Appendix B

Poisson Equation

Poisson's equation is the second-order, elliptic, partial differential equation

$$\frac{\partial^2 h}{\partial^2 x} + \frac{\partial^2 h}{\partial^2 y} = -F(x, y). \tag{B.1}$$

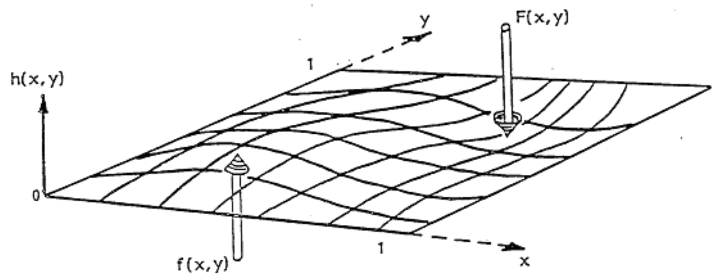

Figure B.1: Equilibrium displacement of a stretched membrane over a square under the force distribution $F(x, y)$.

Consider the solution in the square region $0 \leq x \leq 1$, $0 \leq y \leq 1$, subject to the (Dirichlet) boundary condition $h = 0$ on the boundary of the square. The equation with this boundary condition solves the problem of the equilibrium displacement of a stretched membrane over the square boundary when subjected to a force distribution per unit area proportional to $F(x, y)$ in a direction normal to the (x, y) plane. Of course, the boundary condition specifies zero displacement along the square boundary and the equation itself holds for small displacements of the membrane. The membrane analogy is useful as it allows us to use our intuition on how such a membrane would deform under a given force distribution (e.g., maximum displacement where the force is a maximum) to anticipate the form of solution without having to solve the equation (Fig. B.1).

As a specific example, suppose there is a point force at the centre $(\frac{1}{2}, \frac{1}{2})$ of the unit square represented by delta-functions,

$$F(x, y) = \delta(x - \tfrac{1}{2})\delta(y - \tfrac{1}{2}). \tag{B.2}$$

We expect the solution to be symmetric about the diagonals of the square as shown in Fig. (B.2).

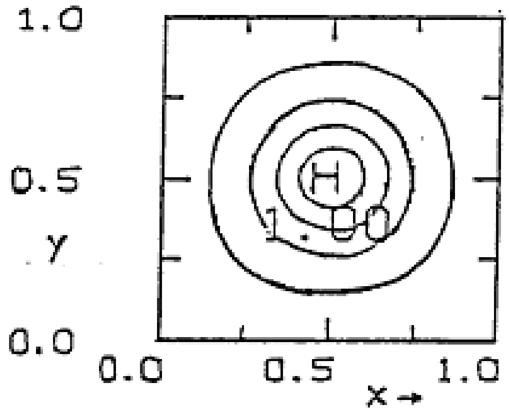

Figure B.2: Isopleths of membrane displacement subject to a point force at the centre point $(\frac{1}{2}, \frac{1}{2})$, giving rise to unit displacement at that point.

In fact, the isopleths of membrane displacement are determined analytically by the Green's function for the centre point, i.e.

$$h(x, y) = \begin{cases} 2\sum_{1}^{\infty} \dfrac{\sinh\,(n\pi x)\ \sinh\,(n\pi/2)\ \sin\,(n\pi y)\ \sin\,(n\pi/2)}{n\pi\ \sinh\ n\pi} & 0 \le x \le \frac{1}{2} \\ 2\sum_{1}^{\infty} \dfrac{\sinh\,(n\pi/2)\ \sinh\,(n\pi(1-x))\ \sin\,(n\pi y)\ \sin\,(n\pi/2)}{n\pi\ \sinh\ n\pi} & 0 \le x \le 1 \end{cases} \tag{B.3}$$

see Friedman, (1956; p. 262, Eq. 12.19), although Fig. B.2 was obtained by solving (B.1) numerically subject to an approximation to (B.2). Note especially that, although the force acts at a point, the response is distributed over the region. Now consider the response of the rectangular membrane $0 \le x \le 3$, $0 \le y \le 1$ due to a point force at the intersection of the diagonals

$(\frac{3}{2}, \frac{1}{2})$. The isopleths of membrane displacement, again normalized so that the maximum diaplacement is unity, are shown in Fig. B.3. Note that in

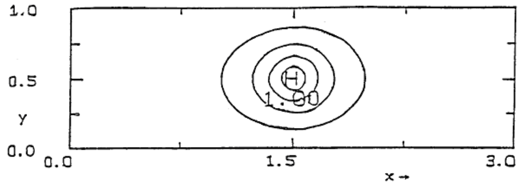

Figure B.3:

this case, the scales of response are set by the smallest rectangle length, in this case $\frac{1}{2}$.

Suppose, that we wish to infer the response of a membrane with nonuniform extension properties, as described for example by the equation

$$N^2 \frac{\partial^2 h}{\partial x^2} + f^2 \frac{\partial^2 h}{\partial z^2} = -F(x, z), \tag{B.4}$$

in the rectangular region $0 \leq x \leq L$, $0 \leq z \leq H$, again with $F(x, z)$ given by a point force proportional to $\delta(x - \frac{1}{2}L)\delta(z - \frac{1}{2}H)$. We consider the case $H << L$ and assume that N and f are constants. The equation may be transformed to one with unit coefficients by dividing both sides by N^2 and making the substitution $z = (f/N)Z$, whereupon

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial Z^2} = -\frac{1}{N^2} \delta \left[x - \tfrac{1}{2}L\right] \; \delta \left[Z - \tfrac{1}{2}L_R\right], \tag{B.5}$$

where $L_R = HN/f$. The equation is valid for the region $0 \leq x \leq L$, $0 \leq Z \leq L_R$. In the case $L > L_R$, as exemplified in Fig. B.3, the response scale is the same, L_R, in both the x and Z directions. The latter corresponds with scale H for z. This last result is important in geophysical applications, for in a strably-stratified rotating fluid characterized by constant Brunt-Väisälä frequency N and constant Coriolis parameter f, we encounter equations of the type (B.4) for the streamfunction $\psi(x, z)$, usually in configurations where the aspect ratio of the flow domain, say H/L, is small. Typically, in the atmosphere, $f/N \sim 10^{-2}$. According to the foregoing results, provided $L > L_R$, then the horizontal length scale of the response is $L_R = HN/f$, which is just the *Rossby radius of deformation*. Since H is typically 10 km, the criterion $L > L_R$ requires that $L > 1000$ km. If $L < L_R$, the horizontal scale of response will be set by L and the vertical scale of response is then Lf/N, sometimes referred to as the *Rossby depth scale*. The former situation, which is usually the case for geophysical flows, is illustrated in Fig. B.4 by numerical solutions of (B.4) in the region where $L = 2000$ km, $H = 10$ km, for four different values of L_R. Again a localized "force" $F(x, y)$ is applied at the point $(\frac{1}{2}L, \frac{1}{2}H)$ and the isopleths of "membrane displacement" are normalized so that the maximum displacement is unity. In all cases, $L > L_R$, but note how the horizontal scale of response decreases as L_R decreases. If other boundary conditions are imposed along all or part of the domain boundary, the foregoing ideas may have to be revised.

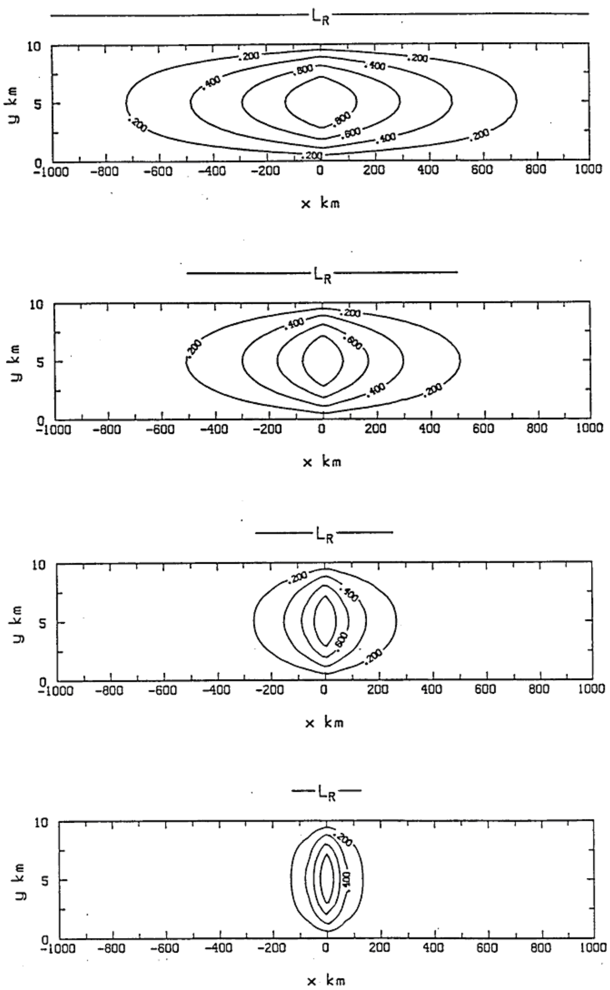

Figure B.4: