A NEW ASPECT OF COALESCENCE THEORY

By J. W. Telford

Commonwealth Scientific and Industrial Research Organization

(Manuscript received 20 January 1955)

ABSTRACT

The process of growth by coalescence is examined from the viewpoint of a discrete rather than a continuous accretion process. It is concluded that, for drops beginning growth at twice the volume of their neighbors, random fluctuations in the times taken for different drops to effect captures can lead to the formation of a complete raindrop spectrum in times shorter than that required for growth to raindrop size by the continuous growth process.

Under typical conditions, drops of 23-microns radius can form in the order of 5 minutes in a cloud of droplets of 10 microns, while smaller numbers of much larger drops can be expected in reasonable times. The latter are likely to be important in chain-reaction theories.

1. Introduction

The growth of raindrops in warm clouds is now a well established fact. Recent work, such as that of Battan (1953) and Murgatroyd (1954), appears to be establishing the coalescence process as one of considerable importance, even in rain formed by clouds whose tops extend considerably above the freezing level.

Bowen (1950) has theoretically investigated the influence of the most important cloud parameters, and Ludlam (1951) made similar computations and concluded that the process might be initiated by salty droplets from sea spray. However, early theoretical work on this problem seems to indicate that the growth of cloud droplets, which arises solely from the collision of drops of different sizes, proceeds too slowly to account for the appearance of raindrops within the observed times.

Thus, various workers set out to establish other phenomena as a basis for faster growth. Cochet (1951) has indicated that charges will have a significant effect, and similar experimental results have been reported by Telford *et al* (1955). Woodcock (1952) has extensively investigated the occurrence and growth of large salt nuclei, but as yet has not established a completely convincing case.

Recently, East and Marshall (1954) have modified Langmuir's (1948) calculation of collection efficiencies to include turbulent accelerations of the air, but the presence of such motion as is required to give a significant effect has yet to be established.

In the paper by Telford *et al* (1955), a new aerodynamic effect involving high collection efficiencies between nearly equal drops is described; it may prove to be of importance in initiating coalescence.

In this article it is hoped to demonstrate that, quite apart from the effects mentioned above, the time required to initiate growth by coalescence is considerably less than indicated by previous treatments. All the work done in the past has been based on a continuous distribution of the water to be collected, whereas in actual fact the collection is restricted in space to the actual positions of the droplets.

Calculations, such as those of Langmuir (1948) and Bowen (1950), all appear to have been based on the production of big drops according to the growth equation

$$\rho \, dV/dt = \pi r^2 U E \omega,$$

where ρ is the density of water, t the time, V, r, U and E are the volume, radius, terminal velocity and collection efficiency, respectively, of the growing drop, and ω is the water content of the cloud through which the growing drops are falling.

This means that for every time, δt, however small, the change in volume of the growing drop is

$$\delta V = (\pi r^2 U E \omega / \rho) \, \delta t,$$

which is, of course, not really the case, since δV must be an integral multiple of the volume of the cloud droplets gathered up.

In other words, in an actual cloud, the water to be collected is not uniformly distributed in space but is grouped in discrete droplets of definite volume.

In this article an attempt will be made to correct for this deficiency, and it will be shown that the correction is very important.

2. General solution

To simplify the situation sufficiently for mathematical treatment, a model will be taken in which the collected droplets are all of one size and do not change in size or concentration with either time or position.

The history of a spectrum of larger drops growing in these smaller droplets will then be deduced under the assumptions that the growing drops do not in any way affect each other, and that the concentration of the droplets available for collection does not change owing to their removal by coalescence. These assumptions will be discussed at the end of the article.

The differential equation describing these conditions can be derived, and solved, as follows.

If the cloud droplets are receding in front of the collecting drop with a terminal velocity u, the growth equation is

$$\delta V = (\pi r^2 E\omega/\rho)(U - u)\,\delta t,$$

where δV is the average volume increase of the growing drop in time δt, and the other symbols are as defined above.

When δV_K is the total volume increase of K growing drops of the same size,

$$\begin{aligned}\delta V_K &= K(\pi r^2 E\omega/\rho)(U - u)\,\delta t\\ &= (n_K + \epsilon)v, \qquad 0 \leqslant \epsilon < 1,\end{aligned}$$

where n_K is an integer, and v is the volume of each of the collected droplets.

When K is made sufficiently large, $\epsilon/n_K \to 0$; thus,

$$\delta V_K \to n_K v.$$

If δt is so small that the chance of a collecting drop acquiring more than one cloud droplet is negligible, the probability δP of any one of the K drops having captured a droplet in this time is

$$\delta P = n_K/K = (\pi r^2 E\omega/\rho v)(U - u)\,\delta t.$$

Now, let the volume of the collecting drop be j cloud droplet volumes, *i.e.*, $V = jv$, where j is an integer. Note that j need not be an integer; but this considerably simplifies the notation and is physically sufficient. Hence,

$$\delta P_j = \frac{3}{4}\frac{E\omega u}{a\rho} j^{2/3}\left(\frac{U}{u} - 1\right)\delta t = A_j\,\delta t, \qquad (1)$$

where a is the radius of the cloud droplet, and δP_j is the probability of a drop of volume jv collecting a cloud droplet in time δt following selection.

Now, consider the change in a distribution after the lapse of a time interval δt.

The number of drops per unit volume of cloud of volume rv at time $t + \delta t$, $N(rv, t + \delta t)$, will consist of those of this volume at time t which have not undergone coalescence in the time interval δt, together with those of volume $(r-1)v$ which have collected a droplet.

Thus,

$$N(r, t+\delta t) = N(r, t)\,(1 - A_r\,\delta t) + N(r-1, t)\,A_{r-1}\,\delta t;$$

and so,

$$\partial N(r, t)/\partial t = A_{r-1}\,N(r-1, t) - A_r\,N(r, t). \qquad (2)$$

The relevant solution to this equation may be obtained by starting at $r = 1$ and working up through increasing r.

Observing that $N(0, t) = 0$, one may write

$$\partial N(1, t)/\partial t + A_1\,N(1, t) = 0;$$

and therefore,

$$N(1, t) = L_1 e^{-A_1 t},$$

where L_1 is the constant of integration. Substituting the initial conditions, we find

$$N(1, t) = N(1, 0)\,e^{-A_1 t}.$$

By repetition of this process,

$$N(2, t) = N(1, 0)\,[A_1/(A_2 - A_1)](e^{-A_1 t} - e^{-A_2 t}) + N(2, 0)\,e^{-A_2 t},$$

and

$$\begin{aligned}N(3, t) = N(1, 0)&\left[\frac{A_1 A_2(e^{-A_1 t} - e^{-A_3 t})}{(A_2 - A_1)(A_3 - A_1)} + \frac{A_1 A_2(e^{-A_2 t} - e^{-A_3 t})}{(A_1 - A_2)(A_3 - A_2)}\right]\\ &+ N(2, 0)\frac{A_3}{A_3 - A_2}(e^{-A_2 t} - e^{-A_3 t})\\ &+ N(3, 0)\,e^{-A_3 t}.\end{aligned}$$

Considering these solutions, let us postulate a solution

$$N(j, t) = \sum_{i=1}^{j}\left[N(i, 0) \times \sum_{l=i}^{j-1}\left\{\frac{\prod_{k=i}^{j-1} A_k}{\prod_{\substack{k=i\\ k\neq l}}^{j}(A_k - A_l)}\right\}(e^{-A_l t} - e^{-A_j t})\right]. \qquad (3)$$

Since the differential equation is linear, a general solution is any linear combination of special solutions which satisfies the boundary conditions. Hence, it is sufficient to prove the above formula for

$$\begin{aligned}N(i, 0) &= 0, \qquad i \neq m,\\ N(i, 0) &= 1, \qquad i = m.\end{aligned}$$

That is to say, we solve the problem for one initial size of the growing drop and find the solution for an initial distribution by adding the separate solutions for each size.

Thus, we have, putting $i = m$,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j-1} A_l\left[\prod_{\substack{k=m\\k\neq l}}^{j}(1-A_l/A_k)^{-1}\right](e^{-A_l t}-e^{-A_j t}).$$

Now, let

$$\prod_{\substack{k=m\\k\neq l}}^{j}[1-(A_l/A_k)]^{-1} = B_l^{m,j}. \tag{4}$$

Therefore,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j-1} A_l B_l^{m,j}(e^{-A_l t} - e^{-A_j t}),$$

and, terminating the sum at j instead of $j-1$,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j} A_l B_l^{m,j}(e^{-A_l t} - e^{-A_j t}),$$

which is permissible as $\exp -A_l t = \exp -A_j t$ when $l = j$. Hence,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j} A_l B_l^{m,j} e^{-A_l t} - \frac{e^{-A_j t}}{A_j}\sum_{l=m}^{j} A_l B_l^{m,j}.$$

It is later proven that

$$\sum_{l=m}^{j} A_l B_l^{m,j} \equiv 0$$

[see (9)]. Hence,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j} A_l B_l^{m,j} e^{-A_l t}.$$

Substituting this in the differential equation, one obtains

$$\frac{\partial N(j,t)}{\partial t} = -\frac{1}{A_j}\sum_{l=m}^{j}(A_l)^2 B_l^{m,j} e^{-A_l t}$$

for the left-hand side. For the right-hand side,

$$A_{j-1}N(j-1,t) - A_j N(j,t) = \sum_{l=m}^{j-1} A_l B_l^{m,j-1} e^{-A_l t} - \sum_{l=m}^{j} A_l B_l^{m,j} e^{-A_l t}.$$

Now, from (4),

$$B_l^{m,j-1} = [1-(A_l/A_j)]B_l^{m,j}.$$

Thus, the right-hand side is

$$\sum_{l=m}^{j-1} A_l[1-(A_l/A_j)]B_l^{m,j} e^{-A_l t} - \sum_{l=m}^{j} A_l B_l^{m,j} e^{-A_l t}$$

$$= -\frac{1}{A_j}\sum_{l=m}^{j} A_l^2 B_l^{m,j} e^{-A_l t} - A_j[1-A_j/A_j)]B_l^{m,j} e^{-A_j t}$$

$$= -\frac{1}{A_j}\sum_{l=m}^{j} A_l^2 B_l^{m,j} e^{-A_l t},$$

which is the value obtained for the left-hand side. Therefore,

$$N(j,t) = \frac{1}{A_j}\sum_{l=m}^{j} A_l B_l^{m,j} e^{-A_l t} \tag{5}$$

is a solution of the equation, and the general solution is

$$N(j,t) = \sum_{i=1}^{j}\frac{N(i,0)}{A_j}\sum_{l=i}^{j} A_l B_l^{i,j} e^{-A_l t}. \tag{6}$$

It is seen in (1) that the terminal velocities of the collecting drop and collected droplet, together with other relevant physical quantities, specify the A_j's. One could evaluate the A_j's, and hence the $B_l^{i,j}$'s, and develop a solution. This would, however, demand a tremendous numerical effort and is not essential to a clear picture of the actual process.

Before useful numerical solutions to the equation are discussed, there is one more mathematical point requiring attention.

The solution above gives a distribution showing the number of drops of a given size per unit volume of cloud at any specified time — as might be observed, for example, in sampling the distribution of drop sizes in rain. However, there is another type of display which is of more use to us here and which can be derived directly from the preceding argument. This is the distribution showing the number of drops which increase from size mv to size $(m+n)v$, say, in time intervals greater than t and less than $t+\delta t$. This distribution is used later to bring out the essential difference between the old and the new theories.

Considering once again only those drops of volume mv at $t = 0$, let us denote the number reaching the volume $(m+n)v$ in a time between t and $t+\delta t$ by $M(m,n,t)\,\delta t$.

These are the drops of size m which have n captures in a time between t and $t+\delta t$ following selection. The number of drops of size greater than $j-1$ is

$$\int_0^t M(m, j-m, t)\,dt.$$

On the basis of (5), this equals

$$\sum_{q=j}^{\infty} N(q,t);$$

and for drops greater than jv in volume mv,

$$\int_0^t M(m, j-m+1, t) = \sum_{q=j+1}^{\infty} N(q,t).$$

Subtraction gives

$$\int_0^t [M(m, j-m, t) - M(m, j-m+1, t)]\,dt = N(j,t).$$

Differentiating and using (2), we obtain

$$M(m, j-m, t) - M(m, j-m+1, t) = \partial N(j, t)/\partial t$$
$$= A_{j-1} N(j-1, t) - A_j N(j, t).$$

Hence,

$$M(m, j-m, t) = A_{j-1} N(j-1, t)$$
$$= \sum_{l=m}^{j-1} A_l B_l^{m, j-1} e^{-A_l t},$$

or

$$M(m, n, t) = \sum_{l=m}^{m+n-1} A_l B_l^{m, m+n-1} e^{-A_l t}. \quad (7)$$

It is interesting to note that (7) can be established on an entirely different basis and is so presented in the appendix.

3. A useful analytic solution

We will now proceed with the numerical development of (6) under different physical assumptions as to the nature of the A_j's.

Let us consider the case where the drops are well above cloud-droplet sizes, say in the radius range of 75 to 750 μ. Here the relative velocity of the drops and the droplets may be taken, to a good approximation, as proportional to the radius of the collecting drop, that is, $U - u = kr$.

Inserting this in (1), we have

$$\delta P_j = \frac{3}{4} \frac{E\omega}{a\rho} j^{2/3} (U-u)\, \delta t = \frac{3}{4} \frac{E\omega}{a\rho} k r j^{2/3}\, \delta t = \frac{3}{4} \frac{E\omega k}{\rho} j\, \delta t,$$

since $j = V/v = (r/a)^3$. Thus, $A_j = (3/4)(E\omega k/\rho) j$.

Substituting in (4) and making E independent of radius, and hence of j, we have

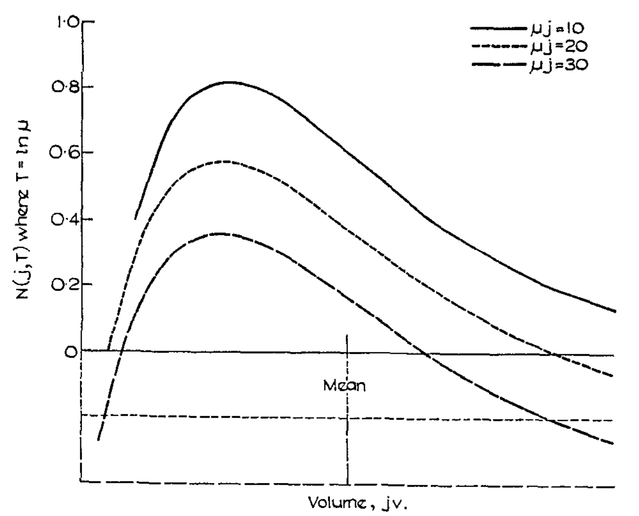

FIG. 1. Graph shows proportion of original drops $N(j, T)$, of volume jv after growth time T. This solution is for case when relative velocities of drops and droplets are proportional to radius of capturing drop, and all droplets are initially twice volume of cloud droplets. Values of T for three graphs have been taken to make mean volume of the drops, $\mu_j v$, 10, 20 and 30 times cloud-droplet volume, v, respectively. Curves have been normalized to bring means into coincidence, and have been displaced vertically for clarity. Curves are almost identical, showing rapid tendency to limiting form for more than 20 captures.

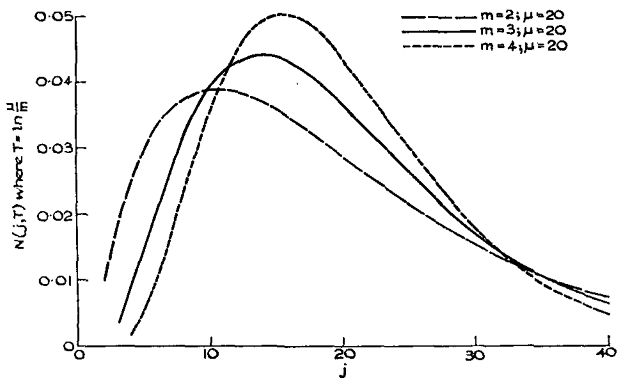

FIG. 2. Graph shows curves equivalent to those of fig. 1 for different initial drop sizes mv, when growth time is taken to give mean μv of curve at 20 cloud-droplet volumes. Curves show number of drops of different volumes and clearly illustrate how standard deviation, σ, becomes less when growth starts at larger sizes. Solution is for same case as fig. 1, except for differing initial volumes.

$$B_l^{m,j} = \prod_{\substack{k=m \\ k \neq l}}^{j} [1 - (l/k)]^{-1}$$
$$= \prod_{\substack{k=m \\ k \neq l}}^{j} \left(\frac{k-l}{k}\right)^{-1} = \prod_{\substack{k=m \\ k \neq l}}^{j} k \Big/ \prod_{\substack{k=m \\ k \neq l}}^{j} (k-l)$$
$$= \frac{j!\,(-1)^{l-m}}{l(m-1)!\,(l-m)!\,(j-l)!}.$$

Thus, (5) becomes

$$N(j, t) = \frac{1}{j} \sum_{l=m}^{j} \frac{j!\,(-1)^{l-m}}{(m-1)!\,(l-m)!\,(j-l)!} \exp -\left(\frac{3}{4}\frac{E\omega k}{\rho}\right) lt.$$

Let $T = (3/4)(E\omega k/\rho)t$. Then

$$N(j, t) = \frac{1}{j} \frac{j!}{(m-1)!\,(j-m)!} \times \sum_{l=m}^{j} (-1)^{l-m} \frac{(j-m)!}{(j-l)!\,(l-m)!} e^{-lT}$$
$$= {}^{j-1}C_{m-1} \sum_{l=m}^{j} (-1)^{l-m}\, {}^{j-m}C_{l-m}\, e^{-lT},$$

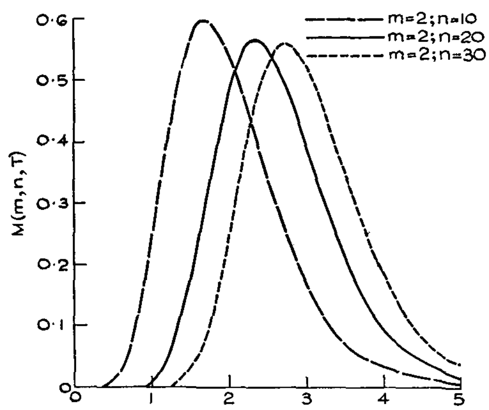

FIG. 3. Curves show functions $M(m, n, T)$, where $M(m, n, T)\,\delta T$ is proportion of drops of volume mv at $T = 0$ to experience n captures in time between T and $T + \delta T$. This is for case where relative velocities of drops are proportional to radius of collecting drop.

where ${}^{j-1}C_{m-1}$ is the appropriate binomial coefficient,

$$= {}^{j-1}C_{m-1}\, e^{-mT} \sum_{l=m}^{j} {}^{j-m}C_{l-m}(-e^{-T})^{l-m}$$

$$= {}^{j-1}C_{m-1}\, e^{-mT}(1 - e^{-T})^{j-m}. \tag{8}$$

It is interesting to note that this is a term of the expansion of $[e^T + (1 - e^T)]^{-m}$. Thus, the distribution is a binomial distribution of negative index.

The time distribution for this case is

$$M(m, n, t) = (m + n - 1)\, {}^{m+n-2}C_{m-1}\, e^{-mT}(1 - e^{-T})^{n-1}.$$

These functions are plotted in figs. 1, 2 and 3 for several values of m and n.

It is easily shown that the mean volume (μ) and the standard deviation (σ) of $N(j, t)$ are $\mu = mve^T$, and $\sigma = \mu(m^{-1} - \mu^{-1})^{\frac{1}{2}}$. Therefore

$$\frac{\sigma}{\mu} = (m^{-1} - \mu^{-1})^{\frac{1}{2}} \approx \frac{1}{m^{\frac{1}{2}}}\left(1 - \frac{m}{2\mu}\right):$$

Thus, as time passes, the mean of the distribution increases exponentially, being equal in this case to the value obtained from the continuous-growth equation. When the drop has increased in volume by a factor of 100, the spread (σ/μ) has increased to within 1 per cent of its limiting value, $m^{-\frac{1}{2}}$. Under these conditions, the smaller the drops are in the beginning the greater will be the spread in the final sizes, and no appreciable spreading occurs after the first few hundred captures.

4. A numerical solution relevant to cloud-droplet sizes

We have just considered the case in which the velocity of the droplet relative to that of the pursuing drop is proportional to the radius of the drop. Let us now examine what happens when both drops are falling at velocities as determined by Stokes' law. This is an accurate representation of the motion of cloud drops and is true, approximately, up to 50-μ radius.

Rewriting of (1) gives

$$A_j = \frac{3}{4}\frac{E\omega u}{a\rho} j^{2/3}\left(\frac{U}{u} - 1\right).$$

Under Stokes' law, $U = (2\rho/9\eta)gr^2$, g being the acceleration due to gravity; so $U/u = r^2/a^2 = j^{2/3}$, since $V/v = j = (r/a)^3$, and

$$A_j = \frac{3}{4}\frac{E\omega u}{a\rho} j^{2/3}(j^{2/3} - 1).$$

We take E independent of radius, as before, and introduce $T = (3/4)(E\omega u/a\rho)t$ as the new time variable. Therefore

$$A_j = j^{2/3}(j^{2/3} - 1), \qquad j = 2, \cdots, 100,$$

are the numbers used to determine the distributions.

With the analytical case as a guide, it is clear that we must examine (5) for drops starting their growth at volumes 2, 3 and 4 times cloud-droplet volumes and experiencing 10 to 100 captures.

From this point onward, we confine our discussion to the form of the equation given by (7), as this gives the times which show so strikingly the importance of the result, and as it also permits the numerical work to be reduced to manageable quantities.

We have, from (7),

$$M(m, n, t) = \sum_{l=m}^{m+n-1} A_l B_l^{m, m+n-1}\, e^{-A_l t}.$$

The numerical work involves development of the numbers

$$A_l B_l^{m,m+n-1} = A_l \prod_{\substack{k=m \\ k\neq l}}^{m+n-1} [1 - (A_l/A_k)]^{-1}$$

$$= \prod_{k=m}^{m+n-1} A_k \prod_{\substack{k=m \\ k\neq l}}^{m+n-1} (A_k - A_l)^{-1}.$$

This involves working to a large number of significant figures, as $\sum_{l=m}^{m+n-1} A_l B_l^{m,m+n-1} \equiv 0$ but $|A_l B_l^{m,m+n}| > 10^5$ for several values at $m = 2$, $n = 30$; thus, owing to the cancellations arising from alternating signs, the accuracy of $M(2, 30, t)$ is about five decimal figures less than the accuracy to which the coefficients of the series $A_l B_l^{2,31}$ must be determined. For example, at $m = 2$, $n = 10$, one obtains the values shown in table 1.

As a computer program is required to operate at a reasonable speed, the program used here was designed to give only twelve useful decimal digits, and it was found necessary to restrict the calculations to values of m and n which did not need more figures. Thus, $M(m, n, t)$ has been calculated for $m = 2$, $n = 10$, 20 and 30; $m = 3$, $n = 9$, 14 and 19; and $m = 4$, $n = 8$ and 13. The distributions for $m = 2$, $n = 10$, 20 and 30 are plotted in fig. 4. These curves should be com-

TABLE 1. Coefficients of series for $M(2, 10, t)$ and the largest coefficients in $M(2, 30, t)$.

l		$A_l B_l^{2,11}$
2	+	4.102635
3	−	22.515130
4	+	58.616401
5	−	94.075034
6	+	101.370859
7	−	75.377139
8	+	38.424268
9	−	12.884861
10	+	2.569506
11	−	0.231505

$\sum_{l=2}^{11} A_l B_l^{2,11} = 0.000000.$

The number $A_{13}B_{13}^{2,31} = -153319.681013$,
and $A_{14}B_{14}^{2,31} = +151952.818327$
etc.

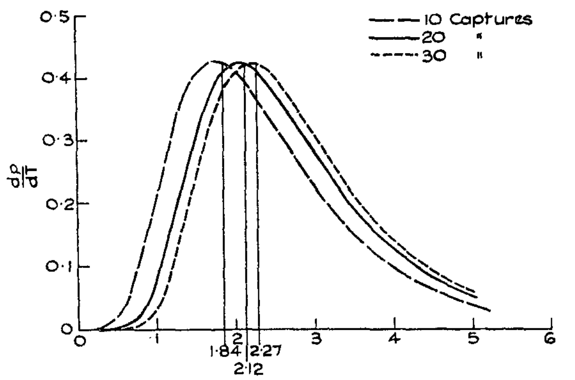

FIG. 4. These show distributions equivalent to those depicted in fig. 3 when velocities of drops obey Stokes' law. Times for same number of captures under previous theories are indicated just below abscissa.

pared with those of fig. 3, which show the corresponding distributions on the assumption about the A_i's taken for the previous section. This was that the rate of growth is proportional to the volume of the collecting drop.

The time measure is $T = (3/4)(E\omega u/a\rho)t$. When $a = 10\ \mu$, $E = 1$, $\omega = 1$ g/m³, and $\rho = 1$ g/cm³, we have $t \approx 1000\ T$ sec.

It can be seen from the figure that, as the number of collisions increases, the greater part of the curve appears to advance towards longer times without much change of shape. This shows clearly that the distribution is virtually settled in the first twenty collisions and, with the exception of very small t, the curve for many more collisions may be obtained simply by adding the average time taken for the 21st to the nth collisions to the statistically distributed times for the first 20 collisions. Alternatively, a specific time may be allocated for growth, and the time remaining after 20 collisions employed in the continuous-growth equation to give a drop-size distribution. This has been done in fig. 5, which shows the drop-size distribution for drops initially 12.6 μ in radius after 4000 sec in a cloud containing 1 g/m³ of 10-μ radius droplets. The size which would have been obtained on the continuous-growth theory is indicated at 0.53 mm. As can be seen, substantial numbers of drops are greater than this figure; in fact, virtually all the drops likely to be recorded as rain are bigger than this. Another point worth noting is that there are quite a number of drops greater than 3 mm in radius. This feature will have a profound effect on chain-reaction theories, where the breakup of such drops is considered.

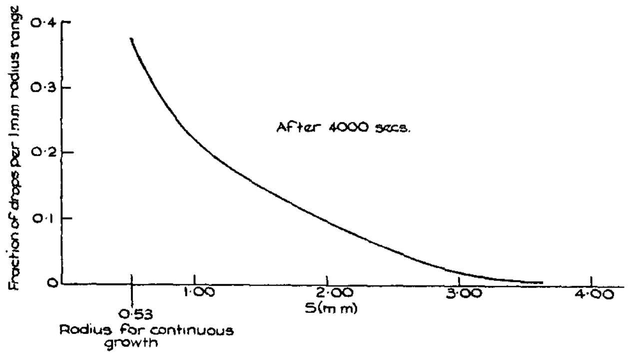

FIG. 5. Graph shows number of drops having radius between S and $S + \delta S$ after 12.6-μ radius drops have grown for 4000 sec in cloud of 10-μ radius drops at concentration of 1 g/m³.

Fig. 6 shows curves similar to those of fig. 4 for larger initial drop sizes; $m = 3$, $n = 9$, 14 and 19; and for $m = 4$, $n = 8$ and 13. As can clearly be seen, the same general behavior occurs here as for $m = 2$ with increasing n. However, the spread in time becomes much less as m increases, in agreement with the analytical case expressed in (8).

5. Contribution of the few, fast growing drops

The growth up to this point has been considered from the point of view of tracing the growth history of drops sufficient in number to provide the concentration observed in rain. There is another approach, however, which appears to be much more significant. Let us now consider the history of drops similar in abundance to those droplets in which they grow. How long does it take sufficient of these to reach raindrop size?

The concentration of drops observed in rain is of the order of 100 drops/m³ whereas the number of drops in a cloud is of the order of 10^8 m^{-3}. Therefore, we are interested in the growth of the 10^{-6} most "fortunate" of the cloud drops.

This requires a detailed investigation of the left-hand tails of the curves shown in figs. 4 and 6. We could examine these by developing (7) to many more significant figures than used in the previous section, but the arithmetic involved is prohibitive. An alter-

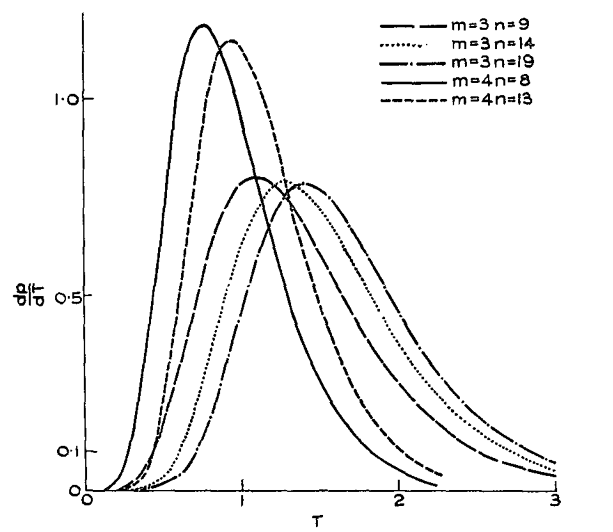

FIG. 6. Curves show $M(m, n, T)$ in Stokes' law region for values of m and n supplementing those in fig. 4. Narrowing of distribution with increasing m is easily seen, and rapid approach to limiting form for $m + n$ about 22 is clearly visible.

native approach lies in expressing the exponentials involved as power series and calculating the coefficients. It so happens that this method gives greatest accuracy near the origin where the direct calculation was inaccurate, but, because of cancellations, requires prohibitive arithmetic to obtain useful accuracy at large t where the former method gives

$$M(m, n, t) \approx A_m B_m{}^{m+n-1} e^{-A_l t}$$

to the accuracy to which $B_m{}^{m,m+n-1}$ and A_m were determined. Therefore,

$$\begin{aligned} M(m, n, t) &= \sum_{l=m}^{m+n-1} A_l B_l{}^{m,m+n-1} e^{-A_l t} \\ &= \sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l\left(1 - A_l t + \frac{1}{2!}(A_l t)^2 - \frac{1}{3!}(A_l t)^3 \cdots + \frac{1}{s!}(-A_l t)^s \cdots\right) \\ &= \sum_{s=0}^{\infty} \frac{(-t)^s}{s!}\left[\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l{}^{s+1}\right]. \end{aligned}$$

Further, from the way $B_l{}^{m,m+n-1}$ is defined in the appendix in (10),

$$\prod_{l=m}^{m+n-1} [1 - (\alpha/A_l)]^{-1} = \sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1}[1 - (\alpha/A_l)]^{-1}.$$

Putting $z = \alpha^{-1}$, one has

$$\begin{aligned} \prod_{l=m}^{m+n-1} A_l z(A_l z - 1)^{-1} &= \sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l z(A_l z - 1)^{-1} \\ &= -z \sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} \\ &\quad \times A_l[1 + A_l z + (A_l z)^2 \cdots + (A_l z)^s \cdots]. \end{aligned}$$

Also,

$$\prod_{l=m}^{m+n-1} A_l z(A_l z - 1)^{-1} = (-z)^n \prod_{l=m}^{m+n-1} A_l \prod_{l=m}^{m+n-1} (1 - A_l z)^{-1}.$$

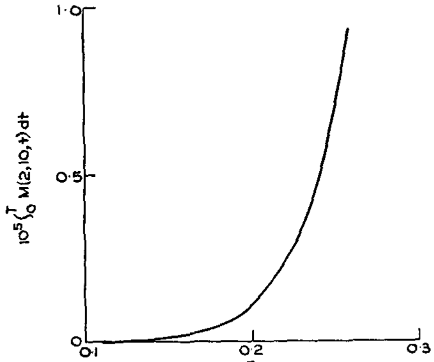

FIG. 7. Fraction of 12.6-μ radius drops greater than or equal to 22.9-μ radius after time T.

Therefore,

$$\begin{aligned} \sum_{s=0}^{\infty} z^s &\left(\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l{}^{s+1}\right) \\ &= (-z)^{n-1} \prod_{l=m}^{m+n-1} A_l \prod_{l=m}^{m+n-1} [1 + A_l z + (A_l z)^2 + \cdots]. \end{aligned}$$

Hence,

$$\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} (A_l)^{s+1} = 0, \qquad \text{for } 0 \leqslant s \leqslant n-2, \tag{9}$$

$$\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} (A_l)^n = (-1)^{n-1} \prod_{l=m}^{m+n-1} A_l,$$

$$\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} (A_l)^{n+1} = (-1)^{n-1} \prod_{l=m}^{m+n-1} A_l \sum_{l=m}^{m+n-1} A_l,$$

$$\begin{aligned} \sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} (A_l)^{n+2} &= (-1)^{n-1}\left(\prod_{l=m}^{m+n-1} A_l\right) \\ &\quad \times \left(\sum_{l=m}^{m+n-1} A_l{}^2 + \sum_{l=m}^{m+n-1} A_l \sum_{z>l}^{m+n-1} A_z\right), \end{aligned}$$

etc. Thus,

$$M(m, n, t) = \sum_{s=n-1}^{\infty} \frac{(-t)^s}{s!}\left[\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l{}^{s+1}\right],$$

the first $n - 1$ terms of the summation being zero. Then, for *very* small t, $M(m, n, t) \propto t^{n-1}$.

The arithmetic procedure is to develop the coefficients

$$\sum_{l=m}^{m+n-1} B_l{}^{m,m+n-1} A_l{}^{n+g}, \qquad g = 0, 1, 2, \cdots,$$

and check the first two for accuracy by using

$$\left(\prod_{l=m}^{m+n-1} A_l\right) \Big/ (n-1)!$$

and

$$\left(\prod_{l=m}^{m+n-1} A_l \sum_{l=m}^{m+n-1} A_l\right) \Big/ n!.$$

This has been done for $m = 2$, $n = 10$, and the result integrated with respect to t. Fig. 7 shows $\int_0^t M(m, n, t)\, dt$ for $m = 2$, $n = 10$.

Now, let us consider a cloud model in which 90 per cent of the water is in 10-μ radius drops and 10 per cent in drops of twice this volume. With a water content of 1 g/m^3, there will be (21.49×10^7) 10-μ drops/m^3, and (1.19×10^7) 12.6-μ drops/m^3.

It is desired to determine the time taken for 100 drops/m^3 to experience their first 10 coalescences. Now, 100 drops/m^3 represent $100/(1.19 \times 10^7) = 8.40 \times 10^{-6}$ of the total number of the larger drops.

Hence, referring to fig. 7, we see that at $m = 2$, $n = 10$, this occurs for $T = 0.257$; when $\omega = 0.9$ g/m^3 and $E = 1$, this is $t = (4a\rho/3E\omega u)T = 5.11$ min.

The time under *continuous growth* for 100 12.6-μ drops/m^3 to undergo this number of collisions, when the total water content and E are the same, is $T = 1.84$, that is, $t = 33.0$ min.

Thus, in a cloud of drops of two size groups, one twice the volume of the other and containing 10 per cent of the water, sufficient of the double drops for the formation of rain undergo 10 collisions in 0.257 time units as compared with 1.84 time units if the same number of double drops grow according to the previous theory. Typically, these times are 5 and 33 min, respectively, giving drops of 23-μ radius.

Ludlam (1951) has reviewed the various factors involved in the formation of rain by coalescence and has concluded that drops of radii greater than 20 μ can readily continue growth to raindrop size, whereas smaller drops may not. He further suggests that such drops may only become available in the form of drops of sea spray in maritime air masses. As can be seen from the preceding arguments, drops of sufficient size are formed within 5 min of the cloud drops' reaching 10-μ radius; thus, many clouds will have a favorable chance of raining without the action of processes other than gravitational coalescence.

Another point made by Ludlam is that very large drops, in far fewer numbers than the raindrops, can significantly alter circumstances by leading to drop rupture and so to chain-reaction processes. The statistical process will produce such drops, and their numbers may easily be sufficient to be of importance.

It is to be noted that in this discussion the collection efficiency has been kept independent of drop size, whereas, almost certainly, this is not so in the region considered. All the indications are that, in this region, collection efficiency increases with drop size from very small values to nearly unity. Langmuir has, in fact, given figures for this function; but because of the approximations in his calculations when the collecting drop nears cloud-droplet size, there is still considerable uncertainty as to their correct values. For this reason, the collection efficiency has been kept at unity. Qualitatively, however, the effect of a collection efficiency increasing with size may be seen quite easily.

For the case in which the probability of a drop experiencing a collision in time δt following selection is $\delta P \propto \delta t$, one gets the Poisson distribution for the number of collisions experienced in a specific time. Hence, $\sigma/\mu = \mu^{-\frac{1}{2}} \rightarrow 0$ as μ increases for large t.

When $\delta P \propto v\,\delta t$,

$$\frac{\sigma}{\mu} = \frac{1}{(m)^{\frac{1}{2}}}\left(1 - \frac{m}{\mu}\right)^{\frac{1}{2}} \rightarrow \frac{1}{(m)^{\frac{1}{2}}} = 0.71,$$

for $m = 2$ and large t.

When δP increases with V at a rate of $(V/v)^{2/3}[(V/v)^{2/3} - 1]$, $\sigma/\mu > 1$, as is seen in fig. 5.

Hence, the inclusion of a collection efficiency increasing with drop size will result in an even greater spread, and the differences brought to light here between continuous and discontinuous growth will become yet greater when the correct values of E are included.

One other point worth considering is the effect of replacing the cloud model used here with a model involving a continuous drop-size distribution. This also will result in an even greater chance for the few drops to get away to an advantageous start. The reasons are that, first, more than 90 per cent of the drops in the model used here can never experience collisions (other than being absorbed), as they are the same size; and, secondly, the large numbers of drops having experienced but few collisions will provide larger and larger increments for the few bigger drops to collect, so giving more opportunities for rapid growth. It is hoped to analyze this situation quantitatively in the near future, to make a comparison with the salt-nuclei theory.

6. Conclusions

The previous theory of rain formation by coalescence has been extended to take account of the random fluctuations in the time intervals between successive coalescences of drops of the same size. It is concluded that:

1. Earlier treatments have greatly overestimated the growth times required.
2. In a cloud consisting initially of identical droplets together with some of double the droplet volume, a proportion of these larger droplets will grow more rapidly than indicated by the "continuous growth" theory, and spread in size during the process to produce a spectrum covering all the drop sizes found in rain.
3. Some of the larger droplets can grow to a size from which further growth to raindrop proportions is likely in but a small fraction of the times previously indicated. It may be unnecessary, therefore, to invoke any other mechanism than free-fall coalescence to account for the formation of rain.
4. A few drops are likely to reach rupture size in reasonable times. These may be important in chain-reaction processes.

Acknowledgments.—The writer wishes to express his thanks to Mr. T. Pearcey for making time available on the Radiophysics Computer, and his appreciation of the assistance rendered by Mr. B. J. McHugh who programmed the problem for the computer and whose help resulted in the solution of the differential equation. Dr. J. A. Roberts was also of assistance in suggesting a general approach to the solution of the differential equation.

APPENDIX

From (1)

$$\delta P_j = A_j\,\delta t,$$

where δP_j is the probability that a drop of volume $V = jv$ will collect a cloud droplet in the time δt following selection.

Let us consider y such intervals of time; then the probability that a drop will *not* have had a collision at the end of time $t = y\,\delta t$ is

$$P_j(t) = \lim_{\delta t \to 0} (1 - A_j\,\delta t)^y$$
$$= \lim \left[(1 - A_j\,\delta t)^{1/A_j\,\delta t}\right]^{A_j y\,\delta t}$$
$$= \left[\lim_{\delta t \to 0} (1 - A_j\,\delta t)^{1/A_j\,\delta t}\right]^{A_j t} = e^{-A_j t}.$$

Therefore,

$$dP_j(t)/dt = -A_j\,e^{-A_j t},$$

and this equals $-M(j, 1, t)$ in the notation introduced earlier in the body of the article; therefore,

$$M(j, 1, t) = A_j\,e^{-A_j t},$$

where $M(j, 1, t)\,\delta t$ is the probability of a drop of volume jv collecting a cloud droplet in a time between t and $t + \delta t$ after selection.

$M(m, n, t)\,\delta t$ must now be found; that is, the probability of a drop of volume mv at $t = 0$ having its nth collision in a time between t and $t + \delta t$.

This time t will be the sum of the times taken for each separate collision.

If t_l is the time for one collision, when the volume is lv,

$$t = \sum_{l=m}^{m+n-1} t_l.$$

The probability distribution of t may most easily be determined by use of moment-generating functions (Hoel, 1947). The moment-generating function, $M(\alpha)$, for the distribution of t is defined so that the rth moment of t about the origin is the coefficient of $\alpha^r/r!$ in the power series expansion of $M(\alpha)$.

Since the A_j's are independent, we may use the statistical theorem which states

$$M(\alpha) = \prod_{l=m}^{m+n-1} M_l(\alpha),$$

where $M_l(\alpha)$ is the moment-generating function for the distribution of t_l.

Now, the moment-generating function $M_l(\alpha)$ is the moment-generating function of the distribution $M(j, 1, t)$, which is

$$M_l(\alpha) = \int_0^\infty e^{\alpha t}\,M(j, 1, t)\,dt,$$

where α must be kept small enough to ensure convergence. Thus,

$$M_l(\alpha) = \int_0^\infty e^{\alpha t}\,A_l\,e^{-A_l t}\,dt$$
$$= A_l \int_0^\infty e^{(\alpha - A_l)t}\,dt$$
$$= \frac{A_l}{\alpha - A_l}\int_0^\infty e^{(\alpha - A_l)t}\,d(\alpha - A_l)t,$$

where α must be kept $< A_l$.

Now,

$$M_l(\alpha) = \frac{A_l}{\alpha - A_l}\left[e^{-(A_l - \alpha)t}\right]_0^\infty$$
$$= -A_l/(\alpha - A_l) = \left[1 - (\alpha/A_l)\right]^{-1}.$$

Therefore,

$$M(\alpha) = \prod_{l=m}^{m+n-1} \left[1 - (\alpha/A_l)\right]^{-1}$$
$$= \sum_{l=m}^{m+n-1} B_l^{m,m+n-1}\left[(1 - (\alpha/A_l)\right]^{-1}. \qquad (10)$$

Using the theory of partial fractions, we have

$$B_l^{m,m+n-1} = \prod_{\substack{k=m \\ k \neq l}}^{m+n-1} \left[1 - (A_l/A_k)\right]^{-1},$$

which was given earlier as (4).

By using the equation defining $M(\alpha)$, one may show that a distribution with the moment-generating function $M(\alpha)$ is

$$\sum_{l=m}^{m+n-1} A_l\,B_l^{m,m+n-1}\,e^{-A_l t}.$$

Thus,

$$M(m, n, t) = \sum_{l=m}^{m+n-1} A_l\,B_l^{m,m+n-1}\,e^{-A_l t},$$

which was arrived at by a different method in (7).

REFERENCES

Battan, L. J., 1953: Observations on the formation and spread of precipitation in convective clouds. *J. Meteor.*, **10**, 311–324.

Bowen, E. G., 1950: The formation of rain by coalescence. *Austral. J. sci. Res.*, A, **3**, 193–213.

Cochet, M. R., 1951: Evolution d'une gouttelette d'eau chargée dans un nuage ou un brouillard à température positive. *Comptes Rendus*, **233**, 190–192.

East, T. W. R., and J. S. Marshall, 1954: Turbulence in clouds as a factor in precipitation. *Quart. J. r. meteor. Soc.*, **80**, 26–47.

Hoel, P. G., 1947: *Introduction to mathematical statistics.* New York, J. Wiley and Sons, pp. 26 and 63.

Langmuir, I., 1948: The production of rain by a chain reaction in cumulus clouds at temperatures above freezing. *J. Meteor.*, **5**, 175–192.

Ludlam, F. H., 1951: The production of showers by the coalescence of cloud droplets. *Quart. J. r. meteor. Soc.*, **77**, 402–417.

Murgatroyd, R. J., 1954: Meteorological Office discussions: Investigations of cumuliform cloud. *Meteor. Mag.*, **83**, 208–215.

Telford, J. W., N. S. Thorndike, and E. G. Bowen, 1955: The coalescence between small water droplets. *Quart. J. r. meteor. Soc.*, **81**, 241–250.

Woodcock, A. H., 1952: Atmospheric salt particles and raindrops. *J. Meteor.*, **9**, 200–212.