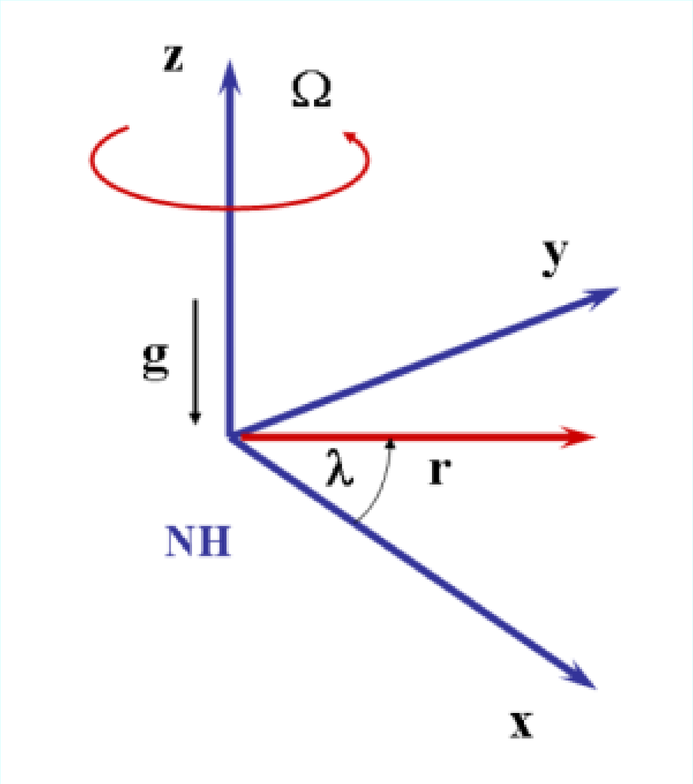
Balanced Dynamical Theory

The primary and secondary circulations

Chapter 2

Axisymmetric Theory



To a good first approximation, a mature TC consists of a horizontal quasi axisymmetric circulation, on which is superimposed a thermally direct overturning circulation

Inviscid equations of motion

in a rotating coordinate system in cylindrical polar coordinates

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \lambda} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \qquad (3.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \lambda} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = -\frac{1}{\rho r}\frac{\partial p}{\partial \lambda}, \qquad (3.2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \lambda} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \qquad (3.3)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho r u}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \lambda} + \frac{\partial \rho w}{\partial z} = 0, \qquad (3.4)$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + \frac{v}{r}\frac{\partial \theta}{\partial \lambda} + w\frac{\partial \theta}{\partial z} = \dot{\theta} \qquad (3.5)$$

$$\rho = p_* \pi^{\frac{1}{\kappa}-1}/(R_d\theta) \qquad (3.6)$$

$\dot{\theta}$ is the diabatic heating rate $(1/c_p\pi)Dh/Dt$ (see Eq. 1.13), and $\pi = (p/p_*)^\kappa$ is the Exner function. The temperature is defined by $T = \pi\theta$.

Absolute angular momentum

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial r} + \frac{v}{r}\frac{\partial M}{\partial \lambda} + w\frac{\partial M}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \qquad (3.7)$$

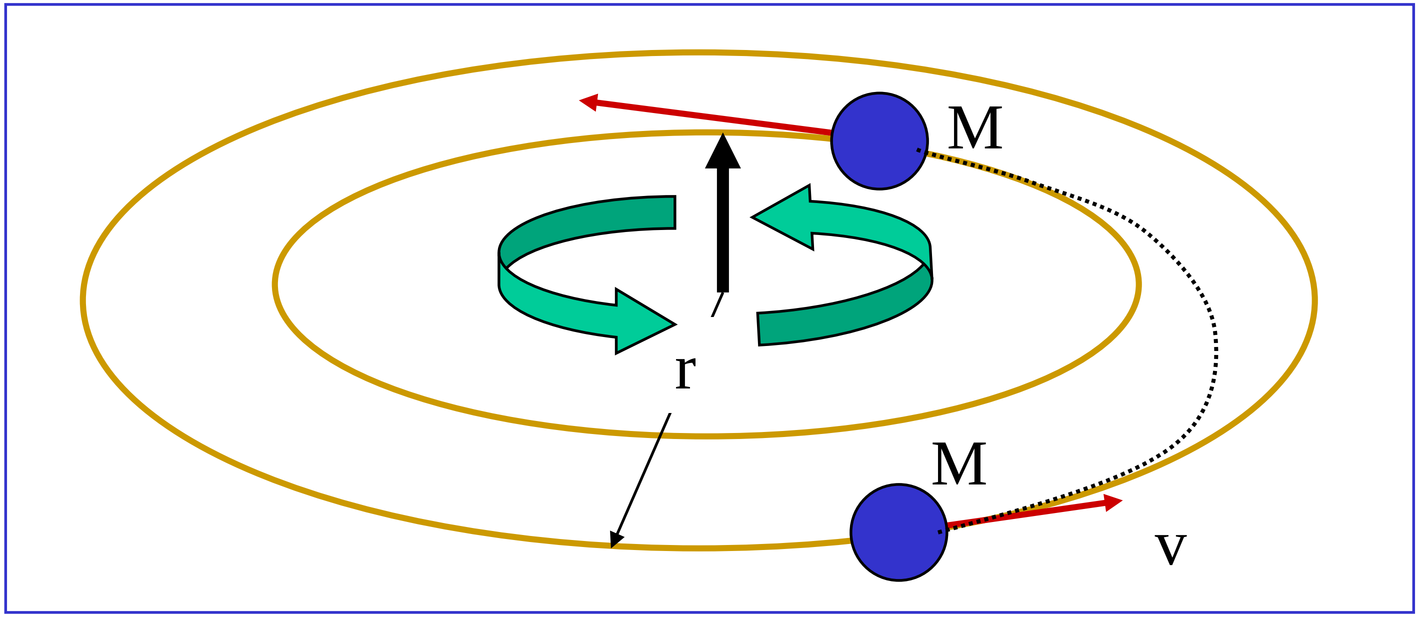
$$M = rv + \frac{1}{2}fr^2, \qquad (3.8)$$

Eq. (3.7) follows from r **times Eq. (3.2)**

Tropical cyclone intensification

Basic principle: conservation of absolute angular momentum:

$$M = rv + \tfrac{1}{2} f r^2$$



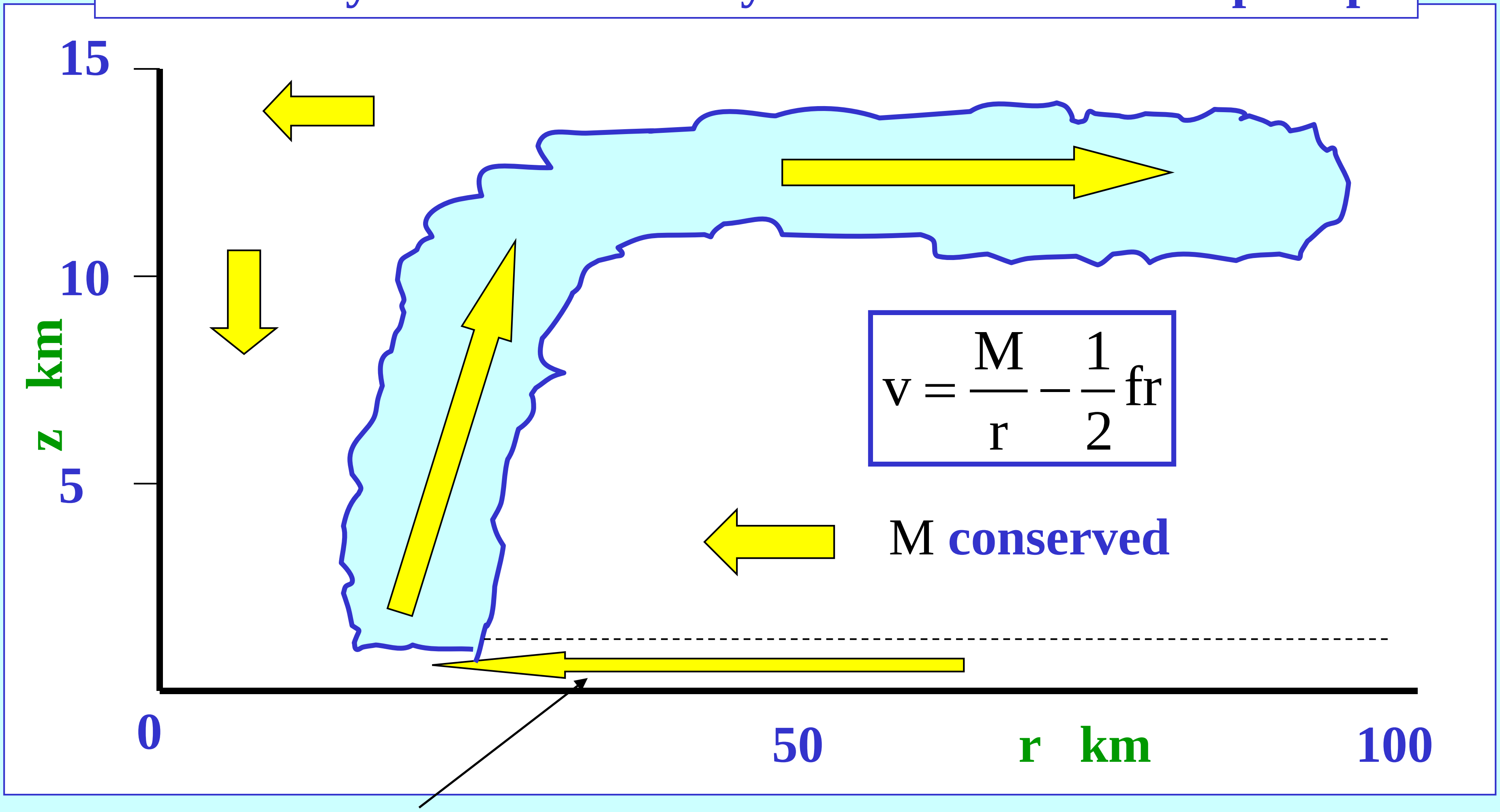
$$v = \frac{M}{r} - \tfrac{1}{2} f r$$

If r **decreases,** v **increases!**

Spin up requires radial convergence

Conventional view of intensification: axisymmetric

Thermally-forced secondary circulation leads to spin up



M not conserved, inflow feeds the clouds with moisture

Is that it? See later for a surprise

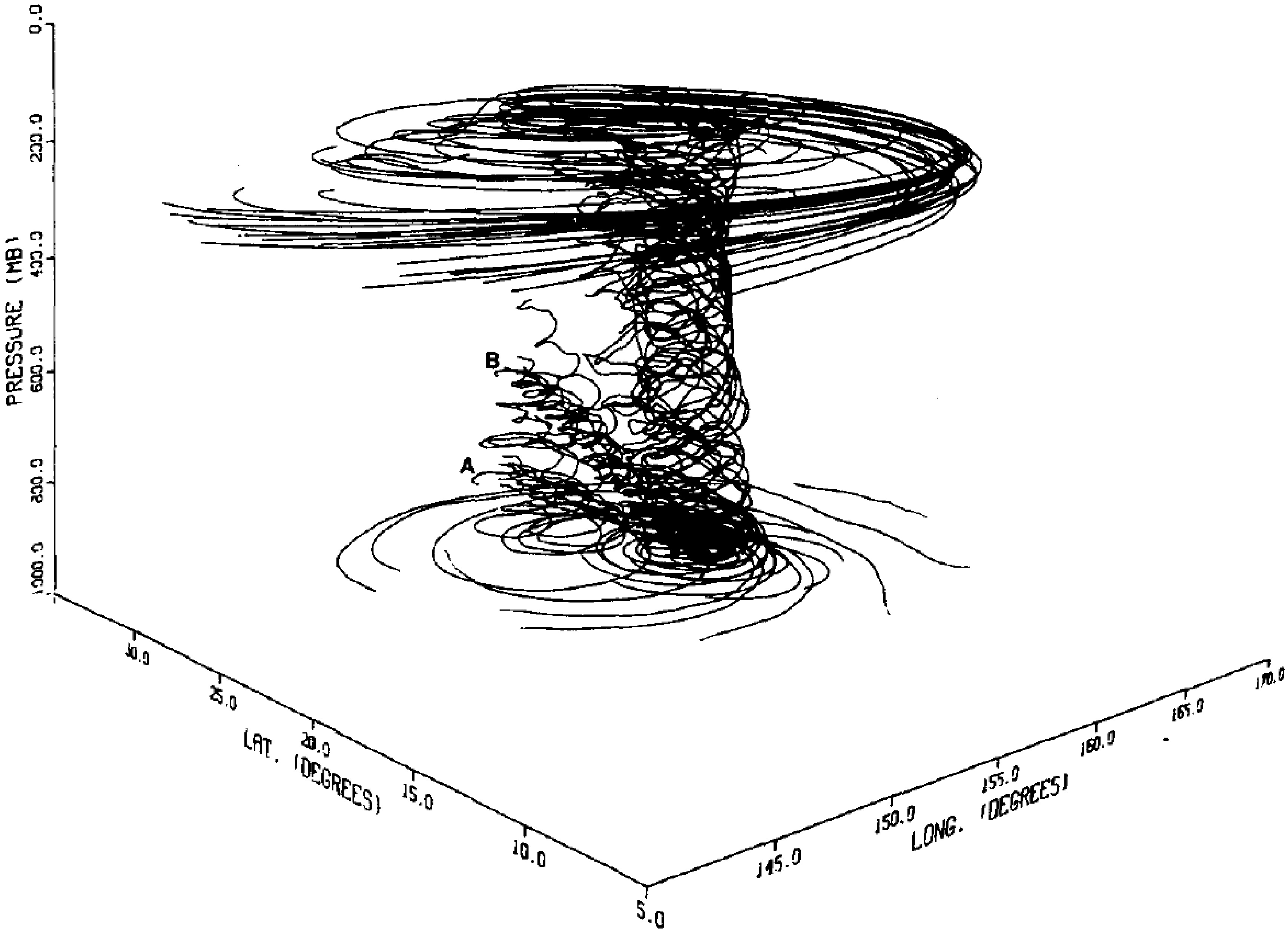
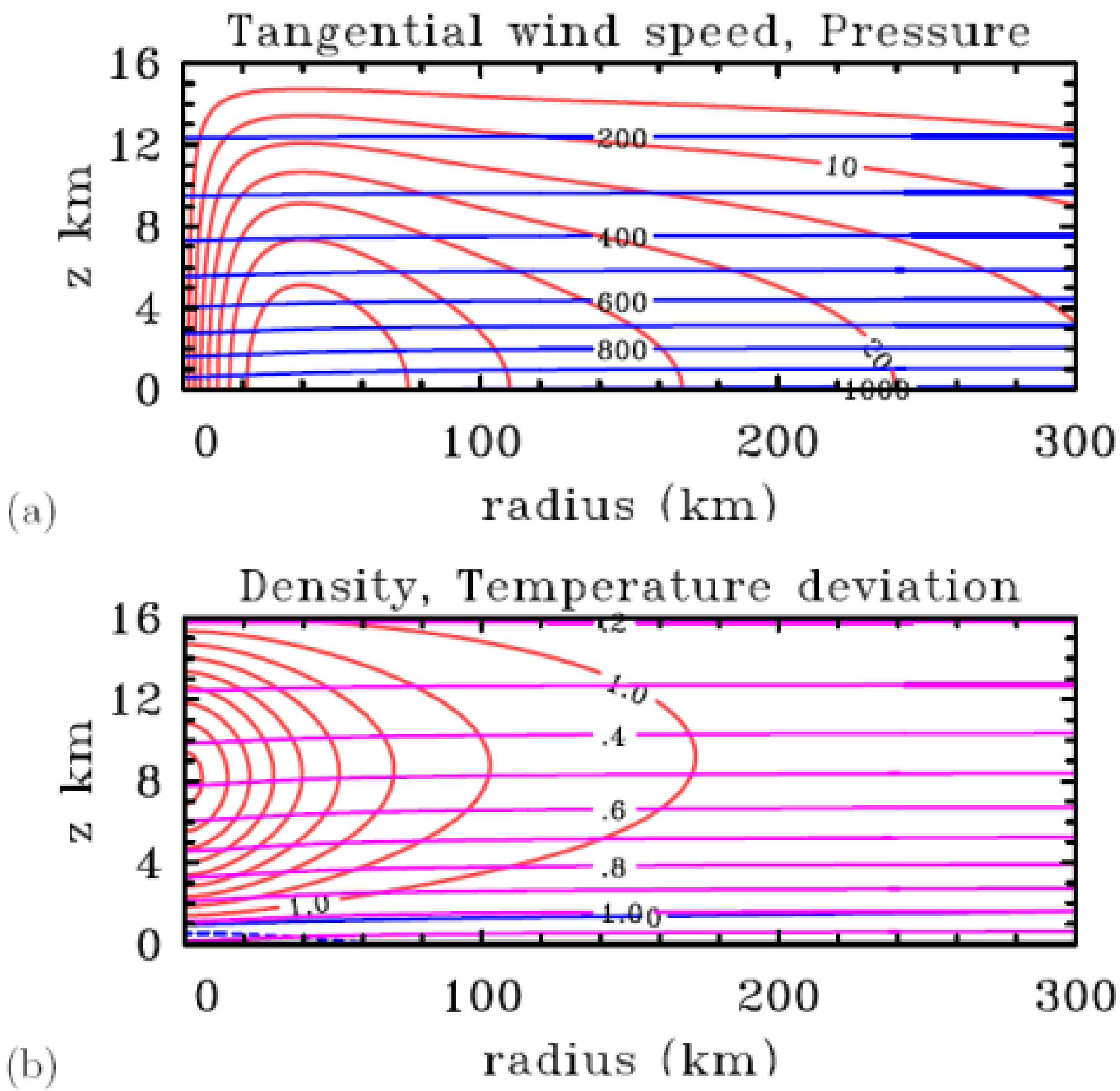


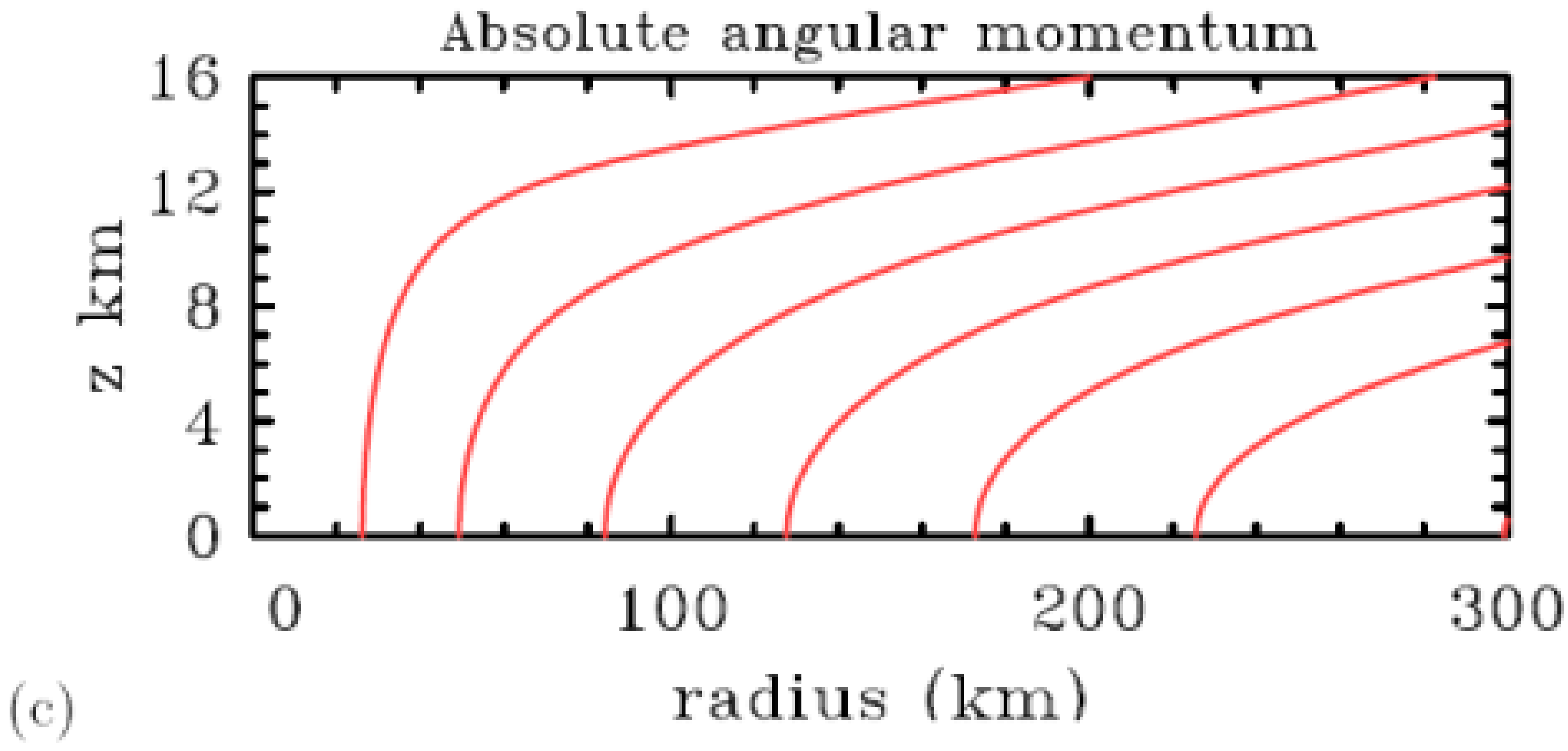
FIG. 9. Trajectories formed by particles released at various radii and pressure levels at $t = 0$. Most particles that reach the outflow level are transported outward by the outflow jet. Most particles released at radii of 20 km (A) and 100 km (B) are "trapped" inside the radius of the maximum wind and only rise slowly and drift toward the NW.

A typical vortex



AAM in a typical vortex

M



Primary circulation

Gradient wind

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} + fv, \tag{3.9}$$

Hydrostatic

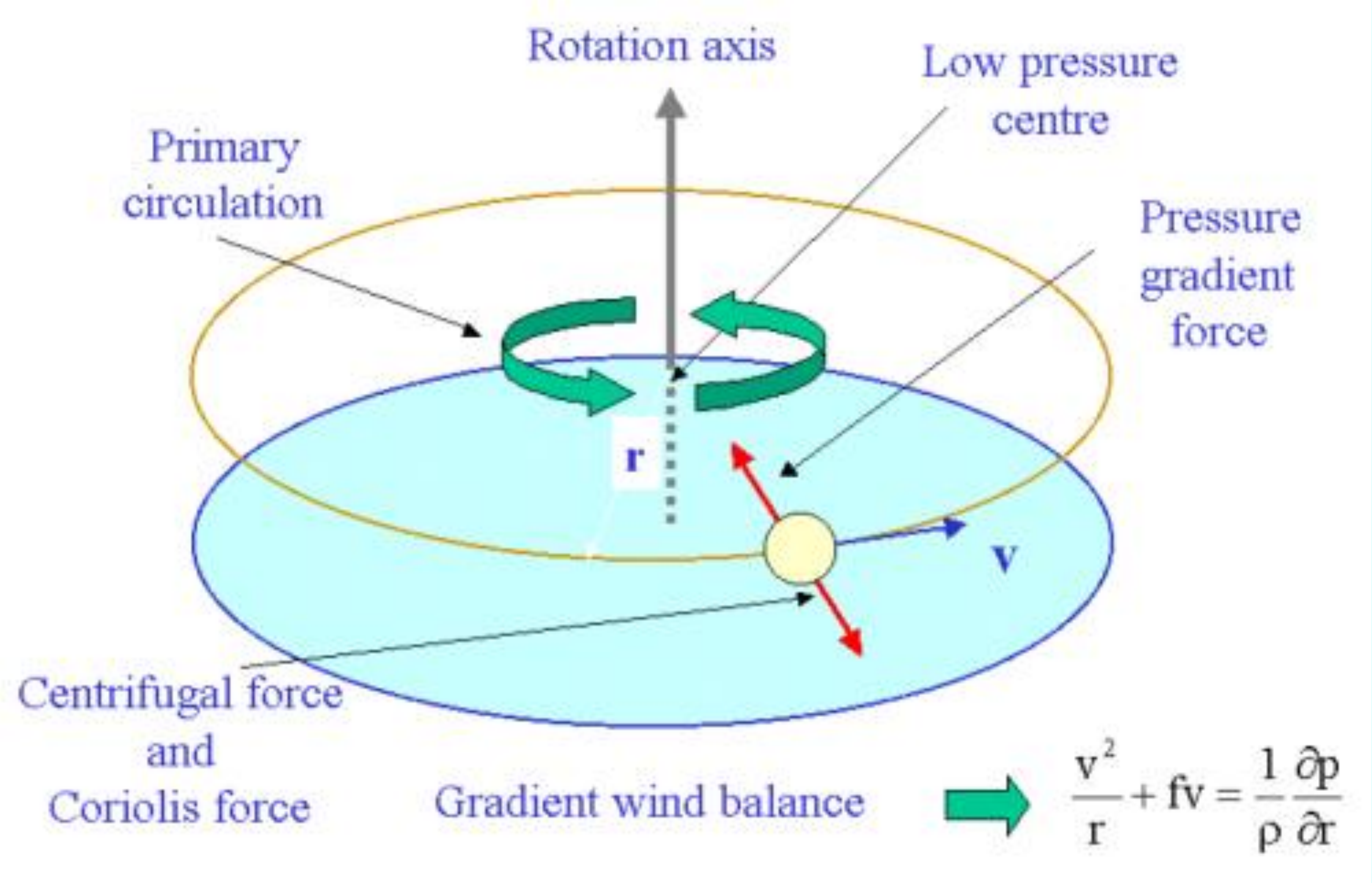
$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, \tag{3.10}$$

Thermal wind equation

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}. \tag{3.11}$$

$$C = \frac{v^2}{r} + fv \tag{3.12}$$

Gradient wind balance



Thermal wind equation

Gradient wind balance **Hydrostatic balance**

Write

$$\frac{\partial p}{\partial r} = \rho\left(\frac{v^2}{r} + fv\right) \qquad \frac{\partial p}{\partial z} = -\rho g$$

Eliminate p **using**

$$\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial r}\right)$$

$$\frac{\partial}{\partial r}\ln\rho + \frac{1}{g}\left(\frac{v^2}{r} + fv\right)\frac{\partial}{\partial z}\ln\rho = -\frac{1}{g}\left(\frac{2v}{r} + f\right)\frac{\partial v}{\partial z}$$

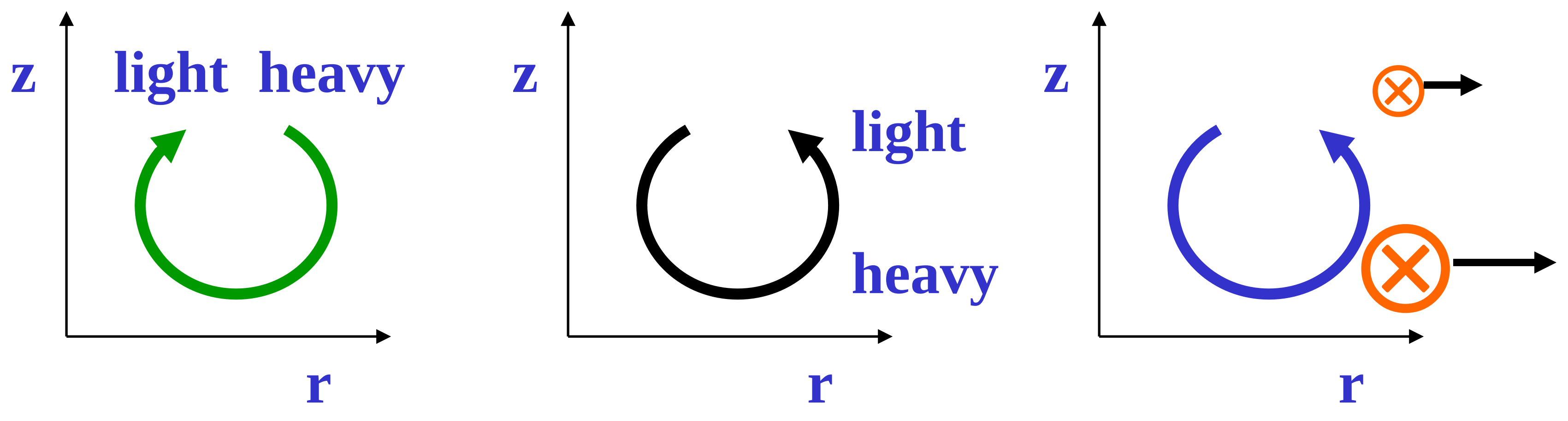
Thermal wind equation

Physical interpretation

Thermal wind equation

$$g\frac{\partial}{\partial r}\ln\rho + \left(\frac{v^2}{r}+fv\right)\frac{\partial}{\partial z}\ln\rho + \left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z} = 0$$

Balance in toroidal circulation tendency



Thermal wind equation

$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}. \quad (3.11)$$

Characteristics

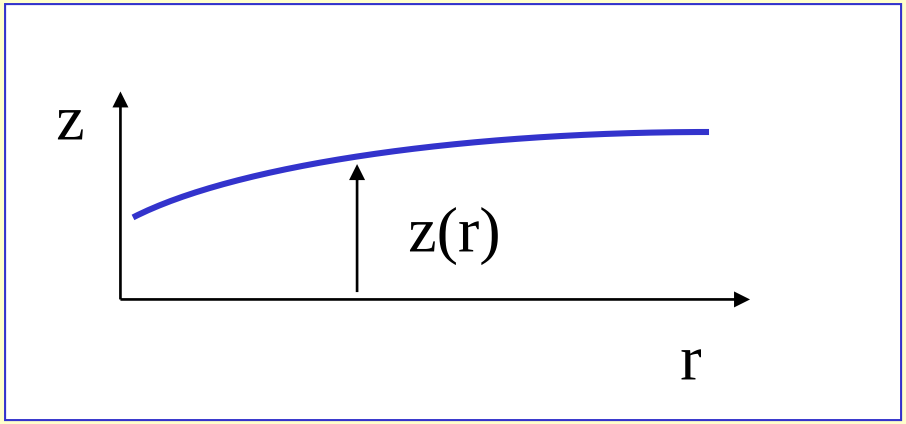
$$\frac{dz}{dr} = \frac{C}{g}.$$

On a characteristic

$$\frac{d}{dr}\ln \rho = -\frac{1}{g}\frac{\partial C}{\partial z}. \quad (3.14)$$

Mathematical solution

$$\frac{\partial}{\partial r}\ln\rho+\frac{1}{g}\left(\frac{v^2}{r}+fv\right)\frac{\partial}{\partial z}\ln\rho=-\frac{1}{g}\left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z}$$



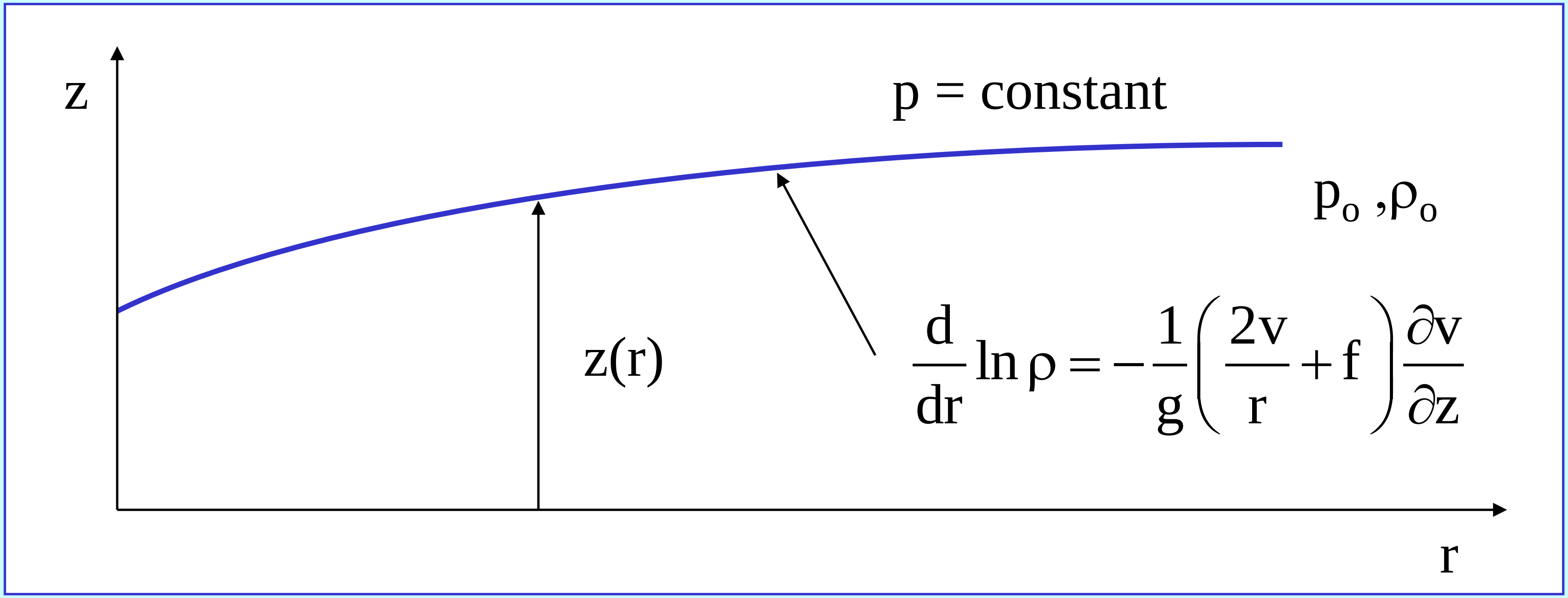
Characteristics

$$\frac{dz}{dr}=\frac{1}{g}\left(\frac{v^2}{r}+fv\right)$$

$$\frac{d}{dr}\ln\rho=-\frac{1}{g}\left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z}$$

Governs the variation of ρ along characteristics

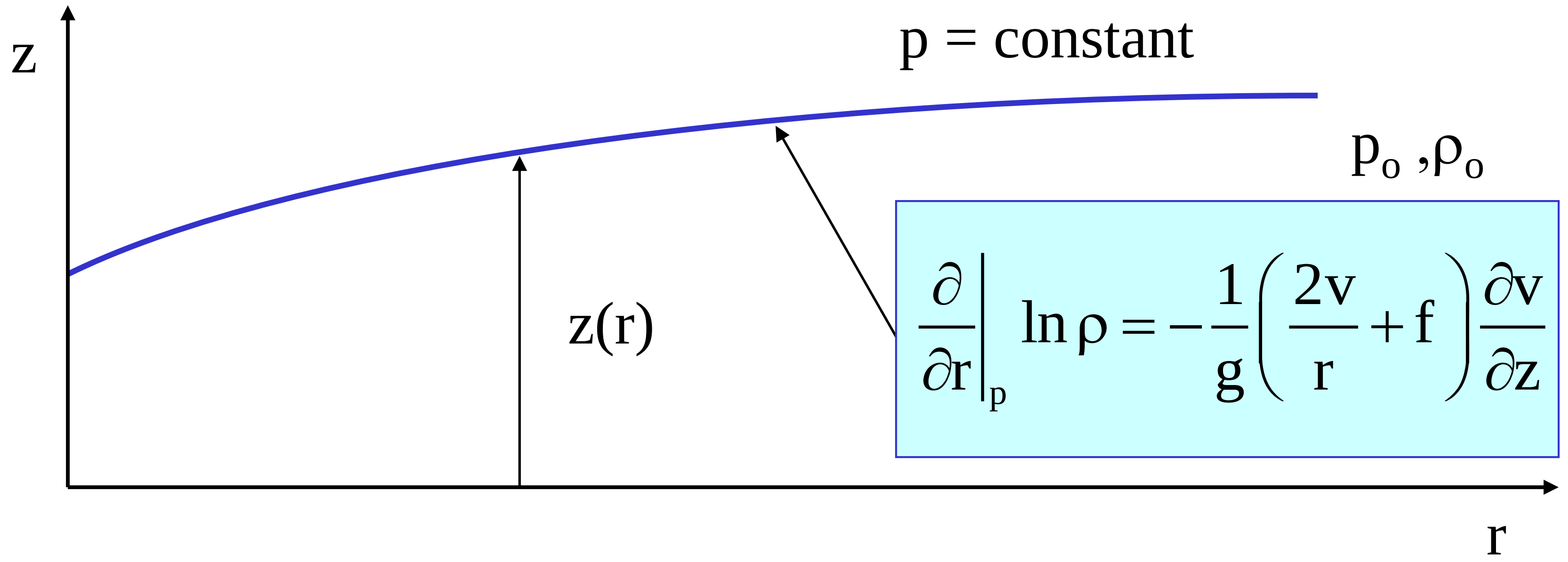
Characteristics are isobaric surfaces



Along a characteristic $$g\,dz = \left(\frac{v^2}{r}+fv\right)dr$$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz = \rho\left(\frac{v^2}{r}+fv\right)dr - \rho g\,dz = 0$$

Inferences



$$\left.\frac{\partial}{\partial r}\right|_p \ln\rho = -\frac{1}{g}\left(\frac{2v}{r}+f\right)\frac{\partial v}{\partial z}$$

Barotropic vortex $\frac{\partial v}{\partial z}=0 \Rightarrow \left.\frac{\partial \rho}{\partial r}\right|_p = 0 \Rightarrow \left.\frac{\partial T}{\partial r}\right|_p = 0$

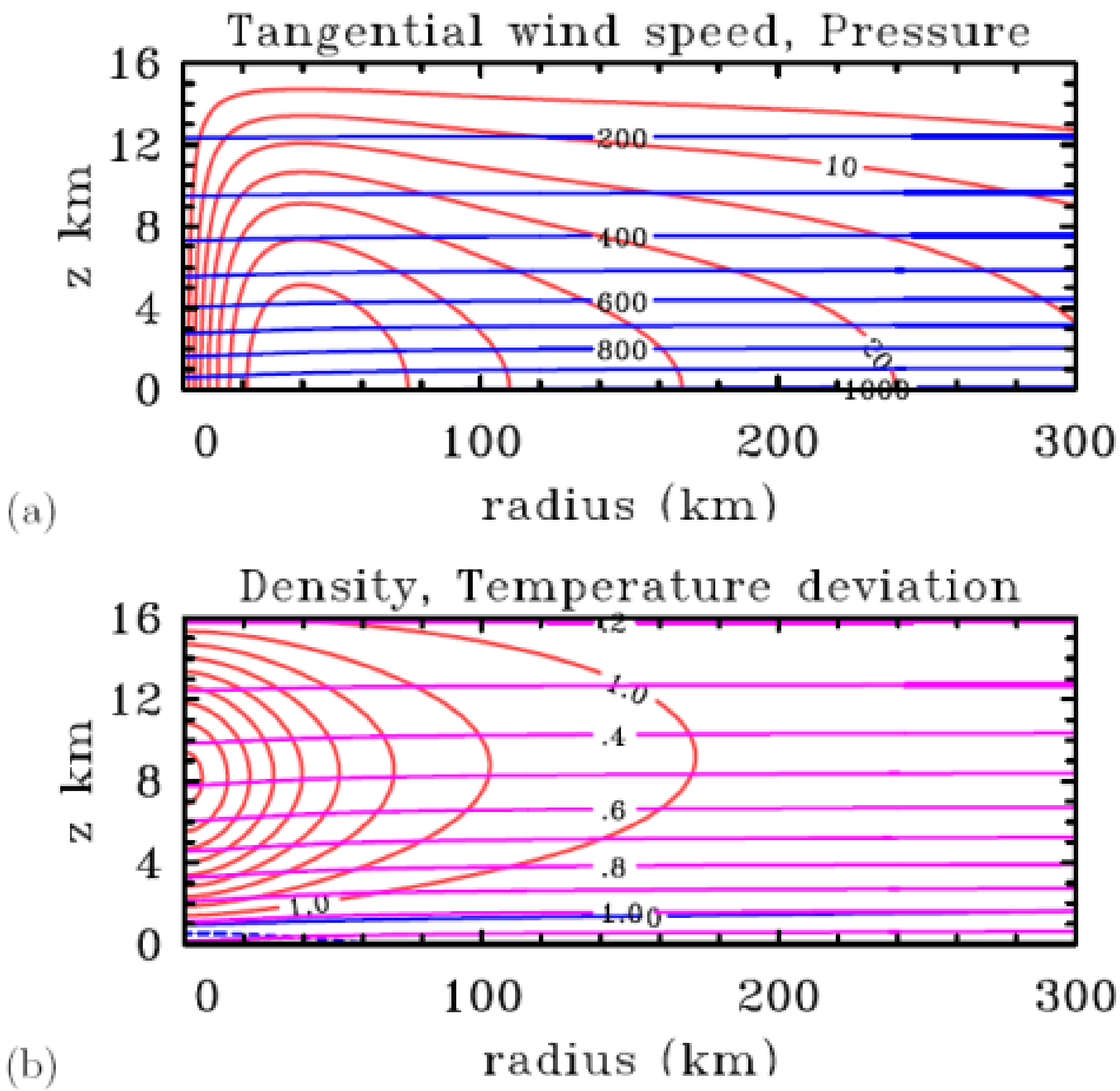
Baroclinic vortex $\frac{\partial v}{\partial z}<0 \Rightarrow \left.\frac{\partial \rho}{\partial r}\right|_p > 0 \Rightarrow \left.\frac{\partial T}{\partial r}\right|_p < 0$

Equation of state $T = p / R\rho$

Summary

- **A barotropic vortex is cold cored if temperature contrasts are measured at constant height.**
- **A baroclinic vortex is warm cored if temperature contrasts are measured at constant height and if** $-\partial v/\partial z$ **is large enough.**

A typical vortex



The secondary circulation

Thermal wind equation

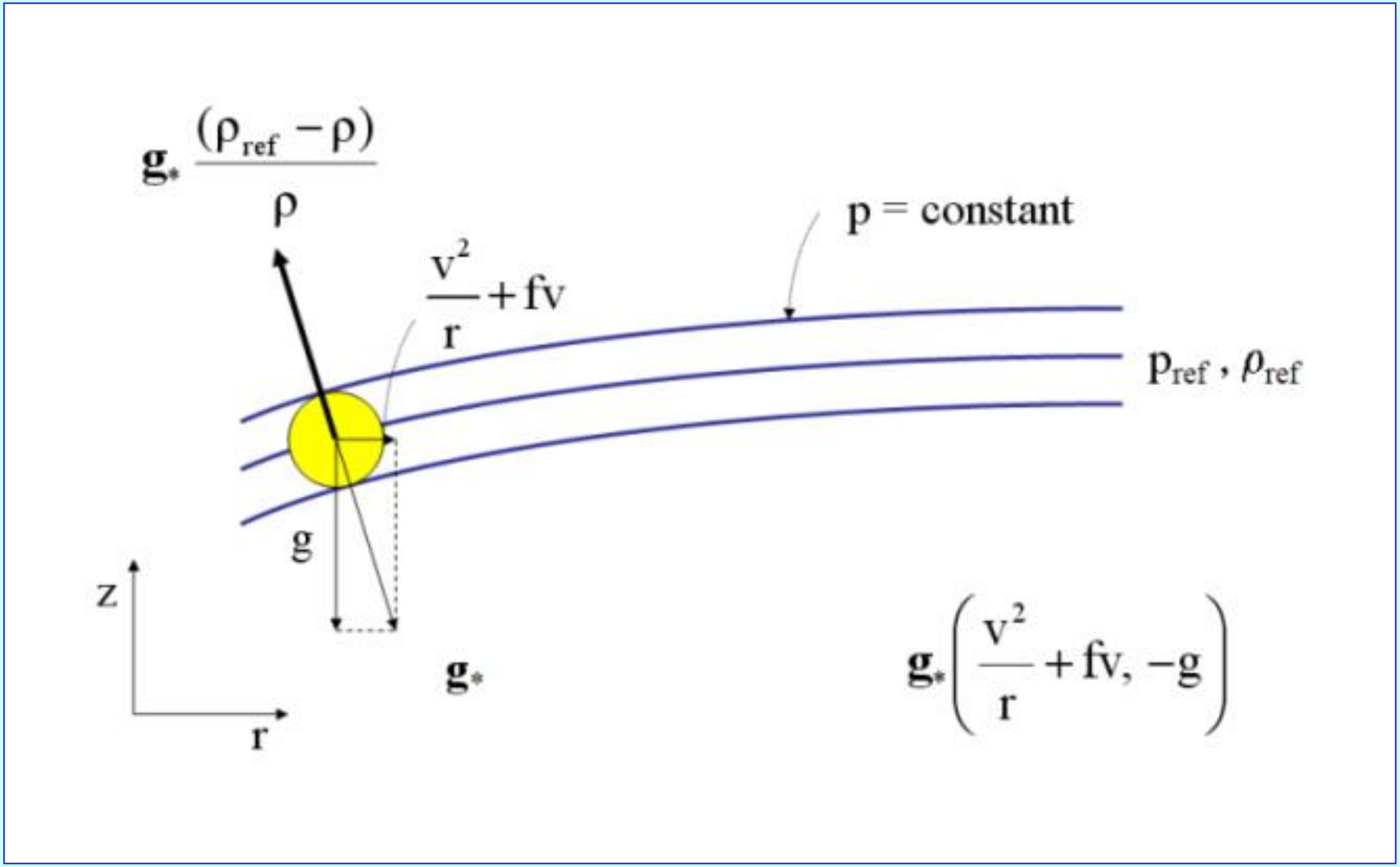
$$g\frac{\partial \ln \rho}{\partial r} + C\frac{\partial \ln \rho}{\partial z} = -\frac{\partial C}{\partial z}$$

Exercise 3.2 Show that Eq. (3.12) may be reformulated as

$$g\frac{\partial (\ln \chi)}{\partial r} + C\frac{\partial (\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}. \quad (3.17)$$

where $\chi = 1/\theta$.

Generalized buoyancy



A scale analysis

continuity

$$\frac{1}{r}\frac{\partial \rho r u}{\partial r} \qquad \frac{\partial \rho w}{\partial z} \qquad (3.4)$$

$$\rho\frac{U}{R} \qquad \rho\frac{W}{Z} \qquad (1)$$

A scale analysis

u-momentum

$$\frac{\partial u}{\partial t} \quad + u\frac{\partial u}{\partial r} \quad + w\frac{\partial u}{\partial z} \quad - \frac{v^2}{r} \quad - fv \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3.1}$$

$$\frac{U}{T} \quad \frac{U^2}{R} \quad W\frac{U}{Z} \quad \frac{V^2}{R} \quad fV \quad \frac{\Delta p}{\rho R} \tag{2a}$$

$$S^2 \quad S^2 \quad S^2 \quad 1 \quad \frac{1}{Ro} \quad \frac{\Delta p}{\rho V^2} \tag{2b}$$

A scale analysis

v-momentum

$$\frac{\partial v}{\partial t} \quad + u\frac{\partial v}{\partial r} \quad + w\frac{\partial v}{\partial z} \quad + \frac{uv}{r} \quad + fu \quad = \tag{3.2}$$

$$\frac{V}{T} \quad U\frac{V}{R} \quad \frac{WV}{Z} \quad \frac{UV}{R} \quad fU \tag{3a}$$

$$S \quad S \quad S \quad S \quad \frac{S}{Ro} \tag{3b}$$

A scale analysis

w-momentum

$$\frac{\partial w}{\partial t} \quad +u\frac{\partial w}{\partial r} \quad +w\frac{\partial w}{\partial z} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad - \ g \qquad (3.3)$$

$$\frac{W}{T} \quad U\frac{W}{R} \quad W\frac{W}{Z} \quad \frac{\Delta p}{\rho Z} \quad g \qquad (4a)$$

$$\frac{WU}{Rg} \quad \frac{UW}{Rg} \quad \frac{WU}{Rg} \quad \frac{\Delta p}{\rho g Z} \quad 1 \qquad (4b)$$

The secondary circulation

A balanced theory: The Sawyer-Eliassen equation

Continuity →

$$u = -\frac{1}{r\rho}\frac{\partial \psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial \psi}{\partial r}. \tag{3.57}$$

Thermal wind →

$$g\frac{\partial(\ln \chi)}{\partial r} + C\frac{\partial(\ln \chi)}{\partial z} = -\frac{\partial C}{\partial z}. \tag{3.17}$$

Prognostic equations

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V} \tag{3.58}$$

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta} \tag{3.59}$$

A balanced theory: The Sawyer-Eliassen equation

$\partial/\partial t$ **(thermal wind equation)**

$$g\frac{\partial}{\partial r}\frac{\partial \chi}{\partial t} + \frac{\partial}{\partial z}\left(C\frac{\partial \chi}{\partial t} + \chi\frac{\partial C}{\partial t}\right) = 0$$

$$\frac{\partial v}{\partial t} + u(\zeta + f) + wS = \dot{V}$$

$$\frac{\partial \chi}{\partial t} + u\frac{\partial \chi}{\partial r} + w\frac{\partial \chi}{\partial z} = -\chi^2\dot{\theta}$$

A balanced theory: The Sawyer-Eliassen equation

∂/∂t (thermal wind equation)

$$g\frac{\partial}{\partial r}\left(u\frac{\partial\chi}{\partial r}+w\frac{\partial\chi}{\partial z}-Q\right)+\frac{\partial}{\partial z}\left[C\left(u\frac{\partial\chi}{\partial r}+w\frac{\partial\chi}{\partial z}-Q\right)+\chi\xi\left(u(\zeta+f)+wS-\dot{V}\right)\right]=0$$

where $\chi = 1/\theta$ and $Q = -\chi^2\dot{\theta}$. Then

$$\frac{\partial}{\partial r}\left[g\frac{\partial\chi}{\partial z}w+g\frac{\partial\chi}{\partial r}u\right]+$$

$$\frac{\partial}{\partial z}\left[(\chi\xi(\zeta+f)+C\frac{\partial\chi}{\partial r})u+\frac{\partial}{\partial z}(\chi C)w\right]=g\frac{\partial Q}{\partial r}+\frac{\partial}{\partial z}(CQ)+\frac{\partial}{\partial z}(\chi\xi\dot{V})$$

or

$$\frac{\partial}{\partial r}\left[g\frac{\partial\chi}{\partial z}w-\frac{\partial}{\partial z}(\chi C)u\right]+$$

$$\frac{\partial}{\partial z}\left[(\chi\xi(\zeta+f)+C\frac{\partial\chi}{\partial r})u+\frac{\partial}{\partial z}(\chi C)w\right]=g\frac{\partial Q}{\partial r}+\frac{\partial}{\partial z}(CQ)+\frac{\partial}{\partial z}(\chi\xi\dot{V}) \qquad (3.60)$$

The Sawyer-Eliassen equation

$$\frac{\partial}{\partial r}\left[g\frac{\partial \chi}{\partial z}\frac{1}{\rho r}\frac{\partial \psi}{\partial r} + \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial z}\right] -$$

$$\frac{\partial}{\partial z}\left[\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial \psi}{\partial z} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right] = g\frac{\partial Q}{\partial r} + \frac{\partial}{\partial z}(CQ) + \frac{\partial}{\partial z}(\chi\xi\dot{V}) \tag{3.61}$$

Discriminant

$$D = -g\frac{\partial \chi}{\partial z}\left(\xi\chi(\zeta + f) + C\frac{\partial \chi}{\partial r}\right) - \left[\frac{\partial}{\partial z}(\chi C)\right]^2 \tag{3.62}$$

Parameters

- the *static stability*

$$N^2 = -g\frac{\partial ln\chi}{\partial z};$$

- the *inertial stability*

$$I^2 = \frac{1}{r^3}\frac{\partial M^2}{\partial r} = \xi(\zeta + f);$$

- the *baroclinicity*

$$B_1 = \frac{1}{r^3}\frac{\partial M^2}{\partial z} = \xi S.$$

The Sawyer-Eliassen equation

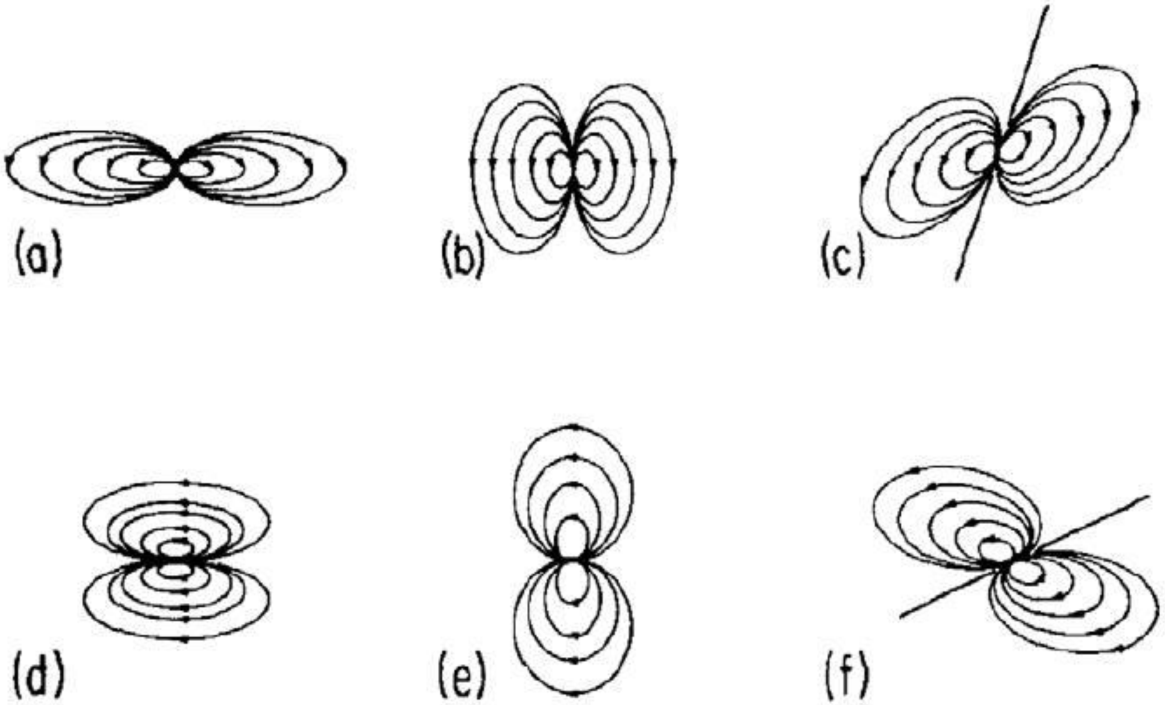


Figure 3.11: Streamfunction responses to point sources of: (a) Heat in a barotropic vortex with weak inertial stability, (b) heat in a barotropic vortex with strong inertial stability, (c) heat in a baroclinic vortex, (d) momentum in a barotropic vortex with weak inertial stability, (e) momentum in a barotropic vortex with strong inertial stability, and (f) momentum in a baroclinic vortex. (Based on Figs. 8, 9, 11, and 12

The Sawyer-Eliassen equation

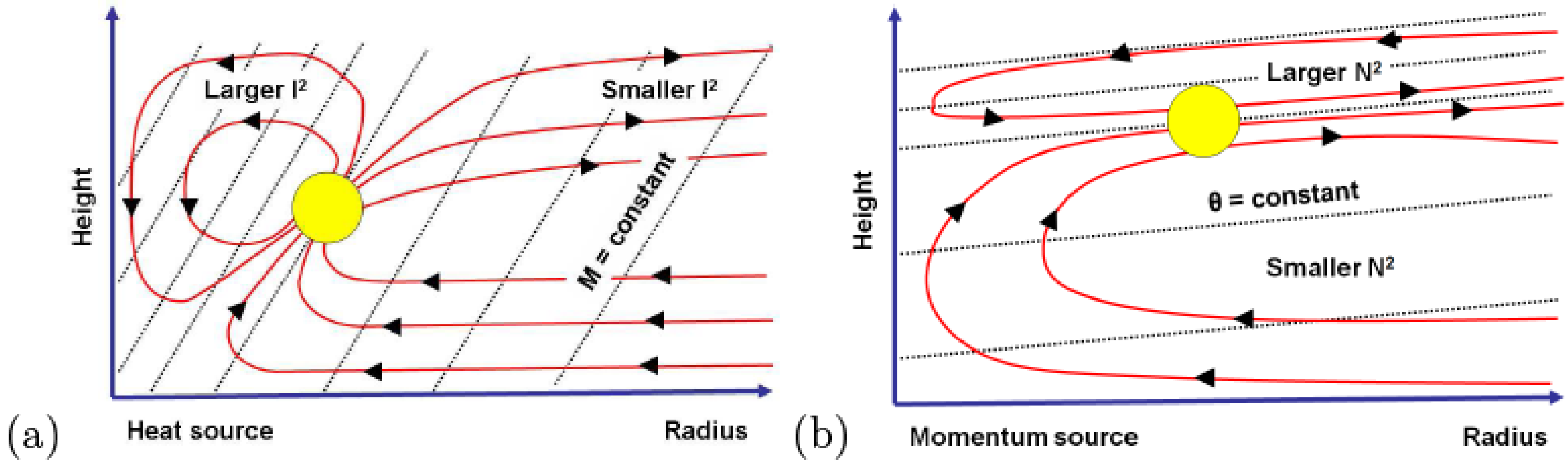


Figure 3.12: Secondary circulation induced in a balanced vortex by (a) a heat source and (b) a cyclonic momentum source showing the distortion induced by variation in inertial stability, I^2 and thermodynamic stability. N^2, and baroclinicity S^2. The strong motions through the source follow lines of constant angular momentum for a heat source and of constant potential temperature for a momentum source. From

Willoughby (1995)

A balance theory for the evolution of an axisymmetric vortex

1. Solve for the initial balanced density and pressure fields corresponding to $v_i(r,z,0)$

$$g\frac{\partial(\ln\chi)}{\partial r} + C\frac{\partial(\ln\chi)}{\partial z} = -\frac{\partial C}{\partial z}.$$

2. Solve for $\psi(r,z,0)$

$$\frac{\partial}{\partial r}\left[g\frac{\partial\chi}{\partial z}\frac{1}{\rho r}\frac{\partial\psi}{\partial r} + \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial z}\right] -$$

$$\frac{\partial}{\partial z}\left[\left(\xi\chi(\zeta+f) + C\frac{\partial\chi}{\partial r}\right)\frac{1}{\rho r}\frac{\partial\psi}{\partial z} - \frac{\partial}{\partial z}(\chi C)\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right] = g\frac{\partial Q}{\partial r} + \frac{\partial}{\partial z}(CQ) + \frac{\partial}{\partial z}(\chi\xi\dot{V}) \tag{3.61}$$

3. Solve for $u(r,z,0)$ **and** $w(r,z,0)$

$$u = -\frac{1}{r\rho}\frac{\partial\psi}{\partial z} \qquad w = \frac{1}{r\rho}\frac{\partial\psi}{\partial r}.$$

4. Predict $v(r,z,\Delta t)$

$$\frac{\partial v}{\partial t} + u(\zeta+f) + wS = \dot{V}$$

5. Repeat to find $v(r,z,\Delta t)$

Issues

- As tropical cyclones intensify, the boundary layer and upper-tropospheric outflow region generally develop regions of zero or negative discriminant ($D < 0$) implying symmetric instability. Then the global balance solution breaks down.

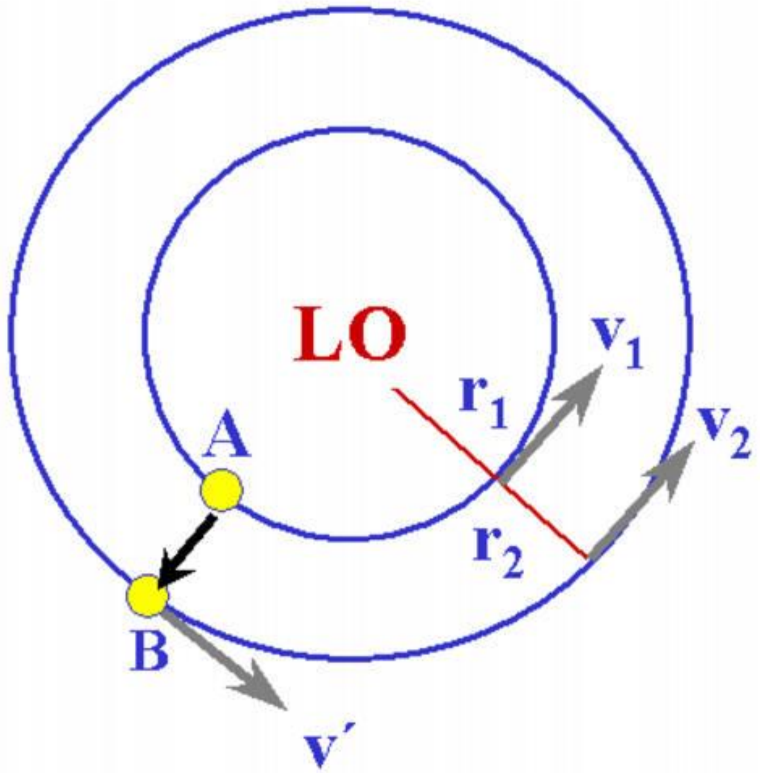
If the regions of $D < 0$ remain localized, one may apply a regularization procedure to keep the Sawyer-Eliassen equation elliptic and thus solvable.

- How to formulate the forcing terms in the Sawyer-Eliassen equation, i.e. the diabatic heating rate and the near-surface frictional force?

The representation of diabatic heating in terms of the evolving flow constitutes the cumulus parameterization problem. Typically, this requires the inclusion of a prediction equation for water vapour and, in more sophisticated representations, for various species of water condensate.

- The boundary layer IS NOT balanced!

Barotropic stability



The parcel at **A** conserves its angular momentum during its radial displacement to **B**

$$r_2 v' + \frac{1}{2} f r_2^2 = r_1 v_1 + \frac{1}{2} f r_1^2,$$

$$v' = \frac{r_1}{r_2} v_1 + \frac{1}{2} \frac{f}{r_2} (r_1^2 - r_2^2) \qquad (3.17)$$

Net radial force on a displaced air parcel

Radial pressure gradient at B

$$\frac{1}{\rho}\frac{dp}{dr}\bigg]_{r=r_2} = \frac{v_2^2}{r_2} + fv_2. \tag{3.18}$$

Net force on parcel at B

$$F = \text{centrifugal} + \text{Coriolis force} - \text{radial pressure gradient}$$

$$= \frac{v'^2}{r_2} + fv' - \frac{1}{\rho}\frac{\partial p}{\partial r}\bigg]_{r=r_2}$$

$$F = \frac{1}{r_2^3}\left[(r_1v_1 + \frac{1}{2}r_1^2 f)^2 - (r_2v_2 + \frac{1}{2}r_2^2 f)^2\right]. \tag{3.19}$$

Net radial force on a displaced air parcel

$$F = \frac{1}{r_2^3}\left[(r_1 v_1 + \frac{1}{2} r_1^2 f)^2 - (r_2 v_2 + \frac{1}{2} r_2^2 f)^2\right]. \qquad (3.19)$$

In the special case of solid body rotation, $v = \Omega r$, and for a small displacement from radius $r_1 = r$ to $r_2 = r + r'$, (3.19) gives

$$F \approx -4(\Omega + \frac{1}{2} f)^2 r' \qquad (3.20)$$


End