Exercise 3: Variational methods

Consider the model for the wind on three height levels:

$$u_i^t(t_{k+1}) = u_i^t(t_k) + \alpha u_i^{t2}(t_k), \qquad i = 1,2,3, \qquad \alpha = 0.2 \tag{1}$$

$$\mathbf{u}^t(t_0) = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} \tag{2}$$

1. **True solution.** Calculate the 'true solution' for $\mathbf{u}^t(t_1)$.

   Proof. Substituting into (1) the value at t_0 we get the value at t_1 as $\mathbf{u}^t(t_1) = \begin{pmatrix} -1.2 \\ 2.8 \\ 7.2 \end{pmatrix}$. □

2. **Tangent linear model.** Write the tangent linear model for this example.

   Proof. Model operator $\mathcal{M}_{k-1,k}$ takes the vector $[u_1 \;\; u_2 \;\; u_3]^T$ into the vector given by $\begin{pmatrix} u_1 + \alpha u_1^2 \\ u_2 + \alpha u_2^2 \\ u_3 + \alpha u_3^2 \end{pmatrix}$.

   Linearization of this model gives us a matrix:

$$\mathbf{M}_{k-1,k} = \begin{pmatrix} 1 + 2\alpha u_1(t_{k-1}) & 0 & 0 \\ 0 & 1 + 2\alpha u_2(t_{k-1}) & 0 \\ 0 & 0 & 1 + 2\alpha u_3(t_{k-1}) \end{pmatrix}. \tag{3}$$

   Tangent linear model takes us from time 0 to time k using $\mathbf{M}_{k-1,k}\mathbf{M}_{k-2,k-1}\ldots\mathbf{M}_{0,1}$. □

3. **Single direct observation** Assume we have the observation $u_2^o(t_1) = 2.8$ with observation error covariance $\sigma_o^2 = 1$ and a background (first guess) $\mathbf{u}^b(t_0) = (-3, 0, 3)^T$ with background error covariance $\mathsf{B} = \sigma_b^2 \mathsf{I}$, $\sigma_b = 2$. Compute the 3DVAR analysis, i.e.:

   (a) Determine $\mathbf{H}$.

   (b) Compute the analysis field $\mathbf{u}^a$.

   (c) Compute the root mean square error of the analysis against the true solution $\sqrt{(\mathbf{u}^a - \mathbf{u}^t(t_0))^T(\mathbf{u}^a - \mathbf{u}^t(t_0))}$.

   Proof. Since we have only one observation of the second component of the vector, observation operator is $\mathbf{H} = [0 \;\; 1 \;\; 0]$. In order to compute the analysis with 3DVar we would minimize the cost function

$$J(\mathbf{u}) = \frac{1}{4}[(u_1 + 3)^2 + u_2^2 + (u_3 - 3)^2] + (2.8 - u_2)^2. \tag{4}$$

   The resulting analysis is $\mathbf{u}^a(t_1) = \begin{pmatrix} -3 \\ 2.24 \\ 3 \end{pmatrix}$. RMS error is 1.0284. □

4. **Indirect observation.** Repeat above assuming the observation $\mathbf{u}_{1.5}^o(t_1) = 0.8$ was made halfway between grid points 1 and 2.

Proof. For this example $\mathbf{H} = [1/2 \;\; 1/2 \;\; 0]$, and the cost function changes to

$$J(\mathbf{u}) = \frac{1}{4}[(u_1+3)^2 + u_2^2 + (u_3-3)^2] + (0.8 - 0.5u_1 - 0.5u_2)^2. \tag{5}$$

The resulting analysis is $\mathbf{u}^a(t_1) = \begin{pmatrix} -1.4667 \\ 1.5333 \\ 3 \end{pmatrix}$. RMS error is 1.8892. □

5. **Sherman-Morrison-Woodbary formula** gives an expression for the inverse of the matrix $\mathbf{A} + \mathbf{U}\mathbf{V}^T$, where $\mathbf{A}$ is $n \times n$ matrix and $\mathbf{U}$, $\mathbf{V}$ are $n \times k$:

$$(\mathbf{A} + \mathbf{U}\mathbf{V}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{A}^{-1}. \tag{6}$$

Using this formula show that

$$(\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} = \mathbf{B} - \mathbf{B}\mathbf{H}^T(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^T)^{-1}\mathbf{H}\mathbf{B}. \tag{7}$$

Proof. Let $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{U} = \mathbf{V} = \mathbf{H}^T\mathbf{R}^{-1/2}$ using SMW formula we get

$$(\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} = \mathbf{B} - \mathbf{B}\mathbf{H}^T\mathbf{R}^{-1/2}(\mathbf{I} + \mathbf{R}^{-1/2}\mathbf{H}\mathbf{B}\mathbf{H}^T\mathbf{R}^{-1/2})^{-1}\mathbf{R}^{-1/2}\mathbf{H}\mathbf{B}. \tag{8}$$

and needed result. □