# Equations of motion for an inviscid fluid

**The equation of motion for a fluid follows from Newton´s second law, i.e.,**

$$\boxed{\text{mass} \times \text{acceleration} = \text{force}}$$

**If we apply the equation to a unit volume of fluid:**

**(i) the mass of the element is $\rho$ kg m$^{-3}$ ;**

**(ii) the acceleration must be that following the fluid element to take account both of the change in velocity with time at a fixed point and of the change with position of the velocity field at a fixed time,**

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} + w\frac{\partial \mathbf{u}}{\partial z}$$

**(iii) the total force acting on the element (neglecting viscosity or fluid friction) comprises the contact force acting across the surface of the element -p per unit volume, and any body forces F, acting throughout the fluid including especially the gravitational weight per unit volume, -gk.**

**For an inviscid fluid, the contact force is a pressure gradient force arising from the difference in pressure across the element.**

**The resulting equation of motion/momentum equation for inviscid fluid flow,**

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{F}, \text{ per unit volume,}$$

**or**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \text{ per unit mass,}$$

**is known as Euler´s equation.**

In rectangular Cartesian coordinates the component equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z,$$

where $\mathbf{u} = (u, v, w)$ and $\mathbf{F} = (X, Y, Z)$ is the external force per unit mass (or body force).

Three partial differential equations in the four dependent variables u, v, w, p and four independent variables x, y, z, t.

The continuity equation gives the fourth equation: $\nabla \cdot \mathbf{u} = 0$, or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

## Equations of motion for an incompressible viscous fluid

It can be shown that the viscous (frictional) forces in a fluid may be expressed as

$$\mu\nabla^2\mathbf{u} = \rho\nu\nabla^2\mathbf{u}$$

where $\mu$ the coefficient of viscosity and $\nu = \mu/\rho$ the kinematic viscosity provide a measure of the magnitude of the frictional forces in particular fluid.

Note: $\mu$ and $\nu$ are properties of the fluid and are relatively small in air or water and relatively large in glycerine or heavy oil.

In a viscous fluid the equation of motion for unit mass is:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu \nabla^2 \mathbf{u}$$

| local acceleration | advective acceleration | pressure gradient force | body force | viscous force |
|---|---|---|---|---|

It is known as the Navier-Stokes' equation.

We require also the continuity equation,

$$\nabla \cdot \mathbf{u} = 0$$

to close the system of four differential equations in four dependent variables: u, v, w, p.

## The continuity equation

There is no equivalent to the continuity equation in either particle or rigid body mechanics, because in general mass is permanently associated with bodies.

In fluids we must ensure that holes do not appear

$$\nabla \cdot \mathbf{u} = 0$$

in the absence of sources or sinks there can be no net flow either into or out of any closed surface.

This equation is not satisfied by a compressible fluid (e.g. a bicycle pump).

- **The Navier-Stokes equation plus continuity equation are extremely important but extremely difficult to solve.**
- **With possible further force terms on the right, they represent:**
  - **the behaviour of gaseous stars,**
  - **the flow of oceans and atmosphere,**
  - **the motion of the earth´s mantle,**
  - **blood flow,**
  - **air flow in the lungs,**
  - **many processes of chemistry and chemical engineering,**
  - **the flow of water in rivers and in the permeable earth,**
  - **the aerodynamics of aeroplanes, and so forth.…**
- **There are probably no more than a dozen or so analytic solutions known for very simple geometries!**

**The difficulty of solution arises from:**

**(i) the non-linear term** $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$

**As a result of this if $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of the equation, $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ (where $c_1$ and $c_2$ are constants) is in general not a solution, so that we lose one of our main methods of solution;**

**(ii) the fact that viscous term is small relative to other terms except close to boundaries, yet it contains the highest order derivatives.**

$$\left(\partial^2\mathbf{u}/\partial x^2,\ \partial^2\mathbf{u}/\partial y^2,\ \partial^2\mathbf{u}/\partial z^2\right)$$

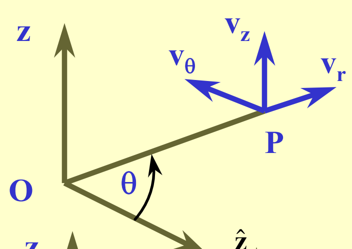
**and hence determines the number of spatial boundary conditions that must be imposed to determine a solution.**

The **Navier-Stokes** equation is too difficult for us to handle at present and we shall concentrate on **Euler´s equation** from which we can learn much about fluid flow.

**Euler´s equation is still non-linear**, but there are clever methods to bypass this difficulty.

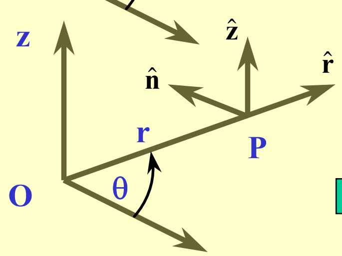
## Equations of motion in cylindrical polar coordinates

$$\mathbf{v} = \left(v_r\hat{\mathbf{r}} + v_\theta\hat{\mathbf{n}} + v_z\hat{\mathbf{z}}\right)$$




More complicated than Cartesian coordinates as $v_r$ , $v_\theta$ change in direction with P.

OP rotates about Oz with angular velocity $v_\theta/r$.




$\hat{\mathbf{z}}$ is fixed in direction, but $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$ rotate in the plane $z = 0$ as **P** moves.

$$\frac{d\hat{\mathbf{r}}}{dt} = \hat{\mathbf{n}}\frac{d\theta}{dt}, \qquad \frac{d\hat{\mathbf{n}}}{dt} == -\hat{\mathbf{r}}\frac{d\theta}{dt}$$

**Now** $\dfrac{d\theta}{dt} = \dfrac{v_\theta}{r}$ **and** $\mathbf{v} = \left(v_r\hat{\mathbf{r}} + v_\theta\hat{\mathbf{n}} + v_z\hat{\mathbf{z}}\right)$

$$\dot{\mathbf{v}} = \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{\mathbf{r}}} + \dot{v}_\theta\hat{\mathbf{n}} + v_\theta\dot{\hat{\mathbf{n}}} + \dot{v}_z\hat{\mathbf{z}} =$$

$$\left(\dot{v}_r - v_\theta^2 / r\right)\hat{\mathbf{r}} + \left(\dot{v}_\theta + v_r v_\theta / r\right)\hat{\mathbf{n}} + \dot{v}_z\hat{\mathbf{z}}$$

**Since** d/dt **must be interpreted here as** D/Dt**, the acceleration is**

$$\frac{D\mathbf{u}}{Dt} = \left[\frac{Dv_r}{Dt} - \frac{{v_\theta}^2}{r}, \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}, \frac{Dv_z}{Dt}\right]$$

**Write** (u,v,w) **in place of** $(v_r, v_\theta, v_z)$

## Euler´s equations in cylindrical polar coordinates

**continuity**

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r ,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + F_\theta ,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z .$$

## Dynamic pressure (or perturbation pressure)

**Euler equation for an incompressible fluid is:**

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

**In a state of rest, $\mathbf{u} = \mathbf{0}$ and** $p = p_0$ ⇒ $$0 = -\frac{1}{\rho}\nabla p_0 + \mathbf{g}$$

**This is the hydrostatic equation and** $p_0$ **the hydrostatic pressure**

$$\nabla p_0 = \rho\,\mathbf{g} \quad \textbf{or} \quad \frac{\partial p_0}{\partial x} = 0, \frac{\partial p_0}{\partial y} = 0, \frac{\partial p_0}{\partial z} = -\rho g$$

**Subtraction gives** $$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla(p - p_0) = -\frac{1}{\rho}\nabla p_d$$

$p_d = p - p_0$ **is the dynamic pressure (or perturbation pressure).**

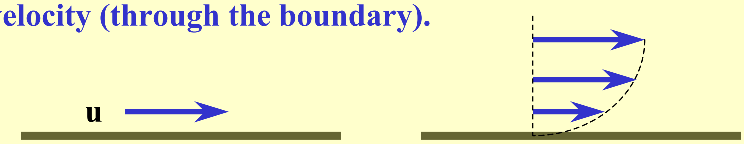
**The dynamic pressure is the excess of total pressure over hydrostatic pressure, and is the only part of the pressure field associated with motion.**

**We usually omit the suffix "$_d$" since it is clear that if g is included we are using total pressure, and if no g appears we are using the dynamic pressure**

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\frac{1}{\rho}\nabla p$$

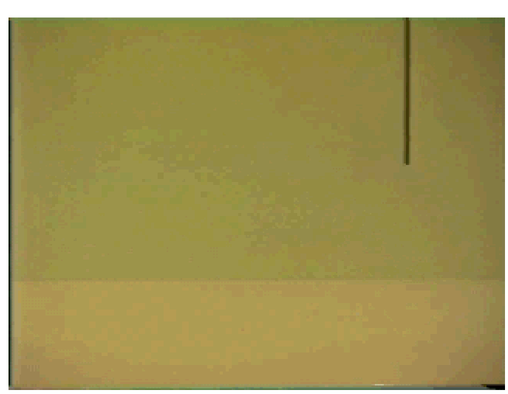
## Boundary conditions for fluid flow

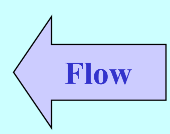
- (i) **Solid boundaries:** there can be no normal component of velocity (through the boundary).




- If friction is neglected there may be free slip along the boundary.
- Friction has the effect of slowing down fluid near the boundary and it is observed experimentally that **there is no relative motion at the boundary**, either normal or tangential to the boundary.
- In fluids with low viscosity this tangential slowing down occurs in a thin **boundary layer**

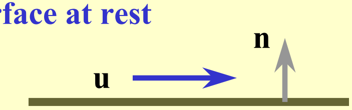
## Illustrating the boundary layer





- In a number of important applications the boundary layer is so thin that it can be neglected and we can say approximately that the fluid slips at the surface.
- In many other cases the entire boundary layer separates from the boundary and the inviscid model is a very poor.
- In summary, in an **inviscid flow** (also called an **ideal fluid**) the fluid velocity must be tangential at a rigid body, and

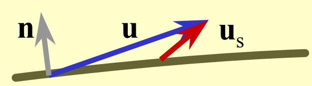
for a surface at rest



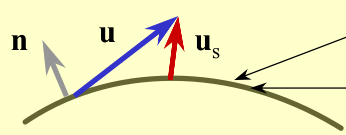

$\mathbf{n}\cdot\mathbf{u} = 0$

for a surface with velocity $\mathbf{u}_s$

$\mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_s) = 0$



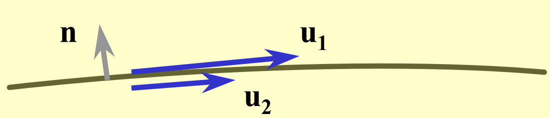

- (i) **Free boundaries**: at an interface between two fluids (of which one might be water and one air) the pressure must be continuous and the component of velocity normal to the interface must be continuous.




p continuous

$\mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_s) = 0$

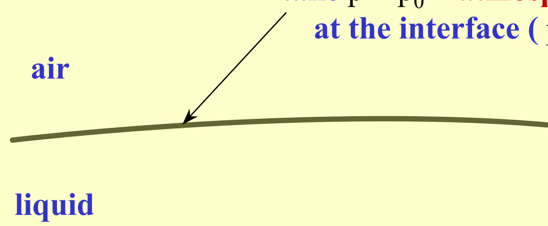
- If the pressure were not continuous there would be an infinite force on an infinitesimally small element of fluid causing unbounded acceleration.
- If viscosity is neglected the two fluids may slip over each other.




$\mathbf{u}_s = \mathbf{0}$

## Air liquid interface

take $p = p_0$ = atmospheric pressure at the interface ( $p_0$ a constant)



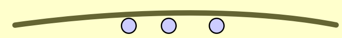

- If surface tension is important there may be a pressure difference across the curved interface.
- This happens in the case of capillary waves.

## Surface tension

A heavier-than-water, double-edged steel razor blade can float on water. Without surface tension it would sink because its weight is greater than the upward force on it due to the water it displaces. A slightly heavier single-edged blade sinks.

## An alternative free surface boundary condition

- The velocity at a boundary of an inviscid fluid must be wholly tangential.



- A fluid particle once at the surface must always remain at the surface.
- Let the surface or boundary have equation $F(x, y, z, t) = 0$.
- If the coordinates of a fluid particle satisfy this equation at one instant, they must satisfy it always.

moving with the fluid at the boundary

$$\frac{DF}{Dt} = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0$$

as F must remain zero for all time for each surface particle.

## Bernoulli´s equation

- For steady inviscid flow under external forces which have a potential $\Omega$ such that $\mathbf{F} = -\nabla\Omega$ the Euler equation is

$$\mathbf{u} \cdot \nabla\, \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Omega$$

For incompressible fluids

$$\mathbf{u} \cdot \nabla\, \mathbf{u} + \frac{1}{\rho}\nabla(p + \rho\Omega) = 0$$

We may regard $p + \rho\Omega$ as a more general dynamic pressure.

For the particular case of gravitation potential, $\Omega = gz$, and

$$\mathbf{F} = -\nabla\Omega = -(0, 0, g) = -g\mathbf{k}$$

Note that

$$\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} = u(\mathbf{u}\cdot\nabla)u + v(\mathbf{u}\cdot\nabla)v + w(\mathbf{u}\cdot\nabla)w$$
$$= \mathbf{u}\cdot\nabla\tfrac{1}{2}\left(u^2 + v^2 + w^2\right)$$
$$= (\mathbf{u}\cdot\nabla)\tfrac{1}{2}\mathbf{u}^2,$$

because $\mathbf{u}\cdot\nabla$ is a scalar differential operator.

$$\mathbf{u}\cdot\left[\mathbf{u}\cdot\nabla\mathbf{u} + \nabla(p/\rho + \Omega)\right] = \mathbf{u}\cdot\nabla\left[\tfrac{1}{2}\mathbf{u}^2 + p/\rho + \Omega\right] = 0$$

$\left(\tfrac{1}{2}\mathbf{u}^2 + p/\rho + \Omega\right)$ is constant along each streamline

Note $\mathbf{u}$ is proportional to the rate of change in the direction $\mathbf{u}$ of streamlines.

## Bernoulli´s Theorem

For steady, incompressible, inviscid flow

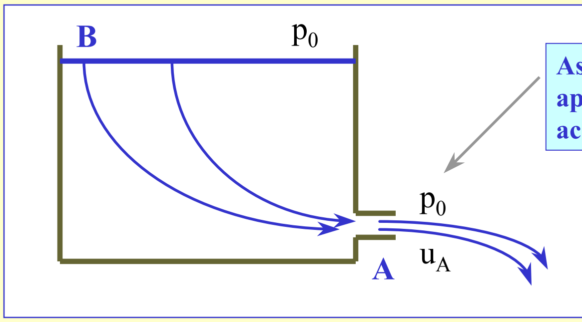
$$\left(\tfrac{1}{2}\mathbf{u}^2 + p/\rho + \Omega\right)$$

is a constant on a streamline, although the constant will generally be different on each different streamline.

Applications

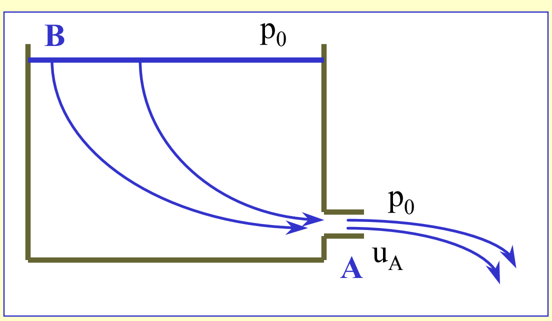
## Applications of Bernoulli´s equation

1. Draining a reservoir through a small hole




Assume $u_A$ and $p_A$ to be approximately uniform across the jet and $p_A = p_0$.

- Assume the draining opening is much smaller in cross-section than the reservoir.
- The water surface in the tank will fall very slowly and the flow may be regarded as approximately steady.

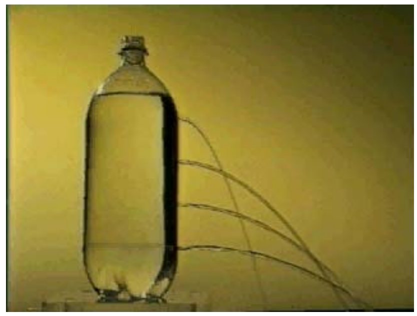



On the streamline AB $\quad \tfrac{1}{2}u_A^{\,2} + \cancel{p_0/\rho} = \tfrac{1}{2}u_B^{\,2} + \cancel{p_0/\rho} + gh$

$u_B << u_A \quad \Rightarrow \quad u_A = \sqrt{2gh}$

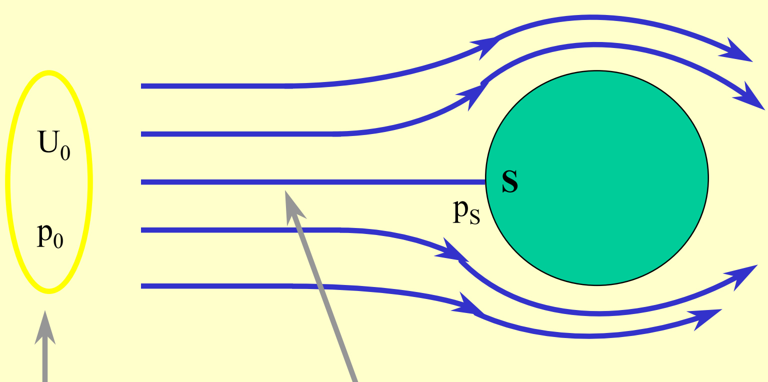
- This result is known as Toricelli´s theorem.
- Note that the outflow speed is that of free fall from B under gravity; this clearly neglects any viscous dissipation of energy.

# Draining flow



**Bernoulli $\Rightarrow$ the velocity is proportional to the square root of the depth.**

## 2. Bluff body in a stream - Pitot tube




**S is a stagnation point where** $u = 0$

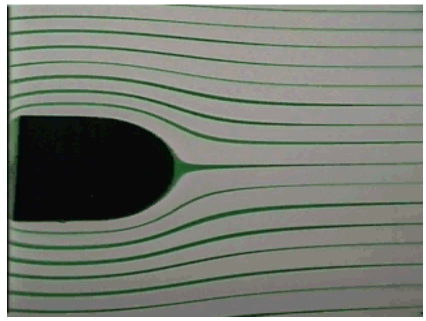
**uniform upstream conditions**

**dividing streamline**

**Bernoulli's theorem along the dividing streamline**

$$\boxed{p_S = p_0 + \tfrac{1}{2}\rho U_0^2}$$

## Flow near a stagnation point



On any body in a flowing fluid is a stagnation point. Some fluid flows "over" the body and some flows "under". The dividing line, the stagnation streamline, terminates at a stagnation point on the body. The flow decelerates as it approaches the stagnation point.

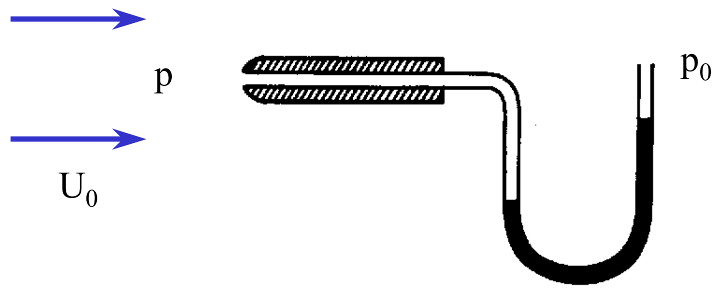
## 3. Pitot tube

The foregoing calculation provides the basis for the Pitot tube in which a pressure measurement is used to obtain the free stream velocity $U_0$.

The pressure $p = p_0 + \frac{1}{2}\rho U_0^2$ is the total or Pitot pressure (also known as the total head) of the free stream

It differs from the static pressure $p_0$ by the dynamic pressure

$$\frac{1}{2}\rho U_0^2$$

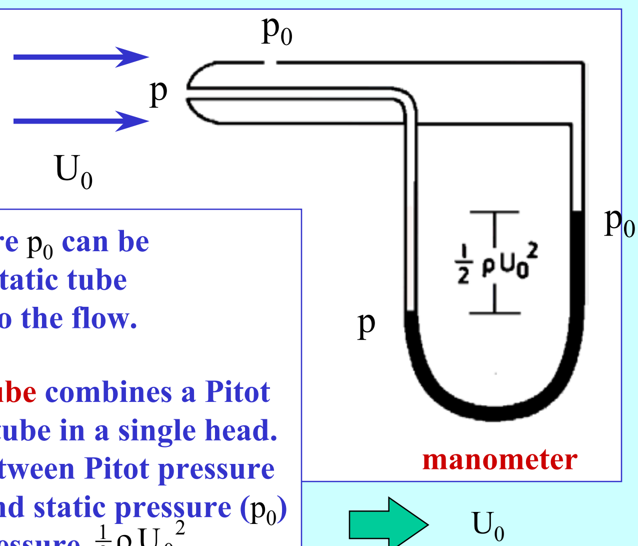



The Pitot tube consists of a tube directed into the stream with a small central hole connected to a manometer for measuring pressure difference $p - p_0$ .

At equilibrium there is no flow through the tube, and hence the left hand pressure on the manometer is the total pressure

$$p = p_0 + \tfrac{1}{2}\rho U_0^{\,2}$$

### 3. Pitot-static tube

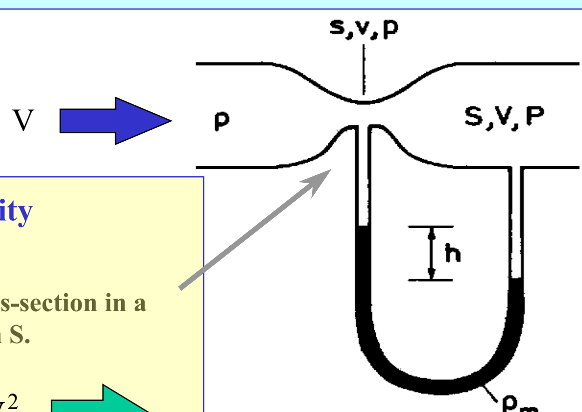



The static pressure $p_0$ can be obtained from a static tube which is normal to the flow.

The Pitot-static tube combines a Pitot tube and a static tube in a single head. The difference between Pitot pressure $(p_0 + \tfrac{1}{2}\rho U_0^{\,2})$ and static pressure $(p_0)$ is the dynamic pressure $\tfrac{1}{2}\rho U_0^{\,2}$ .

The Pitot-static tube can be flown in an aeroplane and used to determine the speed of the aeroplane through the air.

## 4. Venturi tube




**Measures fluid velocity and discharge.**

A restriction of cross-section in a pipe of cross-section S.

$$\frac{p}{\rho}+\frac{1}{2}v^2=\frac{P}{\rho}+\frac{1}{2}V^2$$

$$v^2-V^2=\frac{2}{\rho}(P-p)=\frac{2}{\rho}\rho_m gh=2gh\frac{\rho_m}{\rho}.$$

The discharge $Q = vs = VS$ $\quad\left[\frac{Q}{s}\right]^2-\left[\frac{Q}{S}\right]^2=2gh\frac{\rho_m}{\rho}$

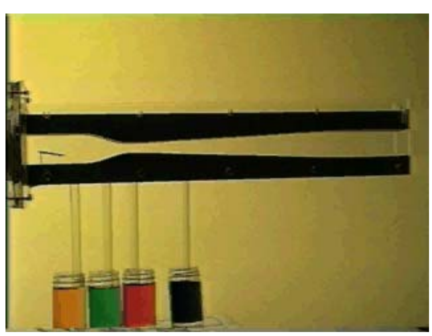
$$\left[\frac{Q}{s}\right]^2 - \left[\frac{Q}{S}\right]^2 = 2gh\frac{\rho_m}{\rho}$$

$$Q = \frac{sS}{\sqrt{S^2 - s^2}}\sqrt{2gh\frac{\rho_m}{\rho}}$$

$$Q = vs = VS$$

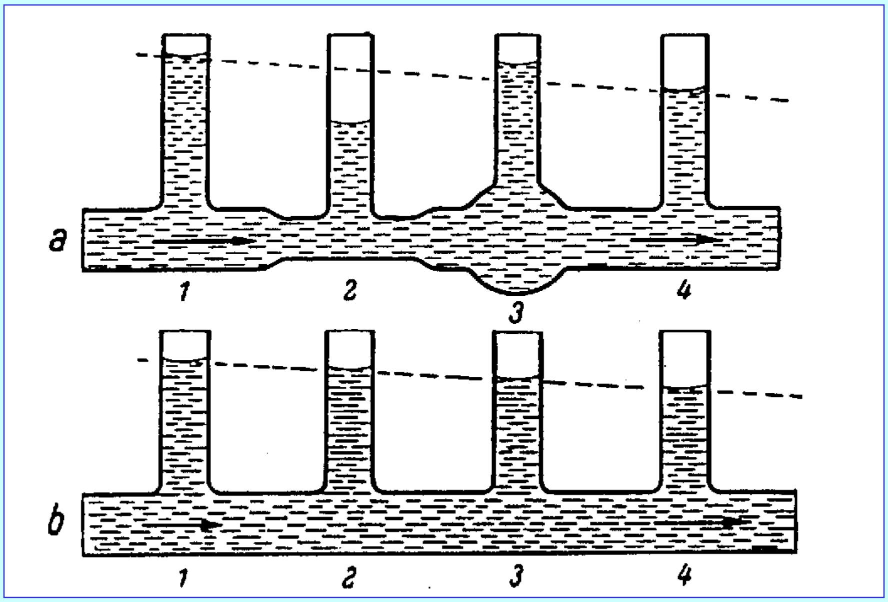
$$V = \frac{Q}{S} = \frac{s}{\sqrt{S^2 - s^2}}\sqrt{2gh\frac{\rho_m}{\rho}}$$

## Venturi meter

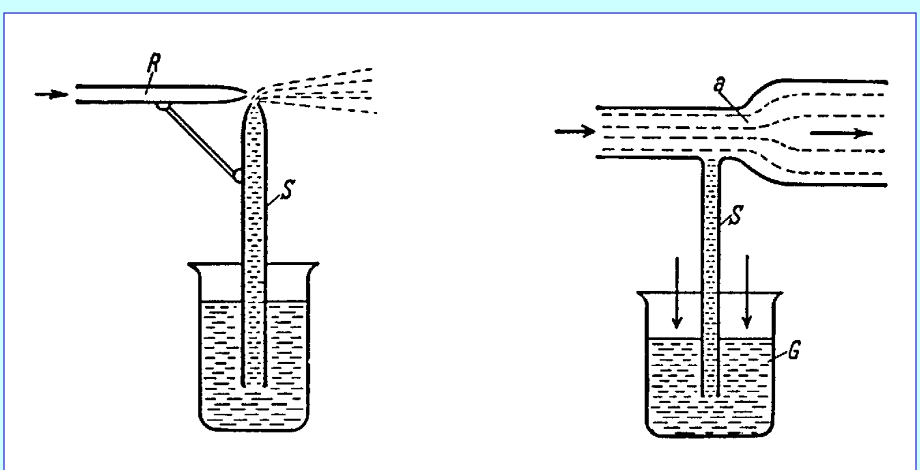


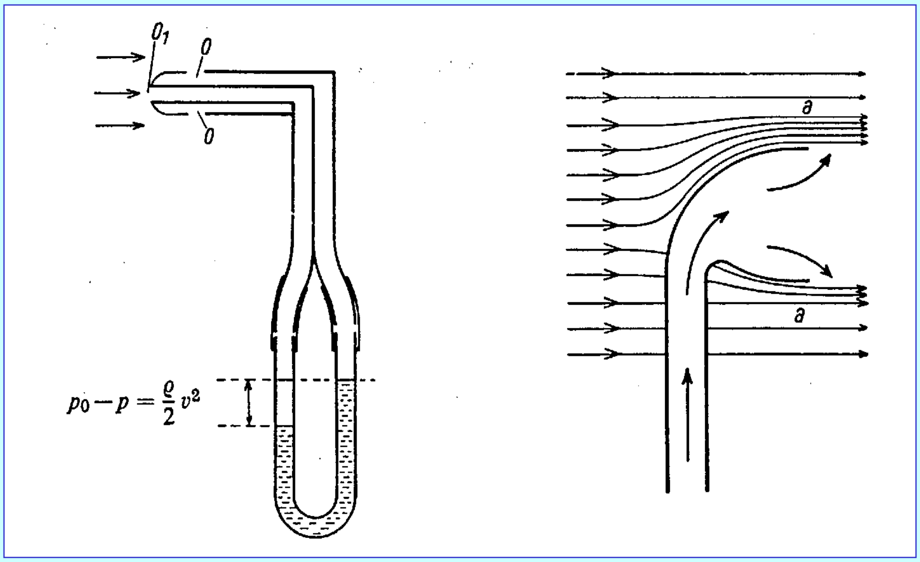
**The same principle can be used in a garden sprayer so that liquid chemicals can be sucked from a bottle and mixed with water in the hose.**

## Other devices




## Other devices






# The End