
$$D = -g\frac{\partial \chi}{\partial z}\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right) - \left[\frac{\partial}{\partial z}(\chi C)\right]^2$$

Can show that

$$g\rho\chi^3\xi P = D$$

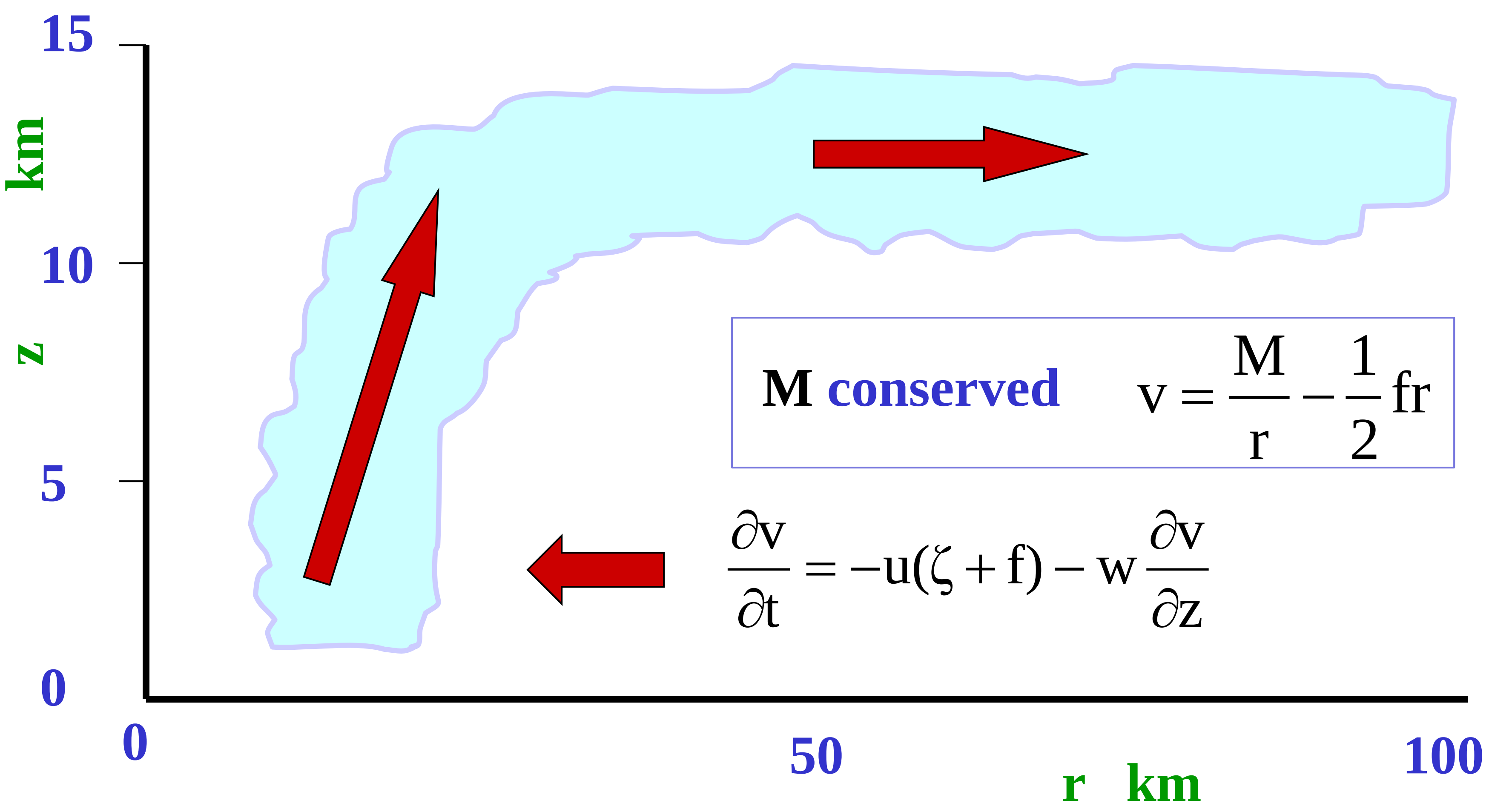


Larger I²
Smaller I²
M = constant
Height
Heat source
Radius



Larger N²
θ = constant
Smaller N²
Height
Momentum source
Radius

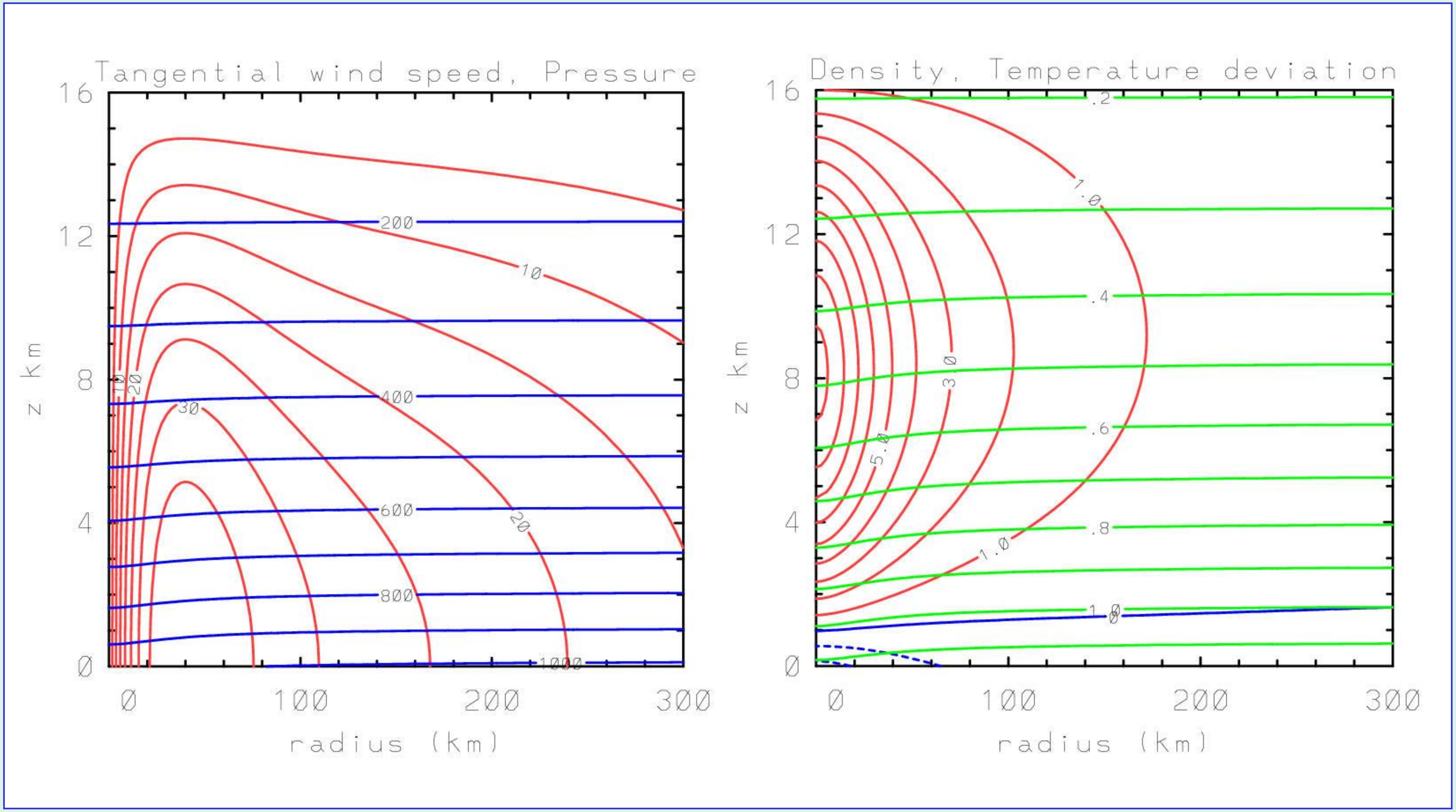
Thermally-forced secondary circulation leads to spin up



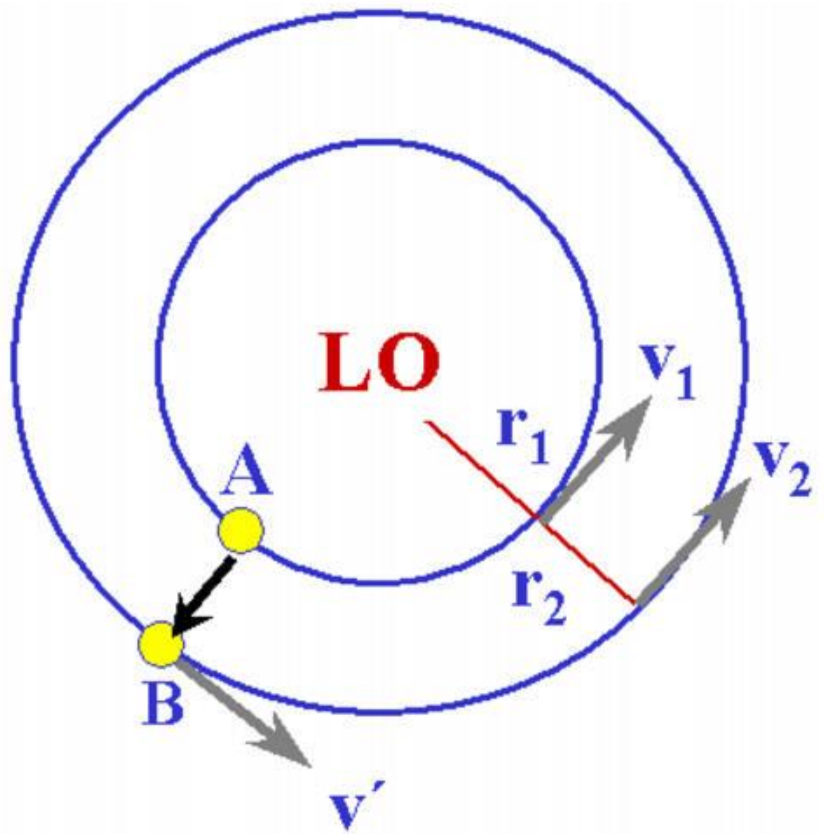
$$v = \frac{M}{r} - \frac{1}{2} f r$$

$$\frac{\partial v}{\partial t} = -u(\zeta + f) - w \frac{\partial v}{\partial z}$$

A warm-cored vortex



Barotropic stability



The parcel at A conserves its angular momentum during its radial displacement to B

$$r_2 v' + \frac{1}{2} f r_2^2 = r_1 v_1 + \frac{1}{2} f r_1^2,$$

$$v' = \frac{r_1}{r_2} v_1 + \frac{1}{2}\frac{f}{r_2}(r_1^2 - r_2^2) \qquad (3.17)$$

Net radial force on a displaced air parcel

Radial pressure gradient at B

$$\frac{1}{\rho}\frac{dp}{dr}\bigg]_{r=r_2} = \frac{v_2^2}{r_2} + f v_2. \tag{3.18}$$

Net force on parcel at B

$$\begin{aligned} F &= \text{centrifugal} + \text{Coriolis force} - \text{radial pressure gradient} \\ &= \frac{v'^2}{r_2} + f v' - \frac{1}{\rho}\frac{\partial p}{\partial r}\bigg]_{r=r_2} \end{aligned}$$

$$F = \frac{1}{r_2^3}\left[\left(r_1 v_1 + \frac{1}{2} r_1^2 f\right)^2 - \left(r_2 v_2 + \frac{1}{2} r_2^2 f\right)^2\right]. \tag{3.19}$$

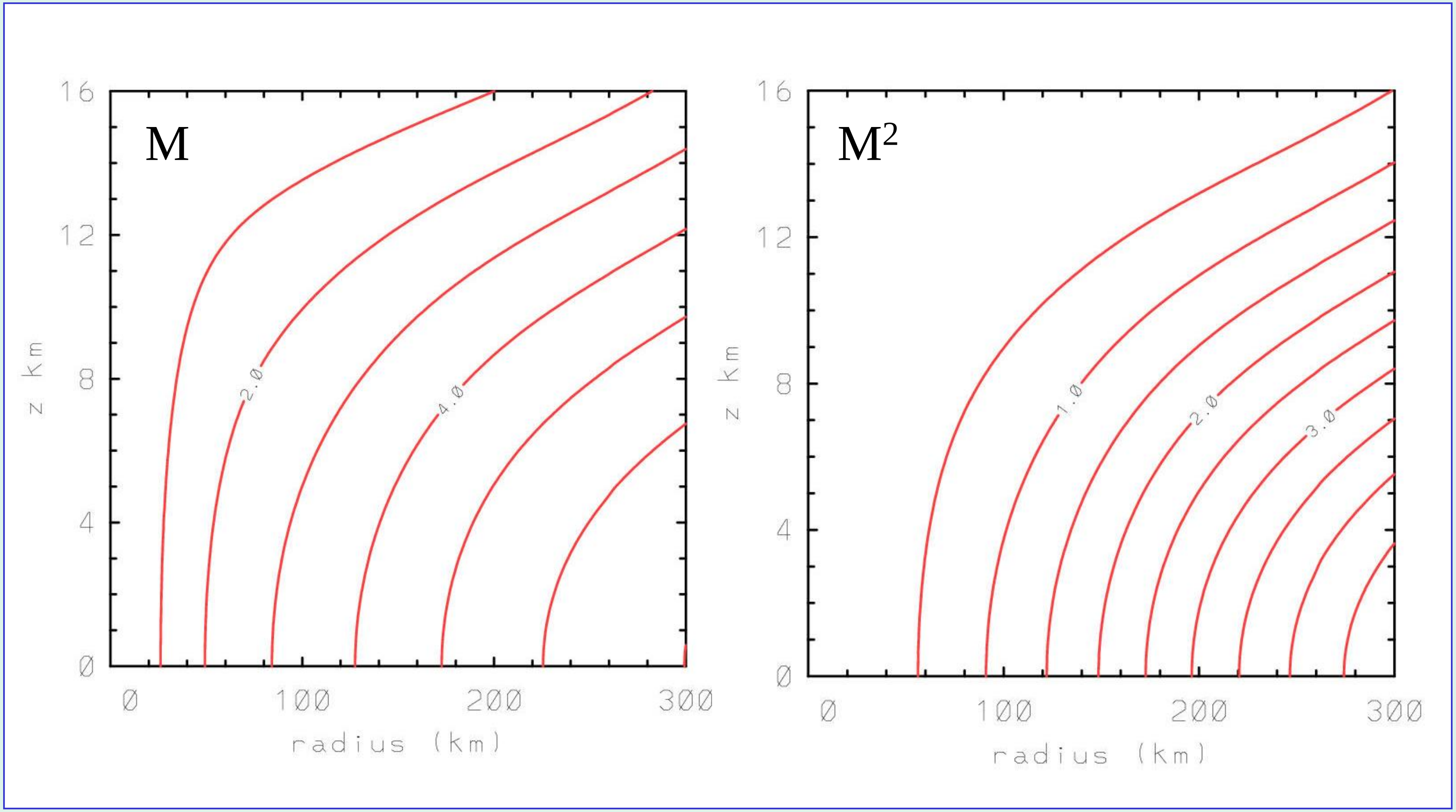
Net radial force on a displaced air parcel

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2} r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2} r_2^2 f)^2\right]. \tag{3.19}$$

In the special case of solid body rotation, $v = \Omega r$, and for a small displacement from radius $r_1 = r$ to $r_2 = r + r'$, (3.19) gives

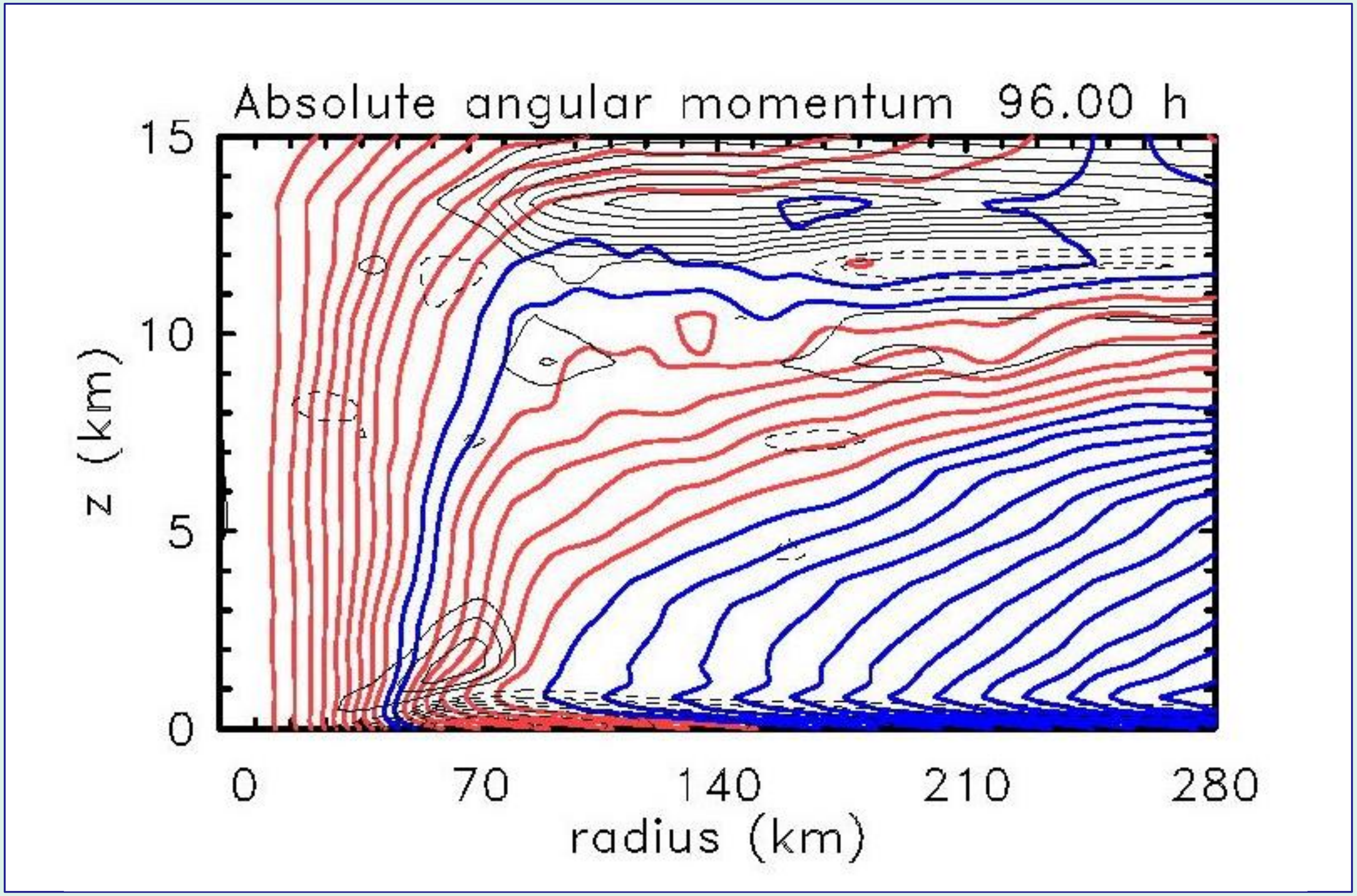
$$F \approx -4(\Omega + \frac{1}{2} f)^2 r' \tag{3.20}$$

AAM in a typical vortex

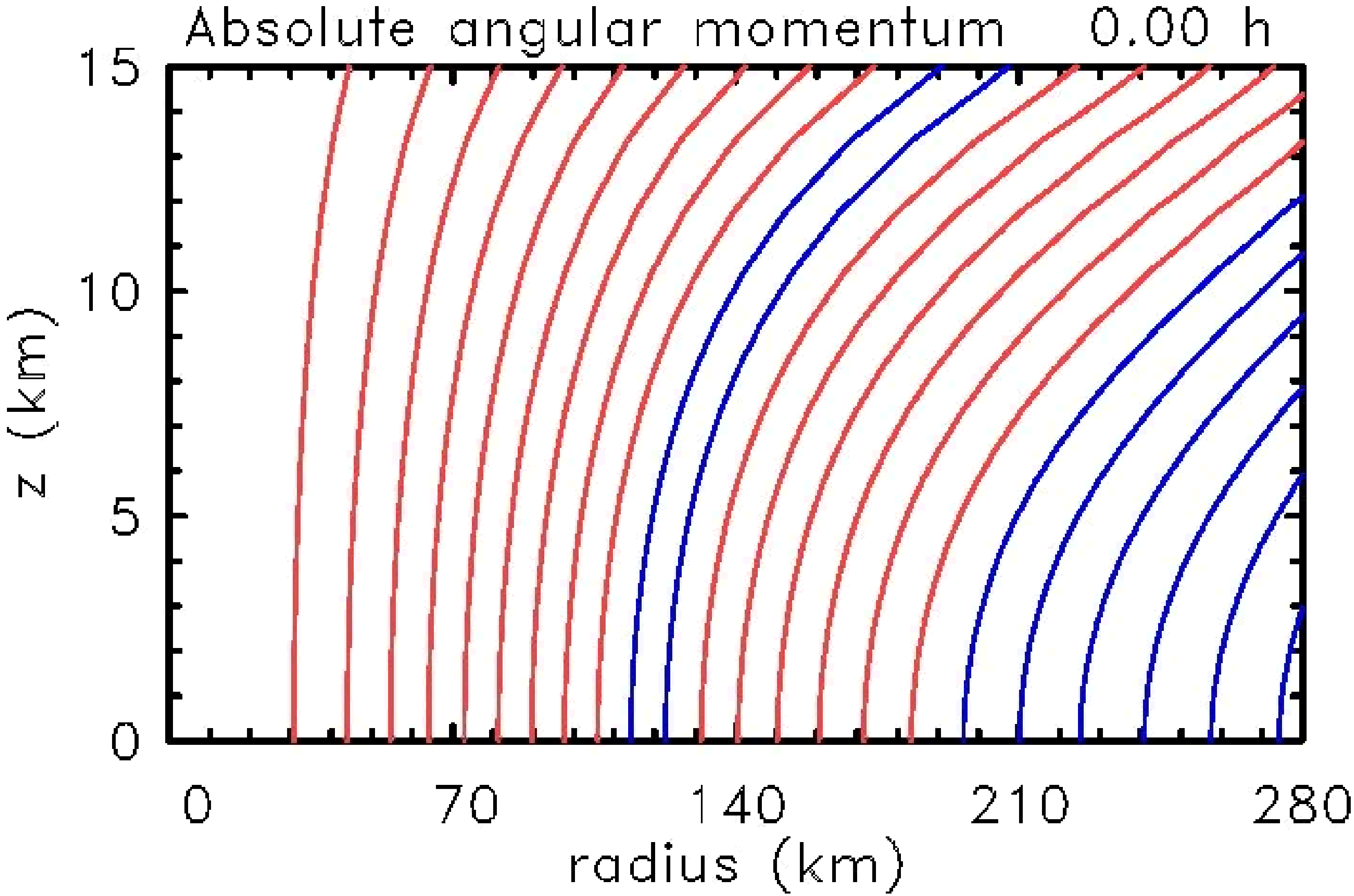


Movie

Time-height sequence of Absolute Angular Momentum

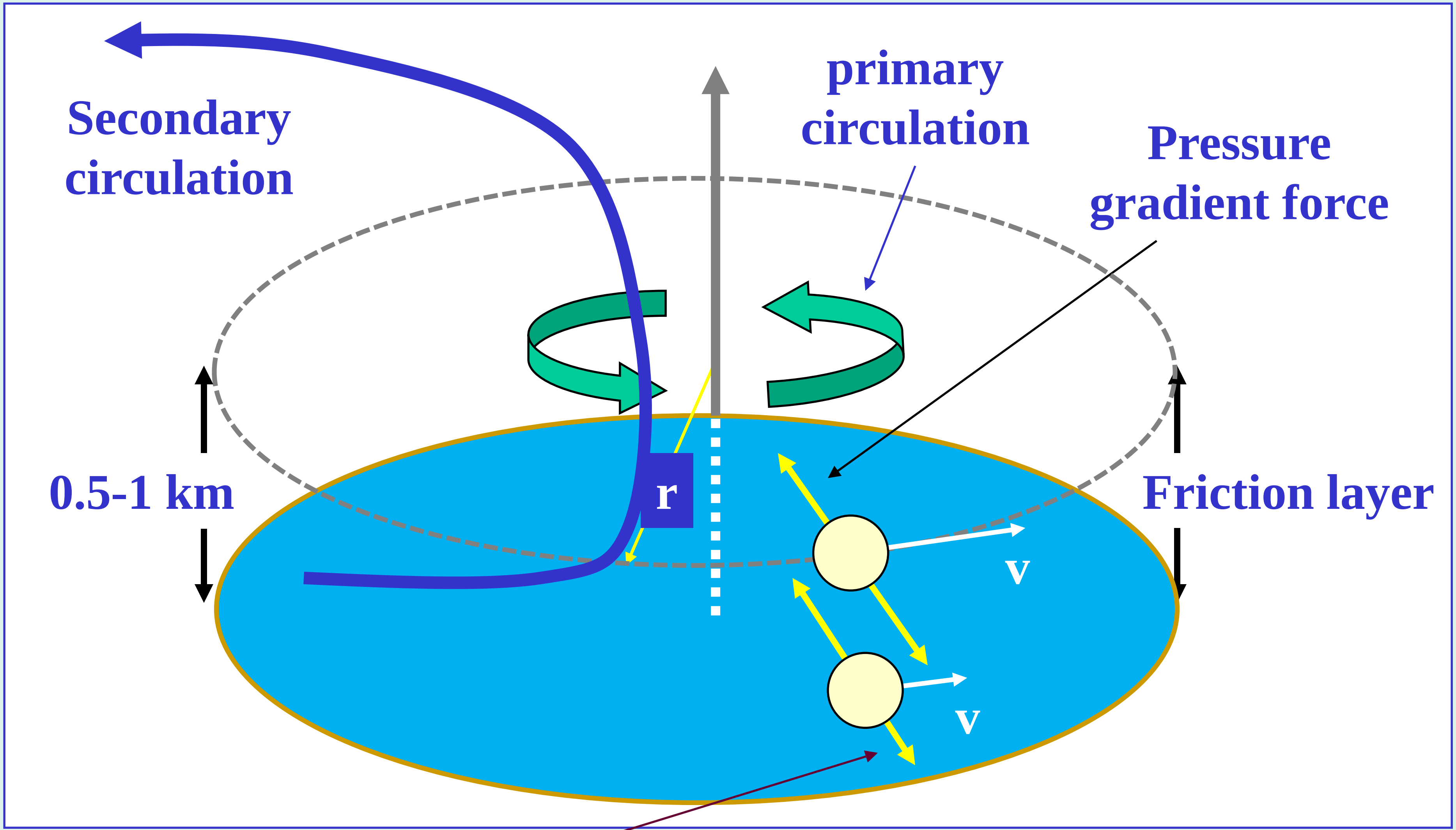


$$M = rv + \frac{1}{2} fr^2 \quad \Rightarrow \quad v = \frac{M}{r} - \frac{1}{2} fr$$


Absolute angular momentum 0.00 h
15
10
5
0
z (km)
0
70
140
210
280
radius (km)

The tropical cyclone boundary layer

Frictional effects on the secondary circulation



Centrifugal force and Coriolis force are reduced by friction

“Tea cup” Experiment

Boundary-layer scaling

Continuity equation

$$\frac{1}{r}\frac{\partial \rho r u}{\partial r} \qquad \frac{\partial \rho w}{\partial z}$$

$$\rho\frac{U}{R} \qquad \rho\frac{W}{Z}$$

w-momentum

$$\frac{\partial w}{\partial t} \quad +u\frac{\partial w}{\partial r} \quad +w\frac{\partial w}{\partial z} \qquad = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad +K\nabla_h^2 w \qquad +K\frac{\partial^2 w}{\partial z^2} \tag{4.3}$$

$$\frac{W}{T} \quad \frac{UW}{R} \quad \frac{WW}{Z} \qquad \frac{\Delta p}{\rho Z} \quad K\frac{W}{R^2} \quad K\frac{W}{Z^2} \tag{3a}$$

$$S^2A^2 \quad S^2A^2 \quad S^2A^2 \qquad \frac{\Delta p}{\rho V^2} \quad SA^3R_e^{-1} \quad SAR_e^{-1} \tag{3b}$$

u-momentum

$$\begin{array}{cccccccccc}
\frac{\partial u}{\partial t} & +u\frac{\partial u}{\partial r} & +w\frac{\partial u}{\partial z} & -\frac{v^2}{r} & -fv & = & -\frac{1}{\rho}\frac{\partial p}{\partial r} & +K\left(\nabla_h^2 u - \frac{u}{r^2}\right) & +K\frac{\partial^2 u}{\partial z^2} & (4.1) \\
\frac{U}{T} & \frac{U^2}{R} & W\frac{U}{Z} & \frac{V^2}{R} & fV & & \frac{\Delta p}{\rho R} & K\frac{U}{R^2} & K\frac{U}{Z^2} & (1a) \\
S_p^2 & S_p^2 & S_p^2 & 1 & \frac{1}{Ro} & & \frac{\Delta p}{\rho V^2} & S_p A R_e^{-1} & S_p A^{-1} R_e^{-1} & (1b)
\end{array}$$

v-momentum

$$\begin{array}{cccccccccc}
\frac{\partial v}{\partial t} & +u\frac{\partial v}{\partial r} & +w\frac{\partial v}{\partial z} & +\frac{uv}{r} & +fu & = & & +K\left(\nabla_h^2 v - \frac{v}{r^2}\right) & +K\frac{\partial^2 v}{\partial z^2} & (4.2) \\
\frac{V}{T} & U\frac{V}{R} & W\frac{V}{Z} & U\frac{V}{R} & fU & & & K\frac{V}{R^2} & K\frac{V}{Z^2} & (2a) \\
S_p & S_p & S_p & S_p & \frac{S_p}{Ro} & & & A R_e^{-1} & A^{-1} R_e^{-1} & (2b)
\end{array}$$

$$\nabla_h^2 = (\partial/\partial r)(r\partial/\partial r)$$

Full BL equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} + \frac{V^2 - v^2}{r} + f(V - v) = K\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = K\frac{\partial^2 v}{\partial z^2}$$

Put $v = V(r) + v'$. Then the equations become:

$$\underline{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}} - \left(\frac{2V}{r} + f\right)v' - \underline{\frac{v'^2}{r}} = K\frac{\partial^2 u}{\partial z^2}$$

$$\underline{\frac{\partial v'}{\partial t} + u\frac{\partial v'}{\partial r} + w\frac{\partial v'}{\partial z} + \frac{uv'}{r}} + \left(\frac{dV}{dr} + \frac{V}{r} + f\right)u = K\frac{\partial^2 v'}{\partial z^2}$$

Nonlinear terms

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \left(\frac{2V}{r} + f\right)v' - \frac{v'^2}{r} = K\frac{\partial^2 u}{\partial z^2}$$

$$\underline{\frac{U^2}{R} \quad \frac{U^2}{R} \quad \frac{U^2}{R}} \quad \frac{VV'}{R} \quad fV' \quad \underline{\frac{V'^2}{R}} \quad \frac{KU}{Z^2}$$

$$\frac{\partial v'}{\partial t} + u\frac{\partial v'}{\partial r} + w\frac{\partial v'}{\partial z} + \frac{uv'}{r} + \left(\frac{dV}{dr} + \frac{V}{r} + f\right)u = K\frac{\partial^2 v'}{\partial z^2}$$

$$\underline{\frac{UV'}{R} \quad \frac{UV'}{R} \quad \frac{UV'}{R} \quad \frac{UV'}{R}} \quad \frac{VU}{R} \quad \frac{VU}{R} \quad fU \quad \frac{KV'}{Z^2}$$

Assume $U \approx V'$

$$\frac{\text{Nonlinear terms}}{\text{Linear terms}} \approx \frac{U}{V} \approx 0.3$$

Nonlinear terms

Linearized equations

$$-v\left(f+\frac{2V}{r}\right)=K\frac{\partial^2 u}{\partial z^2}$$

$$u\left(f+\frac{V}{r}+\frac{\partial V}{\partial r}\right)=K\frac{\partial^2 v}{\partial z^2}$$

Ekman equations

$$-v\left(f+\cancel{\frac{2V}{r}}\right)=K\frac{\partial^2 u}{\partial z^2}$$

$$u\left(f+\cancel{\frac{V}{r}+\frac{\partial V}{\partial r}}\right)=K\frac{\partial^2 v}{\partial z^2}$$

Justification?
Scale analysis

4.3 The Ekman boundary layer

The scale analysis of the u- and v-momentum equations in Table 4.1 show that for small Rossby numbers ($Ro << 1$), there is an approximate balance between the net Coriolis force and the diffusion of momentum, expressed by the equations:

$$f(v_g - v) = K\frac{\partial^2 u}{\partial z^2} \tag{4.1}$$

and

$$fu = K\frac{\partial^2 v}{\partial z^2}, \tag{4.2}$$

Equations (4.1) and (4.2) are linear in u and v and may be readily solved by setting $V = v + iu$, where $i = \surd(-1)$. Then they reduce to the single differential equation

$$K\frac{d^2 V}{dz^2} - ifV = -ifV_g, \tag{4.3}$$

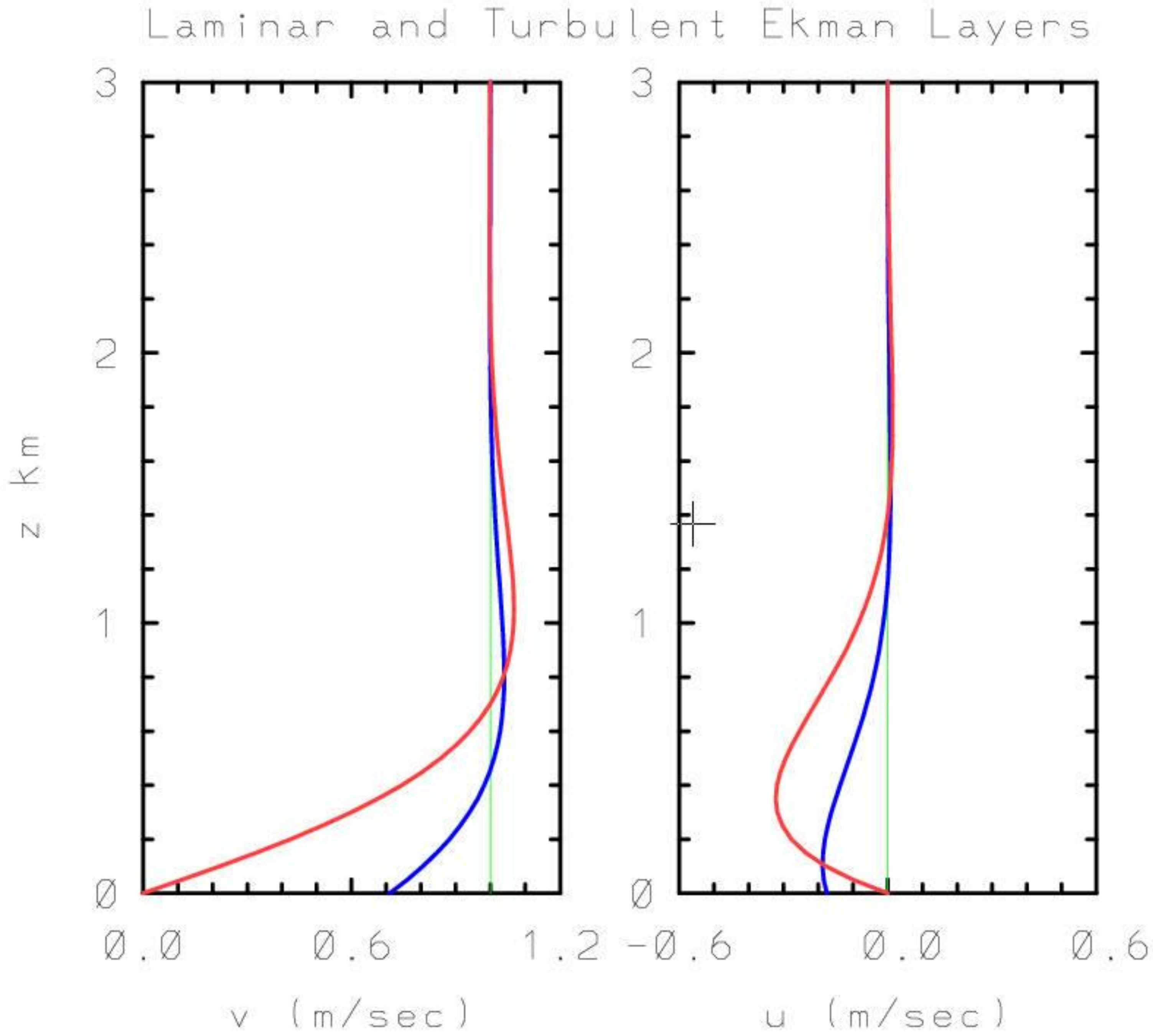
the solution has the form

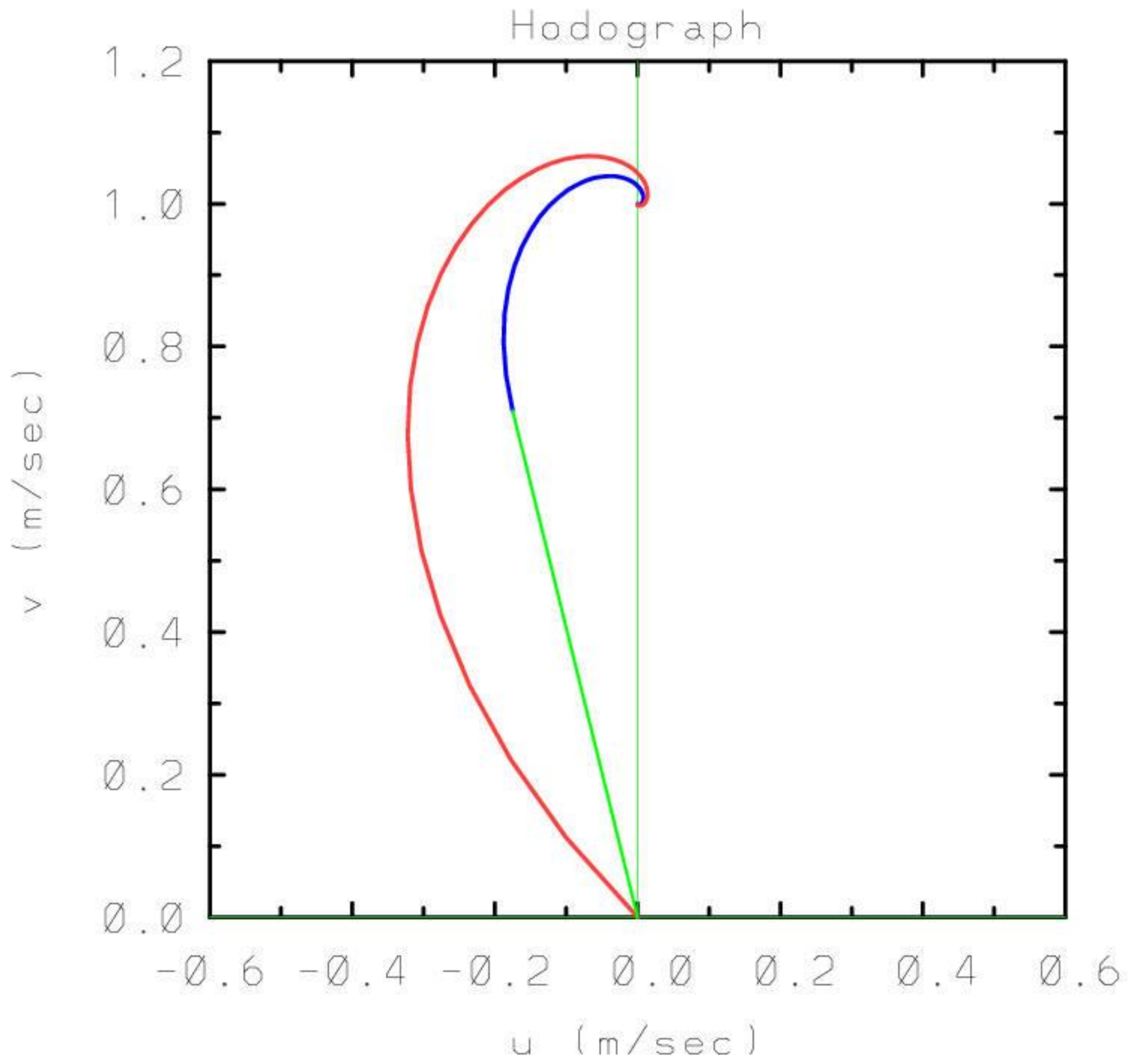
$$V = V_g[1 - A\exp(-(1-i)z/\delta),] \tag{4.4}$$

Representation of frictional stress

$$K \left. \frac{\partial \mathbf{u}}{\partial z} \right|_{z=0} = C_D |\mathbf{u_b}| \mathbf{u_b}$$

$$\mathbf{u_b} = (u_b, v_b)$$

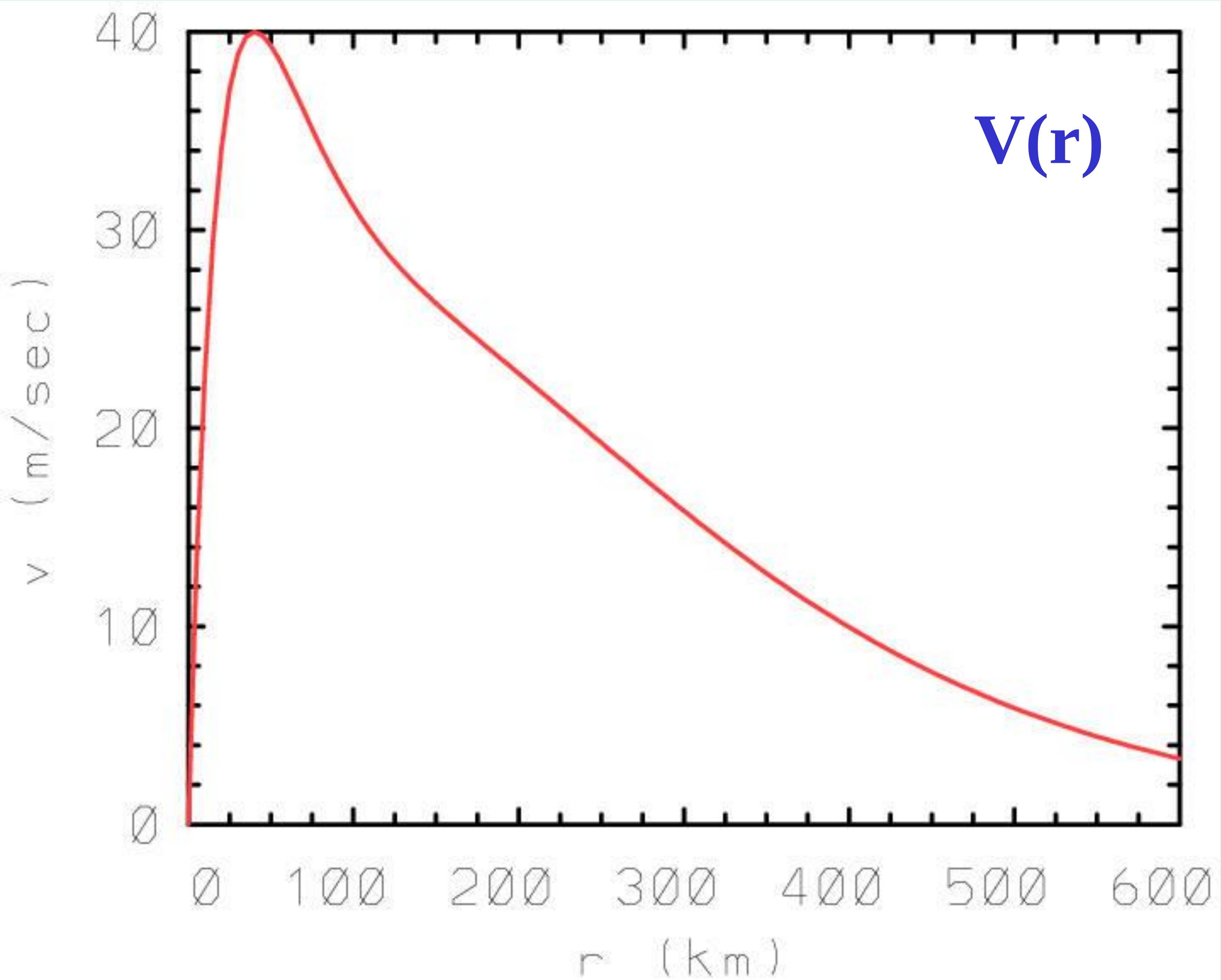

Laminar and Turbulent Ekman Layers
3
2
1
0
z km
0.0
0.6
1.2
v (m/sec)
3
2
1
0
-0.6
0.0
0.6
u (m/sec)


Hodograph
1.2
1.0
0.8
0.6
0.4
0.2
0.0
v (m/sec)
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
u (m/sec)

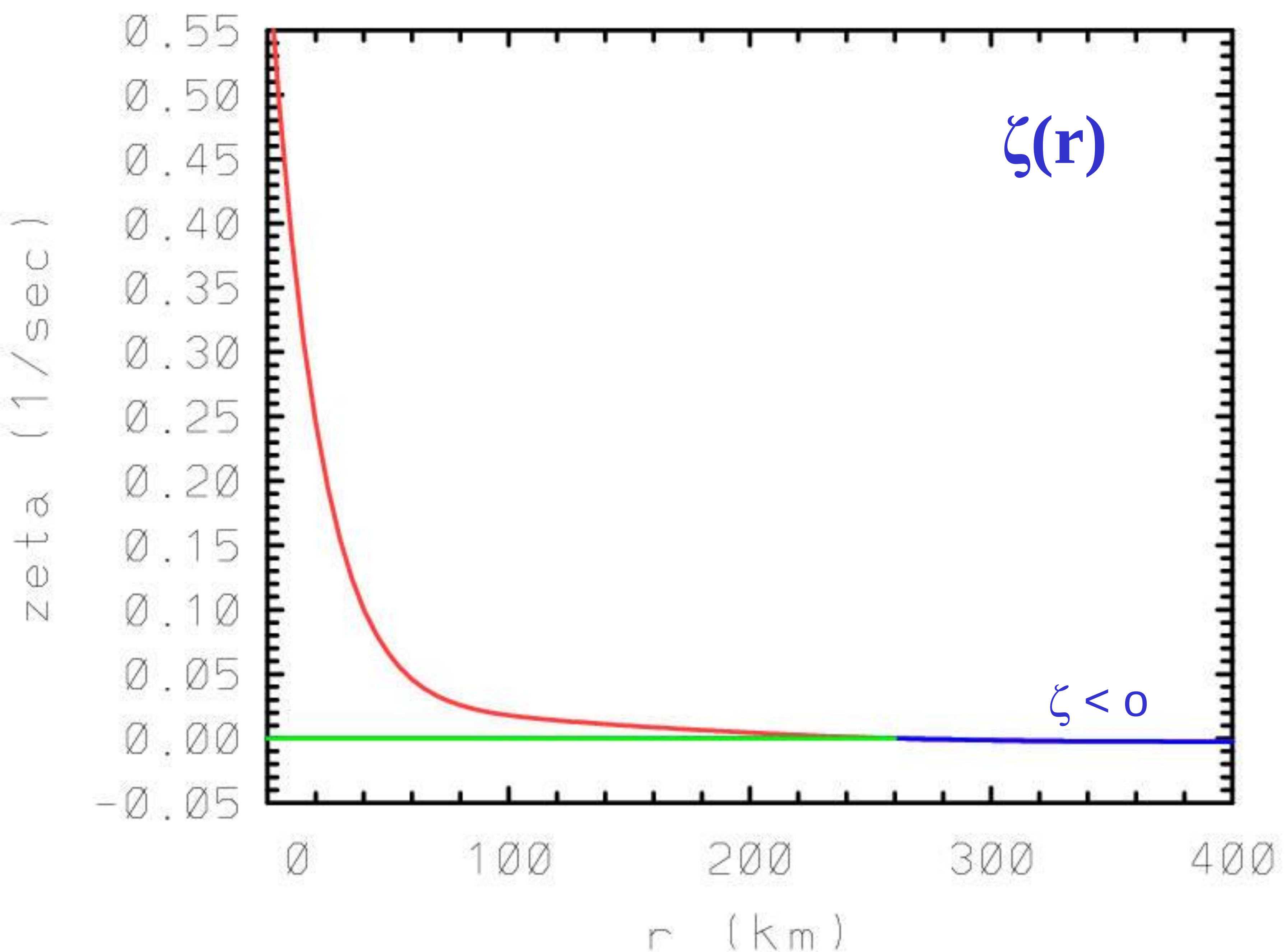
Linear solution

$$-v\left(f+\frac{2V}{r}\right)=K\frac{\partial^2 u}{\partial z^2}$$

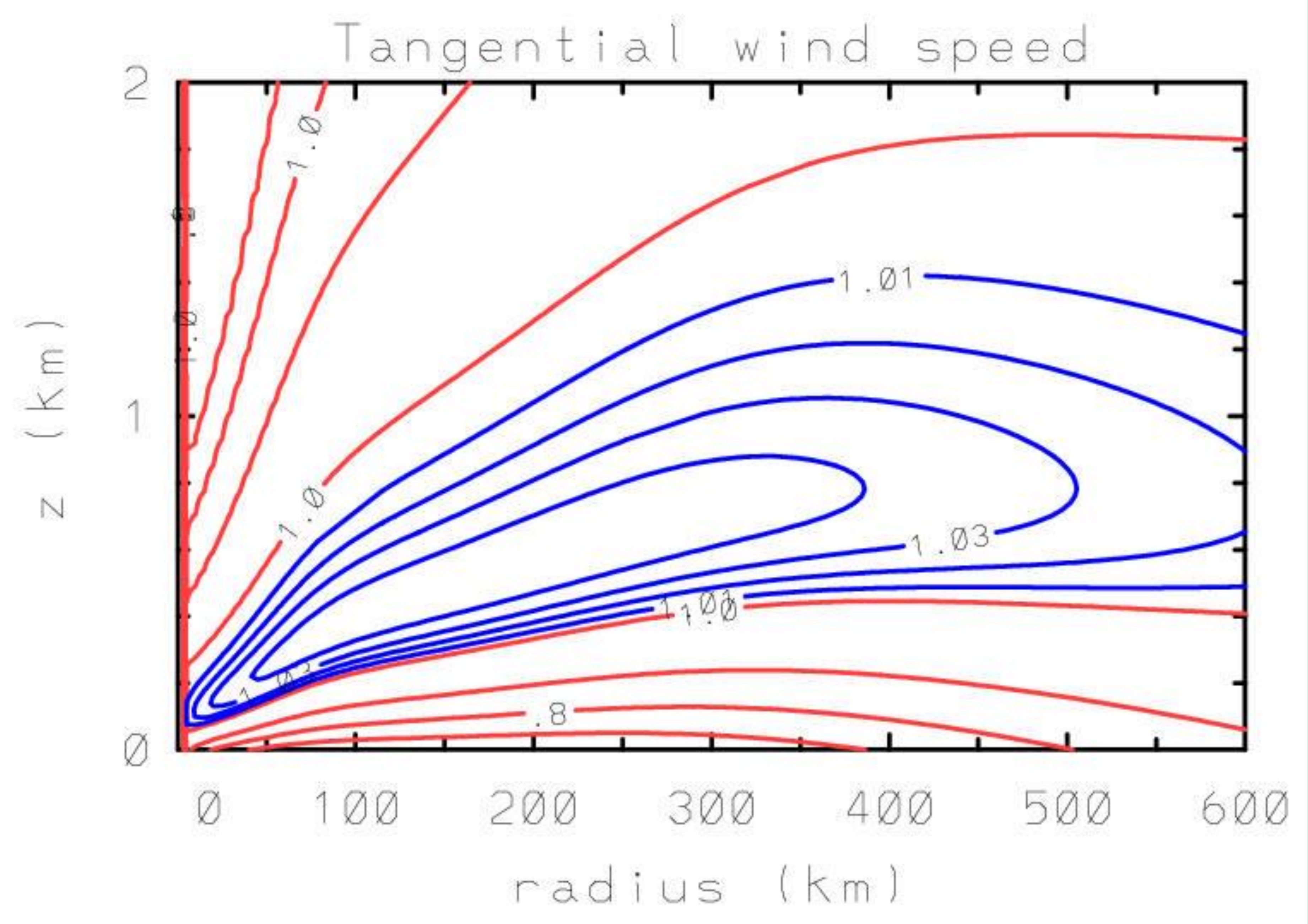
$$u\left(f+\frac{V}{r}+\frac{\partial V}{\partial r}\right)=K\frac{\partial^2 v}{\partial z^2}$$



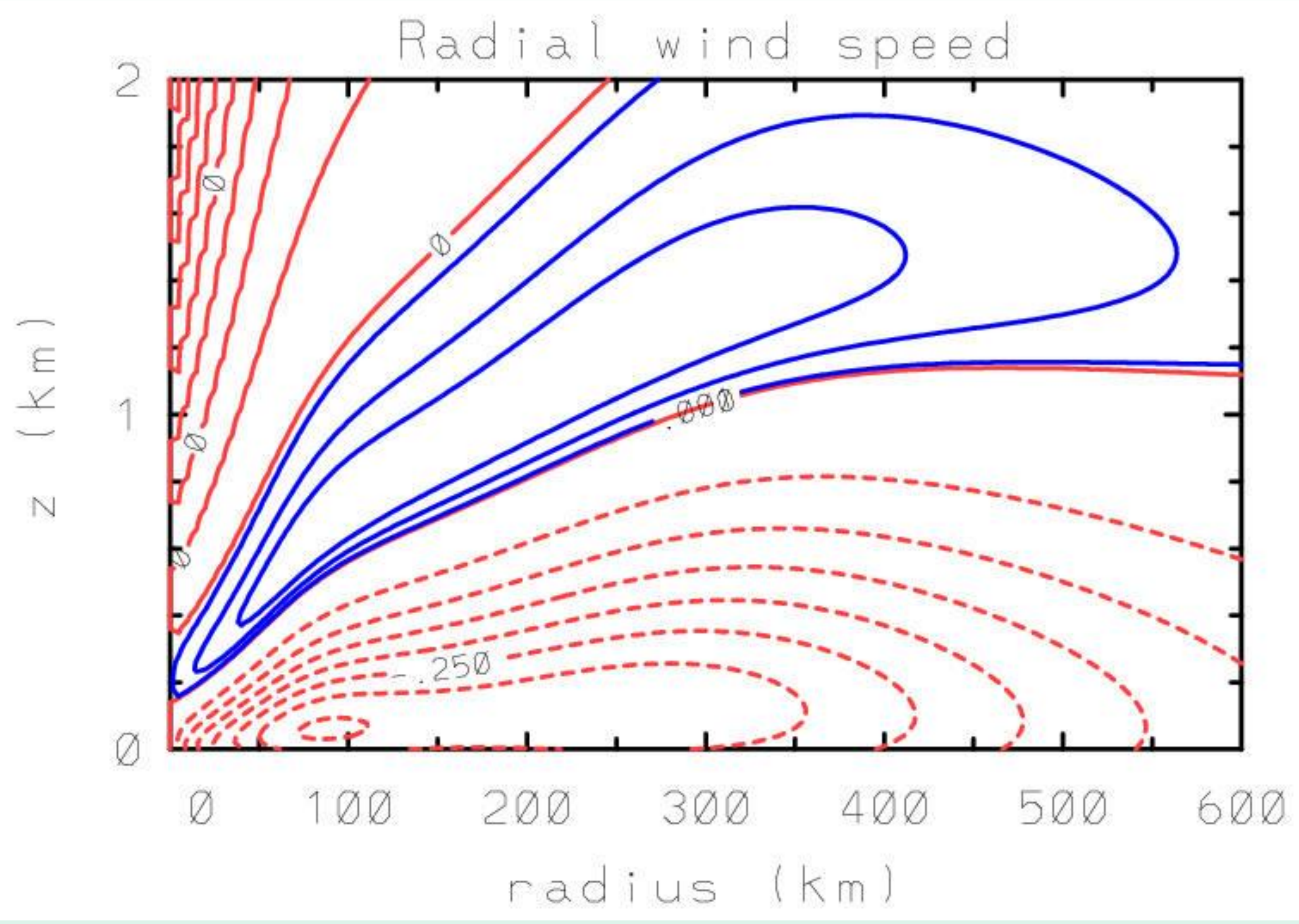
V(r)
40
30
20
10
0
v (m/sec)
0
100
200
300
400
500
600
r (km)


ζ(r)
ζ < o
zeta (1/sec)
r (km)
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0
100
200
300
400

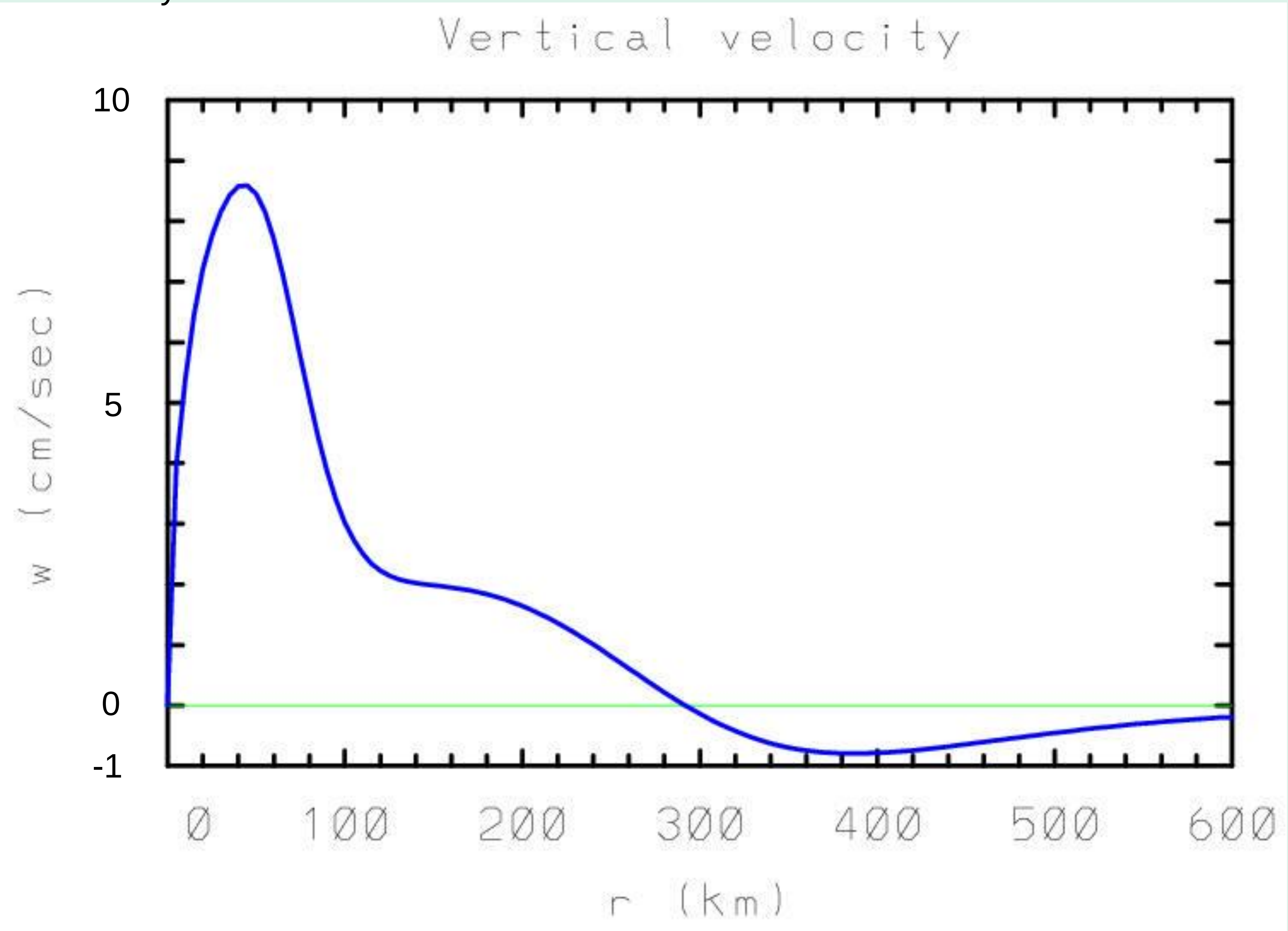
Linear theory



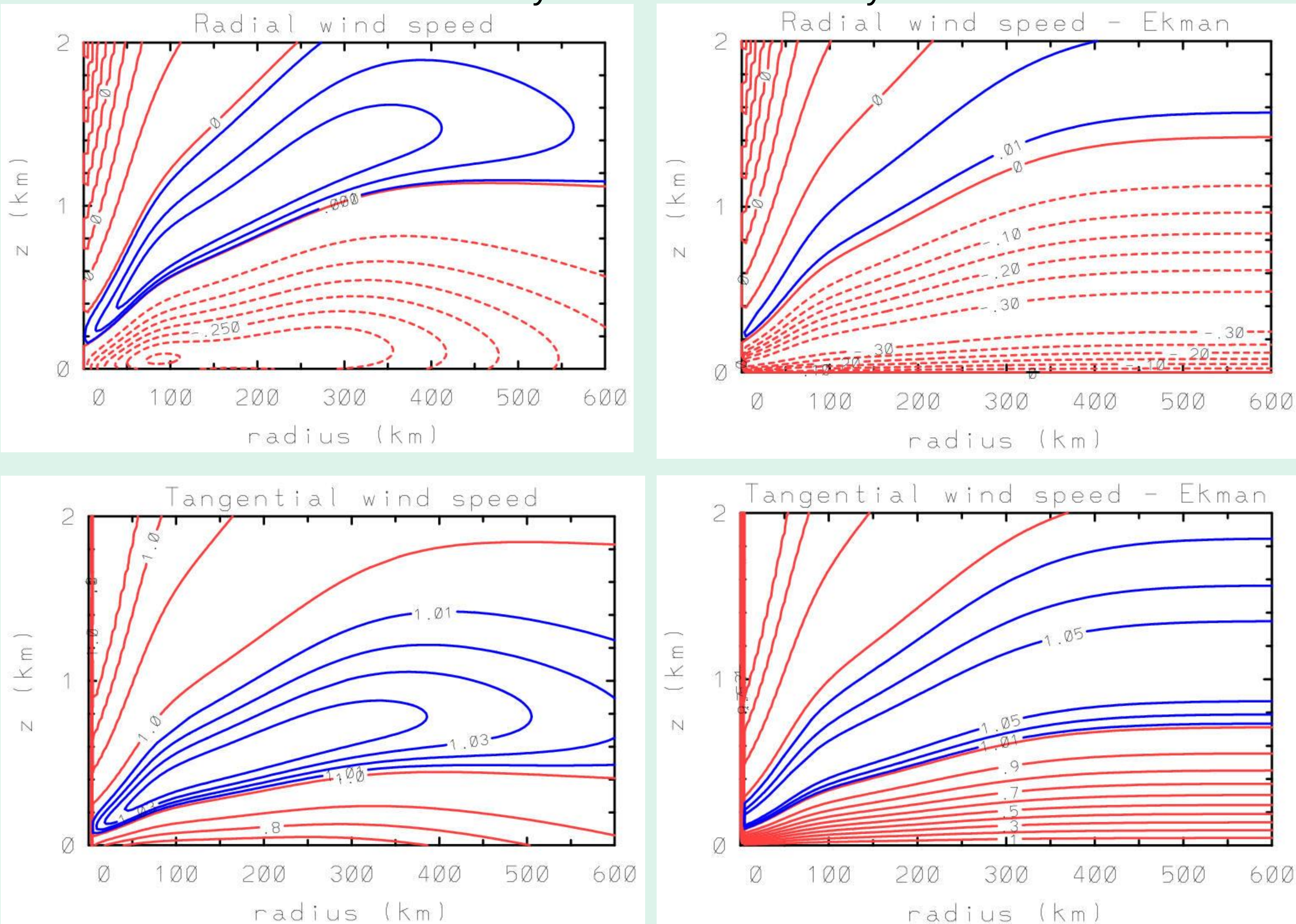
Linear theory

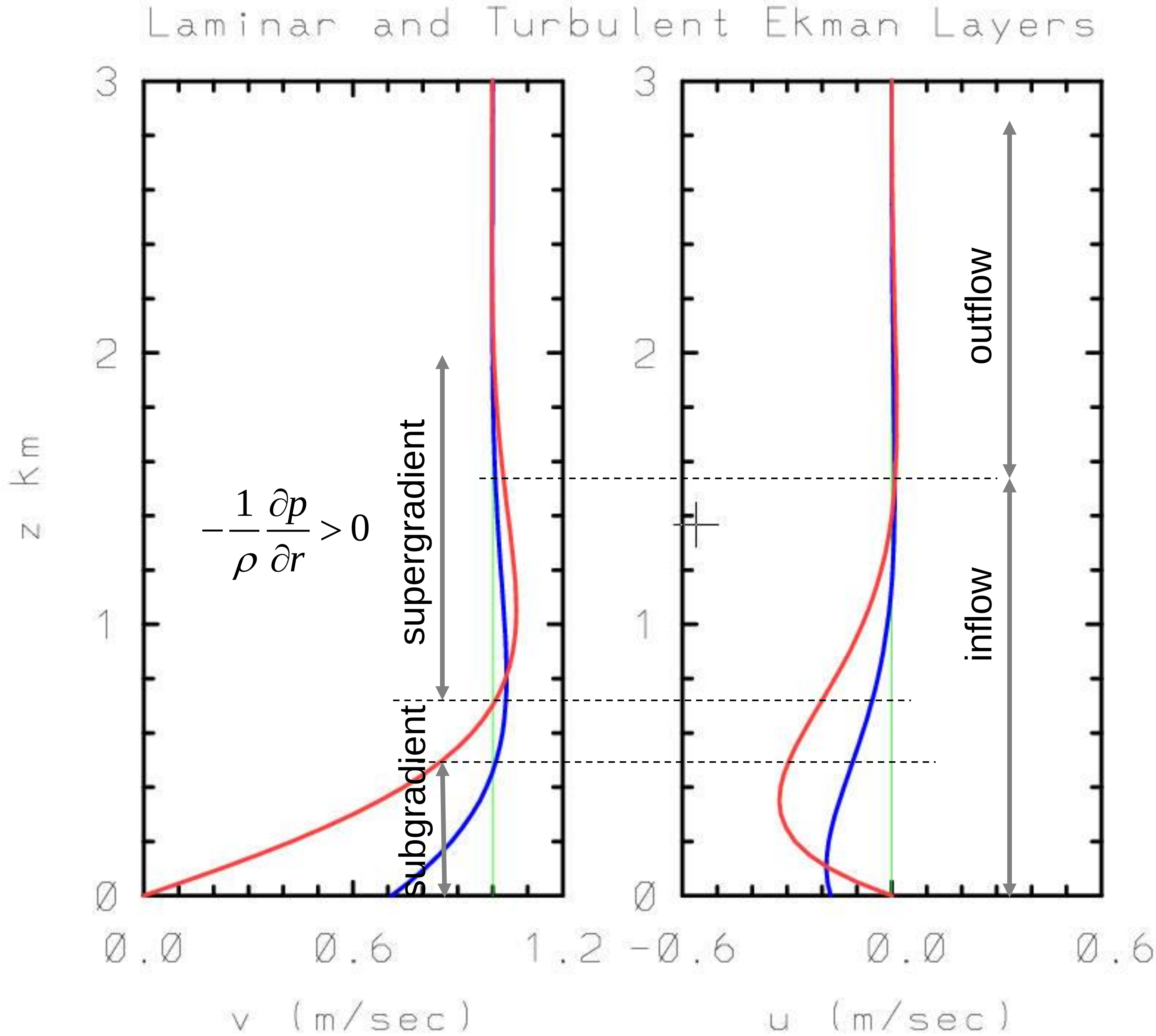


Linear theory

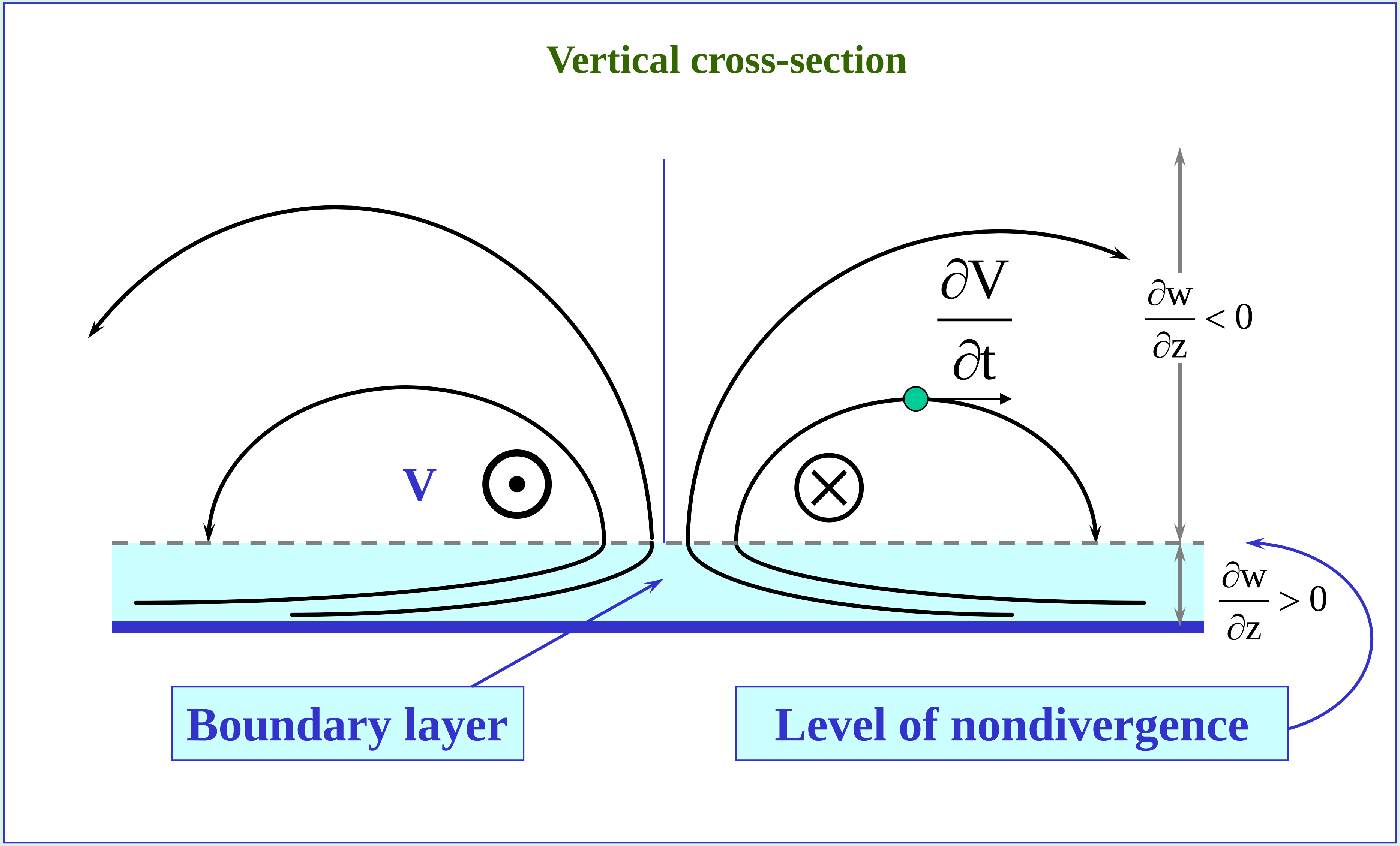


Linear theory versus Ekman theory

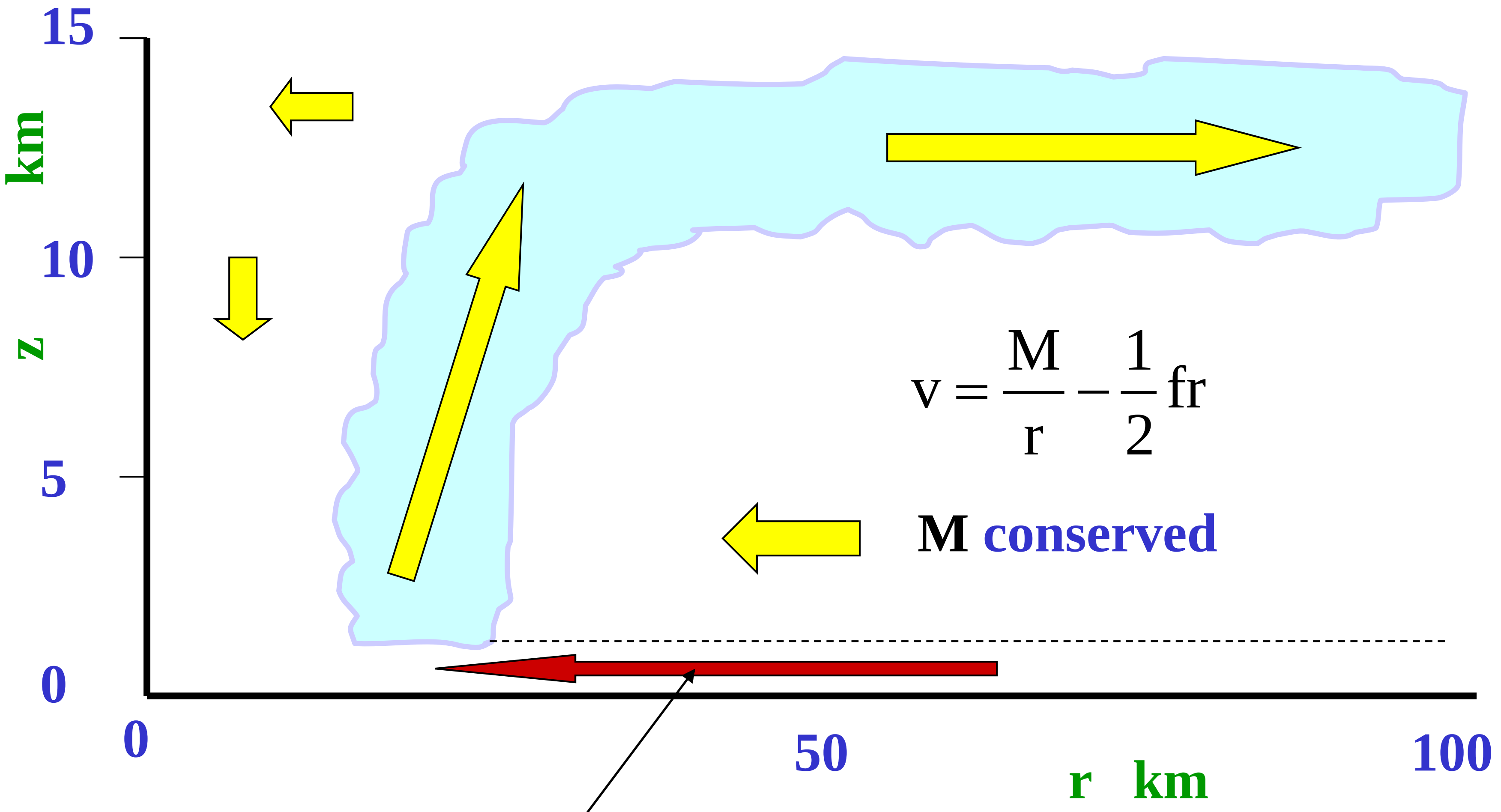



Laminar and Turbulent Ekman Layers
3
2
1
0
z km
0.0
0.6
1.2
-0.6
0.0
0.6
v (m/sec)
u (m/sec)
$-\frac{1}{\rho}\frac{\partial p}{\partial r} > 0$
supergradient
subgradient
outflow
inflow

Dynamics of vortex spindown



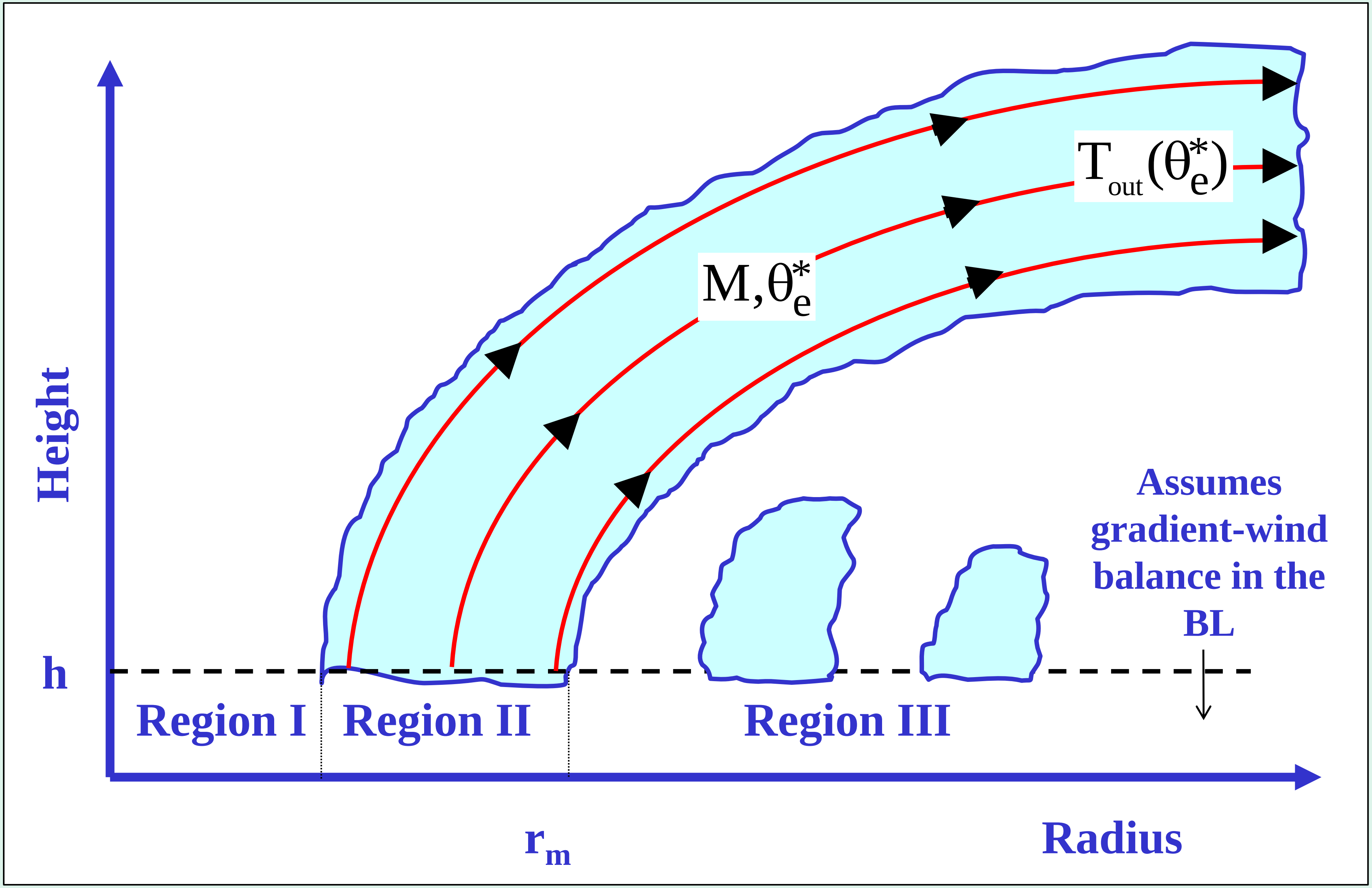
Conventional view of tropical cyclone spin up



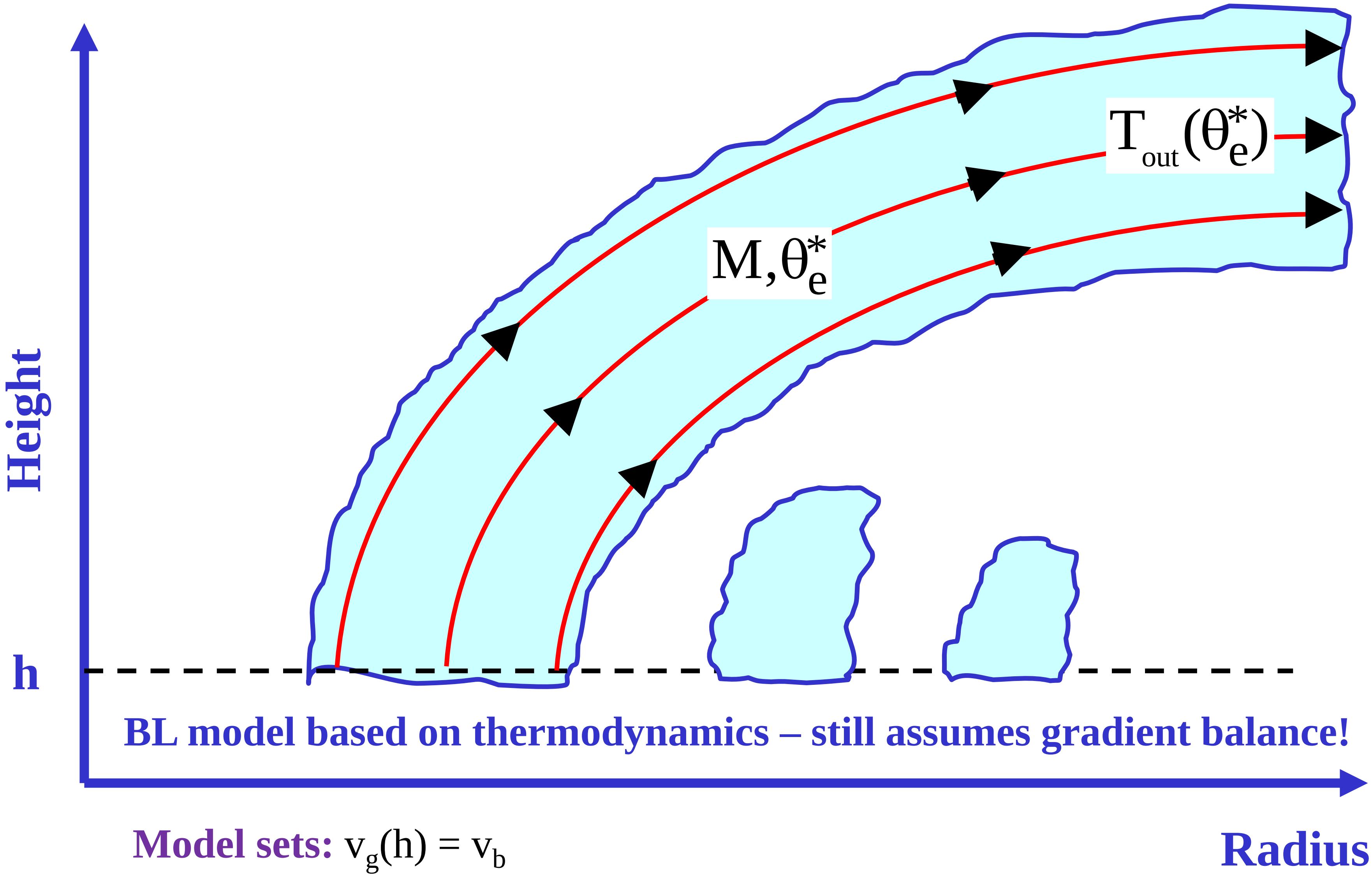
$$v = \frac{M}{r} - \frac{1}{2} fr$$

M not conserved, boundary layer supplies clouds with moisture

Emanuel's 1986 steady-state TC model



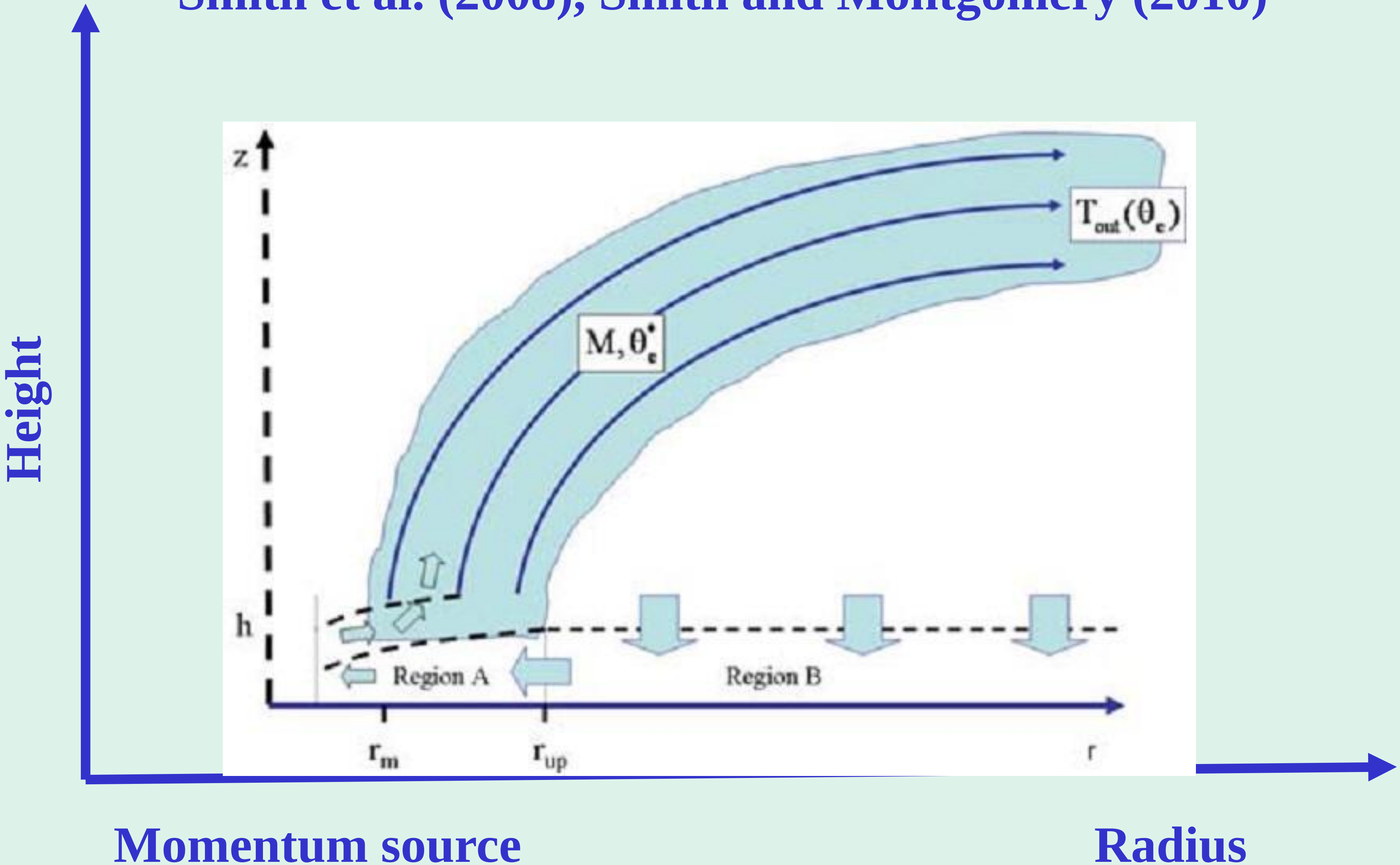
Emanuel's 1997 model for TC spin up



Questions

- **Does the boundary layer determine the tangential wind speed of air that ascends into the updraught?**
- **Consistent only if** $v_g(h) = v_b$**!**
- **Considerations: steady boundary layer equations are parabolic => information travels inwards. Region of ascent out of the boundary layer requires an open boundary condition.**
- **Can one improve the Emanuel model by relaxing the assumption of gradient wind balance in the boundary layer?**

Smith et al. (2008), Smith and Montgomery (2010)



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Tropical cyclone spin-up revisited

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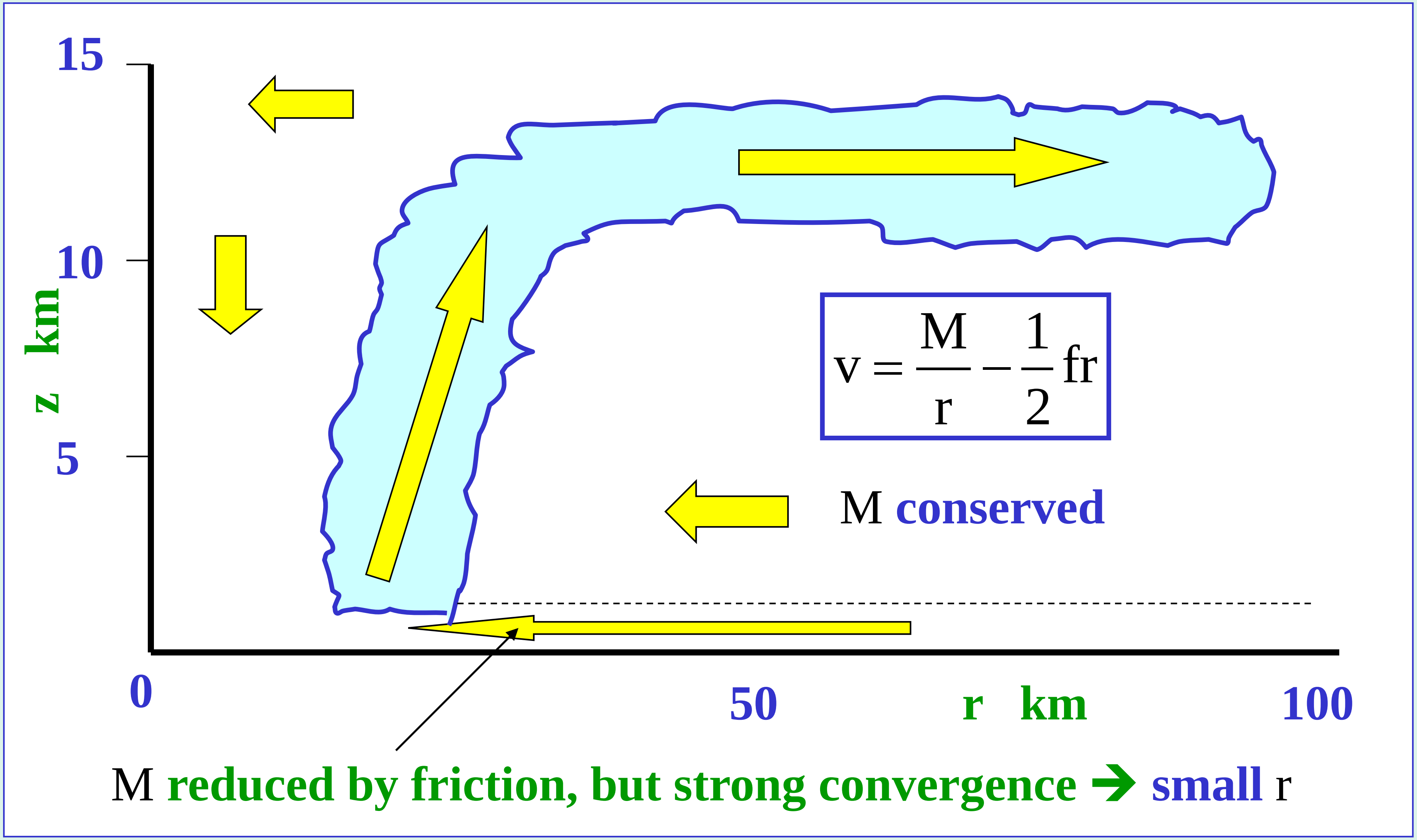
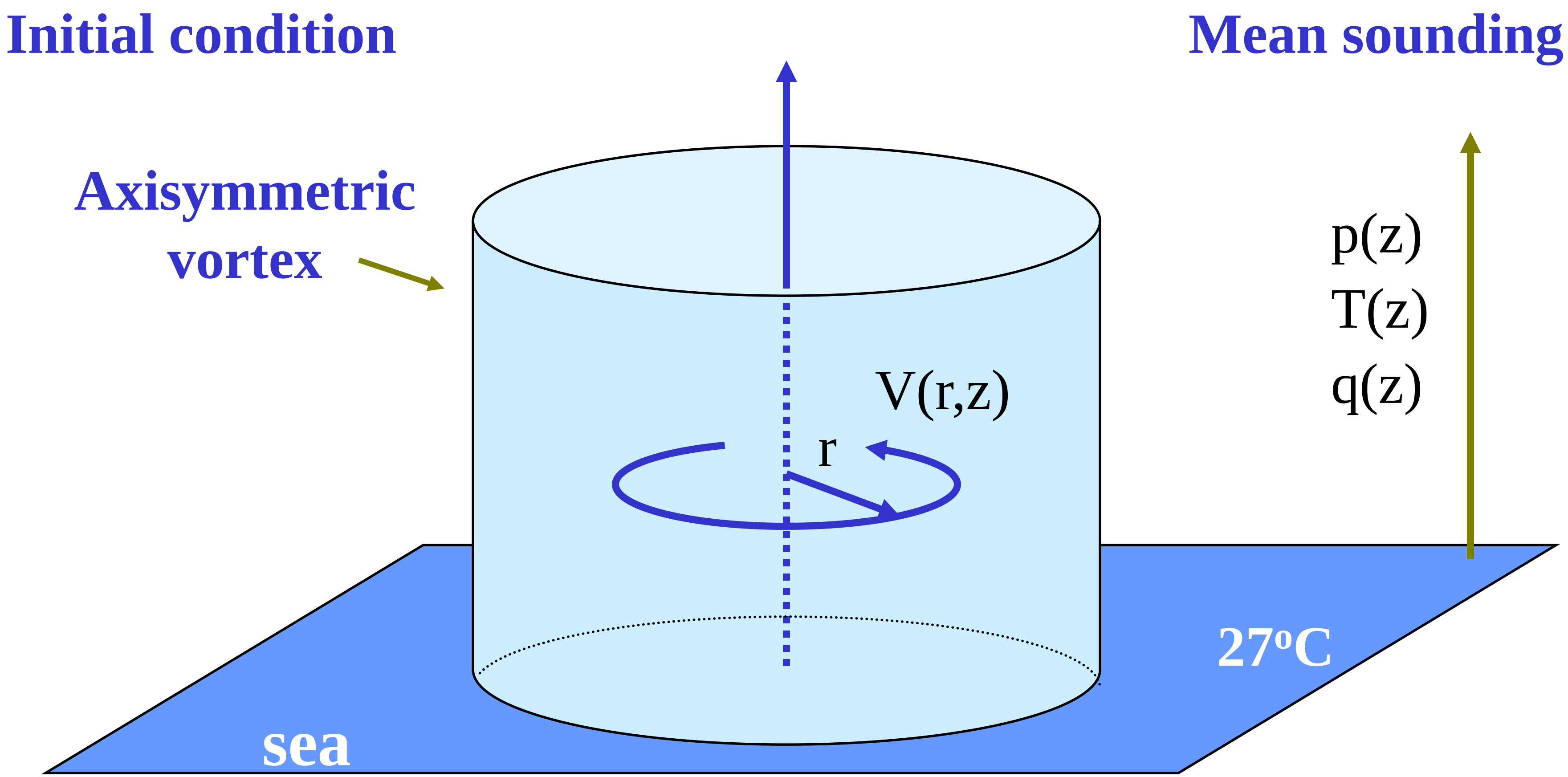
Zhang et al (2001) found that spin up occurred in the BL in Hurricane Andrew.

Revised view of tropical cyclone spin up



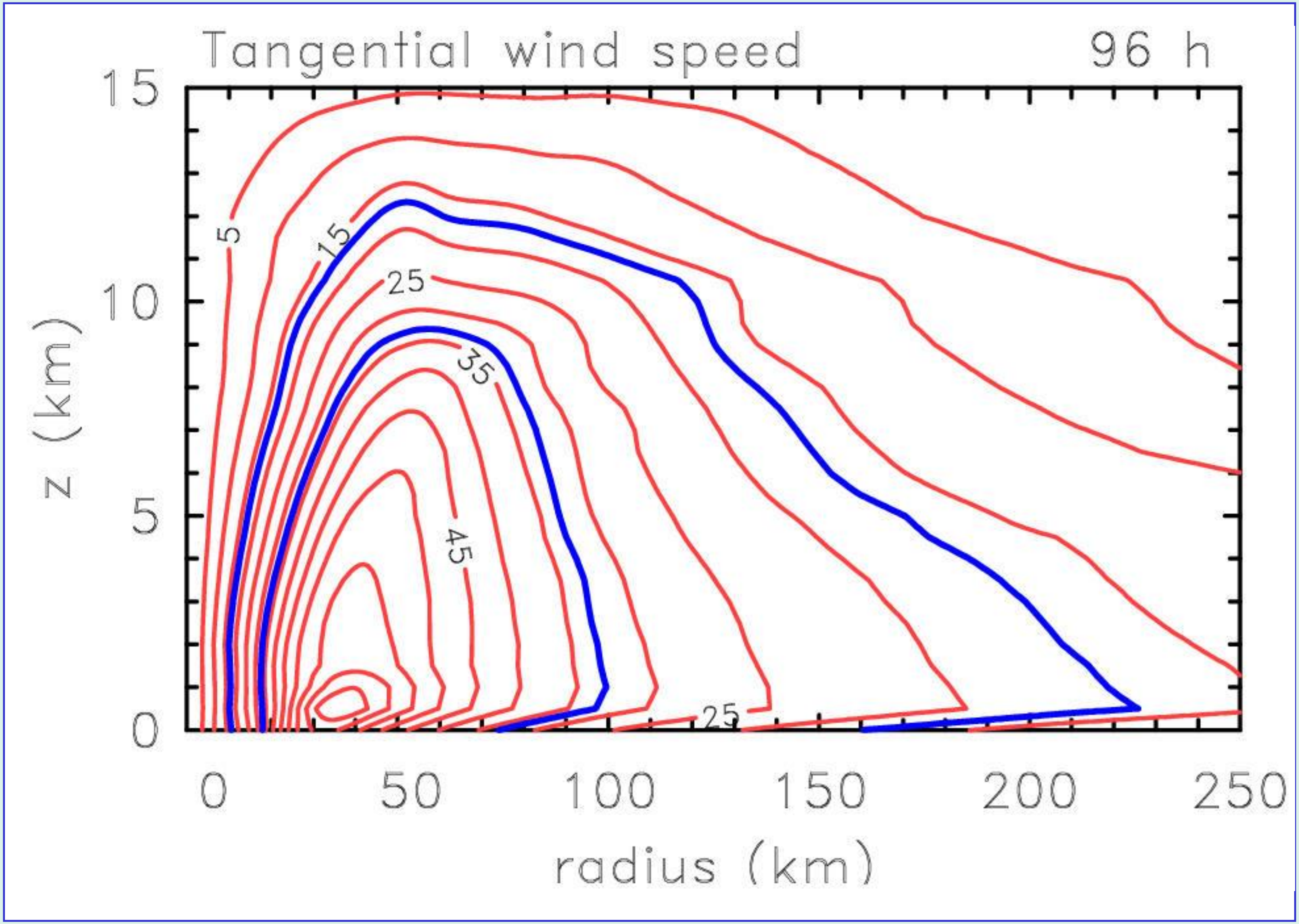
From Montgomery, Nguyen & Smith (2009): QJRMS

The basic thought experiment for intensification

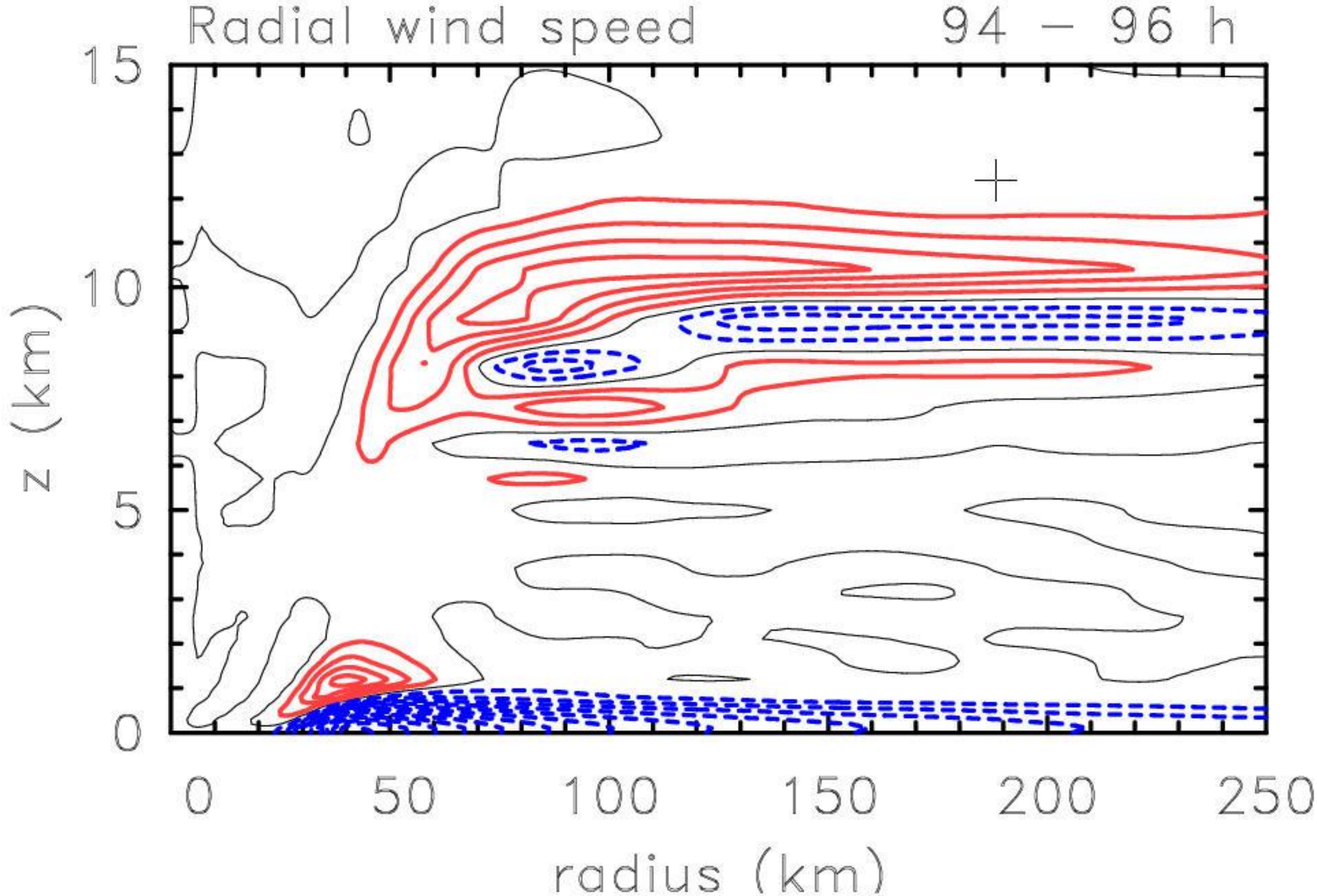


Nguyen, Smith and Montgomery calculation, QJRMS, 2008:

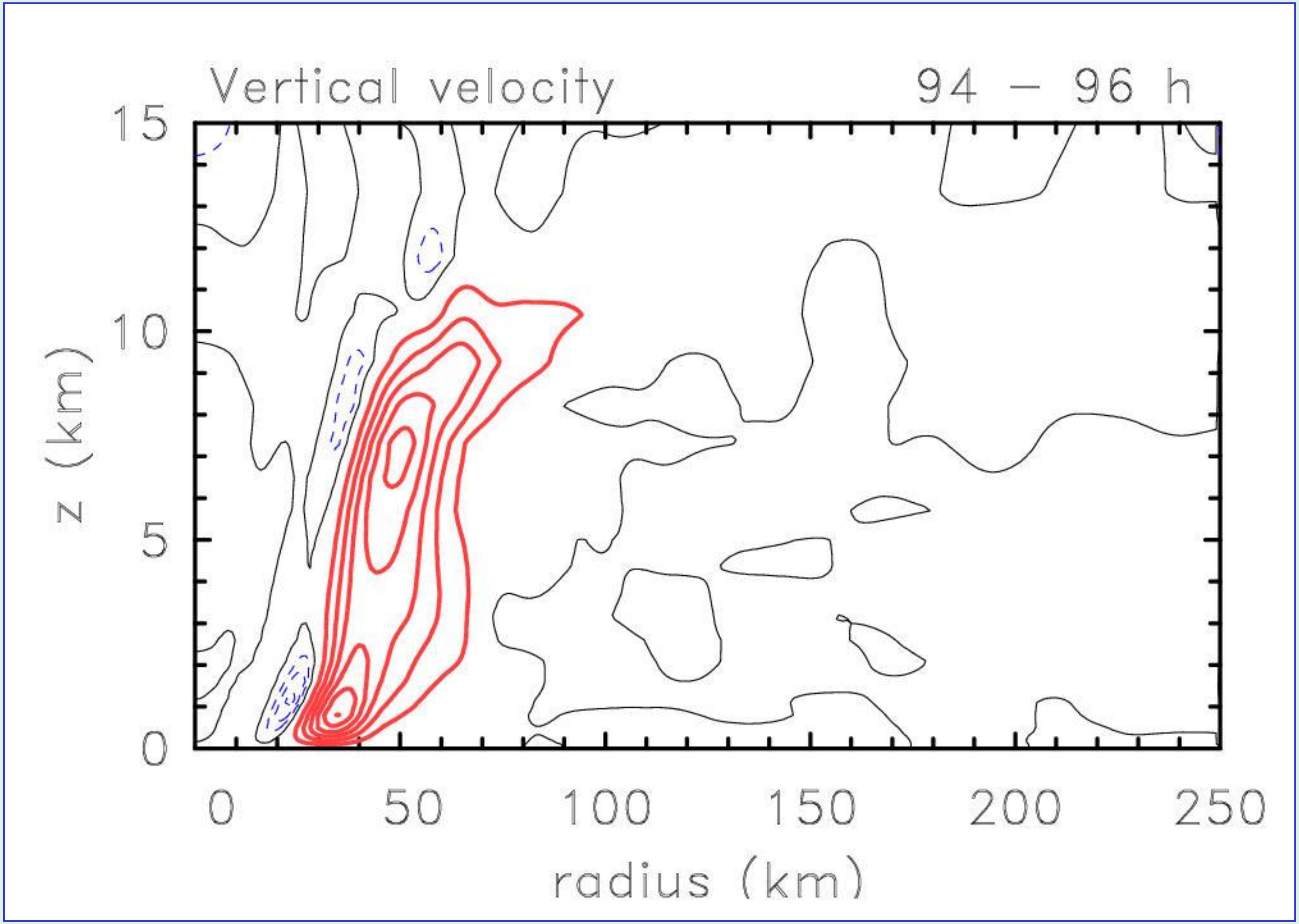
- **Idealized numerical model simulations, simple physics, MM5**
- 5 **km (**1.67 **km) resolution in the finest nest,** 24 **σ-levels**



From Montgomery, Nguyen & Smith (2009): QJRMS



From Montgomery, Nguyen & Smith (2009): QJRMS



From Montgomery, Nguyen & Smith (2009): QJRMS

Hurricane intensification

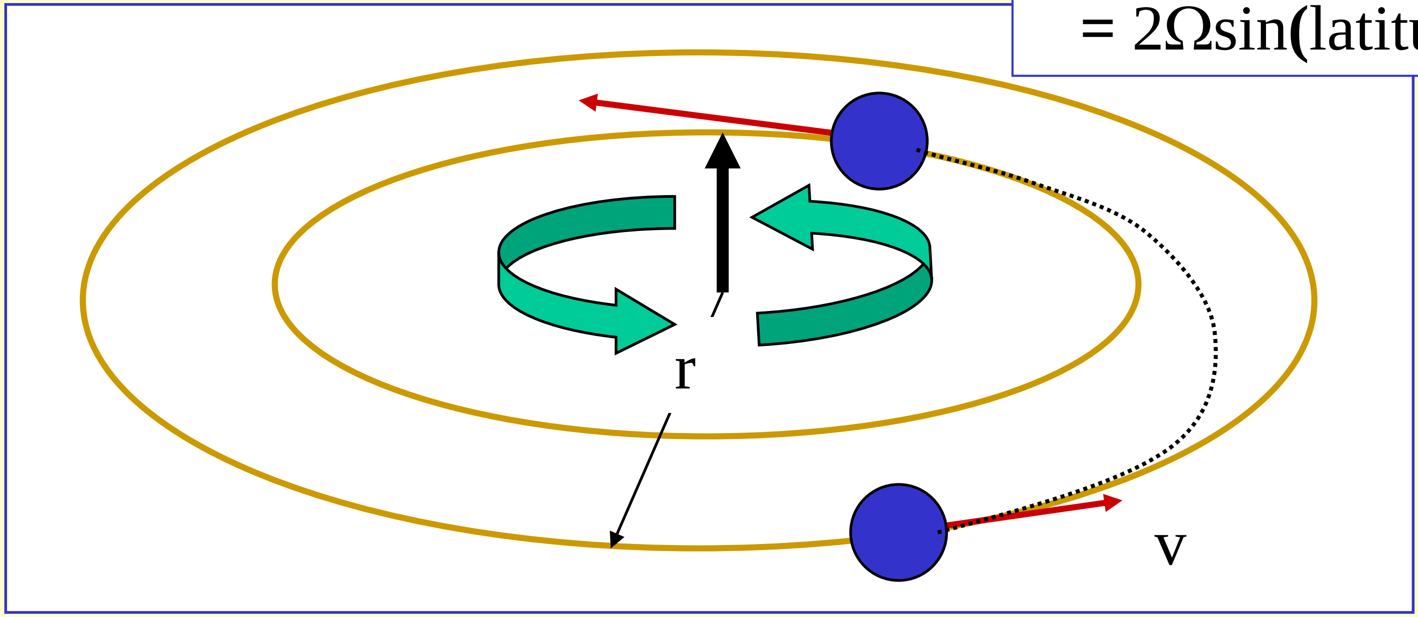
- **Basic principle**

  - Conservation of absolute angular momentum:

$$M = rv + r^2 f/2$$

f **= Coriolis parameter**

= $2\Omega \sin$**(**latitude**)**



$v = M/r - rf/2$ ⇨ **If** r **decreases,** v **increases!**

⇨ **Spin up requires radial convergence**

Thank you for your attention