Exercise: Variational methods

Consider the model for the wind on three height levels:

$$u_i^t(t_{k+1}) = u_i^t(t_k) + \alpha {u_i^t}^2(t_k), \qquad i = 1, 2, 3, \qquad \alpha = 0.2 \tag{1}$$

$$\mathbf{u}^t(t_0) = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} \tag{2}$$

1. **True solution.** Calculate the 'true solution' for $\mathbf{u}^t(t_1)$.

2. **Tangent linear model.** Write the tangent linear model for this example.

3. **Single direct observation** Assume we have the observation $u_2^o(t_1) = 2.8$ with observation error covariance $\sigma_o^2 = 1$ and a background (first guess) $\mathbf{u}^b(t_0) = (-3, 0, 3)^T$ with background error covariance $\mathsf{B} = \sigma_b^2 \mathsf{I}$, $\sigma_b = 2$. Compute the 3DVAR analysis, i.e.:

   (a) Determine H.

   (b) Compute the analysis field $\mathbf{u}^a$.

   (c) Compute the root mean square error of the analysis against the true solution $\sqrt{(\mathbf{u}^a - \mathbf{u}^t(t_0))^T(\mathbf{u}^a - \mathbf{u}^t(t_0))}$.

4. **Indirect observation.** Repeat above assuming the observation $\mathbf{u}_{1.5}^o(t_1) = 0.8$ was made halfway between grid points 1 and 2.

5. **Sherman-Morrison-Woodbary formula** gives an expression for the inverse of the matrix $\mathbf{A} + \mathbf{U}\mathbf{V}^T$, where $\mathbf{A}$ is $n \times n$ matrix and $\mathbf{U}$, $\mathbf{V}$ are $n \times k$:

$$(\mathbf{A} + \mathbf{U}\mathbf{V}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^T)\mathbf{A}^{-1}. \tag{3}$$

   Using this formula show that

$$(\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} = \mathbf{B} - \mathbf{B}\mathbf{H}^T(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^T)^{-1}\mathbf{H}\mathbf{B}. \tag{4}$$