TROPICAL CYCLONE RESEARCH REPORT
TCRR **1**: 1–5 (2019)
Meteorological Institute
Ludwig Maximilians University of Munich

Towards understanding the dynamics of spin up in Emanuel's tropical cyclone model

M. T. Montgomery[a1] and Roger K. Smith[b]
[a] *Dept. of Meteorology, Naval Postgraduate School, Monterey, CA, USA*
[b] *Meteorological Institute, Ludwig-Maximilians University of Munich, Munich, Germany*

Abstract:
We seek to understand the mechanism of vortex spin up in Emanuel's 2012 axisymmetric theory for tropical-cyclone intensification in physical coordinates, starting from first principles. It is noted that, while spin up of the maximum tangential wind must occur at low levels, within or at the top of the friction layer, this spin up is unconstrained by a radial momentum equation in this layer. Instead, the spin up is controlled by a parameterization of turbulent mixing in the upper tropospheric outflow layer, which, as is shown, determines indirectly the rate of inward movement of the absolute angular momentum surfaces. Nevertheless, the physics of how upper-tropospheric mixing leads to spin up in the friction layer are unclear and, as discussed, may be irrelevant to spin up in the model.

KEY WORDS Hurricane; tropical cyclone; typhoon; moist entropy; vortex spin up

Date: May 11, 2019; Revised ; Accepted

1 Introduction

In a highly influential paper, Emanuel (1986) presented a closed analytical theory for the structure and intensity of an axisymmetric, steady-state, tropical cyclone. A key feature of the theory was the assumption that surfaces of saturation moist entropy, s^*, and absolute angular momentum, M, are congruent. Both of these surfaces emanate from the boundary layer and flare outwards with height, becoming nearly horizontal in the upper troposphere (his Fig. 1). Inspired by the pioneering tropical cyclone model of Ooyama (1969), the theory incorporates a simple slab-like boundary layer in which the entropy and M surfaces are assumed to be effectively well mixed in the vertical at leading order[1]. Emanuel showed that the maintenance of a tropical cyclone depends exclusively on self-induced latent and sensible heat transfer from the ocean in the form of moist enthalpy fluxes in contrast to ambient conditional instability. An appraisal of the theory was presented by Montgomery and Smith (2017). Limitations relating to unbalanced aspects of the theory were discussed by Smith et al. (2008) and Bryan and Rotunno (2009).

Over the years, the theory has been developed further[2] and extended to account for the outer wind profile (Emanuel et al. 2004) and storm intensification (Emanuel 1997, henceforth E97, Emanuel 2012, henceforth E12). These time dependent theories incorporate the same basic geometry as the steady-state theory including, in particular, the effective slab boundary layer. Deficiencies of the E97 formulation were noted by E12 and Montgomery and Smith (2014).

The most recent time-dependent theory presented in E12 takes a fundamentally different approach to that of E97. In E12 is postulated that small-scale turbulent mixing in the upper troposphere plays a crucial role in determining the spatial distribution of outflow temperature and, in particular, the vertical stratification. This new theory has been invoked to suggest that tropical cyclones may be more prone to rapid intensification in a warmer climate, with the rate of storm intensification scaling as the square of the potential intensity (Emanuel 2017). The basic idea is that since the square of the potential intensity is a more sensitive metric than the potential intensity, itself, the increase in rapid intensification rate would be a more detectable signal of global warming than the potential intensity used previously in climate change assessments.

Despite the inclusion of time dependence in E12, questions remain concerning the unbalanced dynamics as well as the key physical mechanism of vortex spin up, even in the limiting case of strictly axisymmetric balance dynamics. In particular, the precise role of small-scale turbulent mixing in the upper troposphere begs a physical interpretation and a recent study by Montgomery et al. (2019) presented evidence supporting the view that this turbulent mixing is inconsequential to the spin-up process in three dimensions.

Another recent paper, by Peng et al. (2018), has appraised the validity of some key assumptions of the E12 theory, based mainly on simulations using an axisymmetric nonhydrostatic numerical model designed to mimic as close

[1]Correspondence to: Prof. M. T. Montgomery, Department of Meteorology, Naval Postgraduate School, Monterey, CA USA. E-mail: mtmontgo@nps.edu
[1]See Emanuel (1986), p593, and Emanuel and Rotunno (2011), top line, left column on p2240.
[2]For a brief review see Montgomery and Smith (2017), section 5.

as possible the E12 model. Starting from a relatively strong initial vortex (maximum tangential wind 20 m s^{-1}) with no secondary circulation and a saturated initial sounding, they identified two phases of evolution. In Phase I, the M- and s^*-surfaces evolve from nearly orthogonal to almost congruent, while in Phase II, these surfaces remain approximately congruent, supporting a key assumption of the E12 model. Here, s^* refers to the saturation specific entropy, s^*, and M to the absolute angular momentum. These quantities are defined in the usual way with $s^* = c_p ln\theta_e{}^*$ and $M = rv + \frac{1}{2}fr^2$, where c_p is the specific heat of dry air, θ_e^* is the saturation equivalent potential temperature (assuming pseudo-adiabatic ascent in which all condensed water instantly precipitates), v is the tangential velocity component, and f is the Coriolis parameter, assumed constant.

Peng et al. (2018) argue that, compared with their nonhydrostatic cloud model, the theoretical E12 model "possesses the chief virtue of transparency". However, neither E12 nor Peng et al. (2018) explained physically how small-scale turbulent mixing in the upper troposphere "drives" an amplification of the system-scale maximum tangential velocity at the top of the friction layer as encapsulated in Eq. (16) of E12. In the latter equation, the effects of upper-level mixing can be interpreted as a force per unit mass acting on the gradient wind at the top of the friction layer (see Eq. (A17) in the appendix of Montgomery et al. 2019 and related discussion). It is further unclear how the spin-up process relates to the classical intensification paradigm of Ooyama (1969) and the rotating convection paradigm reviewed by Montgomery and Smith (2014, 2017) and Smith and Montgomery (2016).

In the classical axisymmetric paradigm, for example, inflow in the lower troposphere induced by entraining deep convection in the central region of the vortex circulation is argued to draw absolute angular momentum surfaces inwards. Above the frictional boundary layer, M, is approximately conserved so that the inward movement of the M-surfaces implies a local spin up of the tangential velocity component. The relationship of intensification in the E12 model to Ooyama's articulation is unclear as the flow above the layer of friction is assumed to be approximately parallel to the M-surfaces. Because the E12 model is formulated with M as radial coordinate, any component of flow normal to the M-surfaces above the boundary layer is implicit.

The present paper seeks to explore issues surrounding the spin up of the low-level tangential winds at the top of the friction layer in the E12 model.

2 Summary and critique of the E12 model

As a preamble, we present here our understanding of the mathematical formulation of the E12 model and the physical constraints embodied in it. In particular, we seek to articulate our understanding of how spin up in the model comes about.

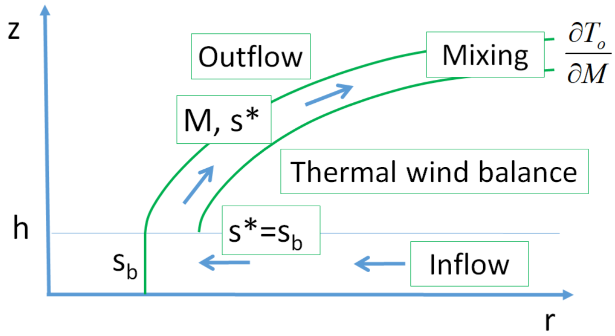

Figure 1. Geometric configuration of the axisymmetric Emanuel (2012) formulation for an intensifying tropical cyclone in cylindrical polar coordinates (r, z). In the schematic, h denotes the depth of the layer influenced by friction, which is assumed constant, s_b is the specific entropy in the boundary layer, s^* is the saturation specific entropy above the boundary layer, and M is the absolute angular momentum. It is assumed that s_b is independent of height. The arrows indicate the secondary circulation with radial inflow in the friction layer and outflow above it. Mixing by shear-stratified turbulence in the outflow layer of the developing tropical cyclone is assumed to determine the thermal stratification of the outflow ($\partial T_o/\partial M$, where T_o is the outflow temperature, our insertion).

The assumed flow configuration in radius-height coordinates (r, z) is sketched in Fig. 1. Air is assumed to converge in a shallow frictional boundary layer of constant depth h, acquiring moisture from the surface as it does so. As air parcels ascend out of this layer at inner radii, they are assumed to flow upwards and radially outwards into the upper troposphere, conserving their values of s^* and M. By construction, the model assumes a sufficiently well developed vortex that is saturated in its core, with an accompanying anticyclone near the tropopause (not depicted in the schematic). The M and s^* surfaces are assumed to flare outwards with height and not fold over[3], whereupon the flow remains everywhere centrifugally (or inertially) stable.

The master prognostic equation is one for the moist entropy, s_b, in the thermodynamic boundary layer, which is assumed to have depth h also. It is assumed further that $s^* = s_b$ at $z = h$. In turn, M and its time derivative are constrained by the following assumptions above the boundary layer:

(1) the flow there is in hydrostatic and gradient wind balance and therefore thermal wind balance;
(2) the M- and s^*-surfaces are congruent; and
(3) a closure assumption in the upper-tropospheric outflow relating the partial derivative of outflow temperature T_o with respect to M, i.e., $\partial T_o/\partial M$, to the derivative of s^* with respect to M, i.e. $\partial s^*/\partial M$.

[3]It may be significant that, in axisymmetric balance calculations in which the heating rate is specified along M surfaces centered around the radius of maximum tangential wind, which is qualitatively equivalent to assuming the s^* is constant along M surfaces, the M surfaces are found to turn over, e.g., Fig. 8 of Smith et al. (2018). Some hours after this overturning occurs, the balance calculations break down.

The closure assumption, in essence a parameterization for $\partial T_o/\partial M$, is based on the premise that "the thermal stratification of the outflow ($\partial T_o/\partial M$, our insertion) is set by small-scale turbulence that limits the Richardson number" to a critical value (E12, p988).

Significantly, the E12 model does not include a classical boundary layer in which both the tangential and radial components of flow satisfy Newton's equations of motion. Rather, the mean radial inflow in the layer influenced by friction, which in physical coordinates would be required to predict the time evolution of s_b, is determined by integrating vertically the tendency equation for M, across the layer. It is assumed that both horizontal velocity components, and therefore M, are essentially uniform through the depth of the layer. This formulation is represented by Eq. (A13) in Peng et al. (2018), which gives the inflow as a function of the temporal and radial derivatives of M together with the surface torque.

Because the equation for M is used to determine the radial flow, it is no longer available to determine the tendency of M in the layer of friction. Rather, the tendency of M is determined by the tendency of s_b in conjunction with the assumed constraint that the M- and s^*- surfaces are congruent. In essence, the M-surfaces are dragged in with the s_b surfaces on account of this assumed congruence. In this sense, the dynamics are slaved to the thermodynamics in the inner-core region where there is ascent out of the friction layer. It is unclear what is assumed at larger radii where there is subsidence into the friction layer.

The main outcomes of the theory are an equation for the intensification rate, $\partial V_m/\partial \tau$, of the maximum tangential wind, V_m, given by

$$\frac{\partial V_m}{\partial \tau} = \underbrace{\frac{C_D}{2h}\frac{Ri_c}{r_t^2}M^2}_{\text{turbulent mixing term}} - \frac{C_k}{2h}V_m^2, \qquad (1)$$

(see Eq. (A17) of Montgomery et al., 2019) and, with some further approximations, including the assumption that $V_m = 0$ at $\tau = 0$, an analytical solution for V_m of the form

$$V_m(\tau) = V_{max}\tanh\left(\frac{C_k V_{max}}{2h}\tau\right). \qquad (2)$$

In these equations, τ is the time, C_D and C_k are the surface drag and enthalpy coefficients, both assumed constant, Ri_c is the critical Richardson number that determines the onset of turbulent mixing in the upper-tropospheric outflow, r_t is the radius at which the upper-tropospheric mixing begins, and V_{max} is determined by Eq. (18)[4] in E12.

[4]This equation is a formula for V_{max} in terms of C_D, C_k, T_b, T_t, s_0, s_e^*, where T_b is the absolute temperature evaluated at the top of the inflow layer, T_t is equal to the outflow temperature T_o corresponding to the M-surface passing through the radius of maximum winds and is assumed equal to the initial tropopause temperature, s_0 "is the environmental (constant) saturation entropy of air at sea surface temperature and ambient surface pressure", and s_e^* "is the value of s^* in the undisturbed environment" (E12, p991-2).

It follows from Eq. (1) that spin up occurs only if the first term on the right-hand-side, which contains the effect of the parameterized turbulent mixing, is sufficiently large. This is because the second term on the right is negative definite. Invoking the classical paradigm for spin up, which refers to physical coordinates, spin up in the model can occur in the layer of friction only if the M surfaces therein move inwards at a sufficient rate that the radial advection of M exceeds the azimuthal torque per unit depth. Since the radial flow above the friction layer is radially outwards, spin up of the flow above the friction layer has to occur by the vertical advection of M from the friction layer. It follows that, in physical coordinates, spin up of the maximum tangential wind in the E12 model must occur within or at the top of the friction layer.

While the mathematical constraints leading to vortex spin up in the E12 model as outlined above are reasonably clear, the physical processes are not. Despite the fact that the mixing parameterization relating $\partial s^*/\partial M$ to $\partial T_o/\partial M$ in the upper-tropospheric outflow layer is a crucial element of the theory, without which the vortex will not spin up (see Montgomery et al. 2019, Appendix), the physics of spin up brought about by this mixing are mysterious, at least to us.

As pointed out by a reviewer of our manuscript (D. Raymond, personal communication), the parameterization of turbulent mixing in the E12 model introduces a parameter r_t in the tendency equation for the maximum gradient wind, the radius at which the upper-tropospheric mixing begins. This radius is unknown *a priori*. It turns out that the positive term in this tendency equation (Eq. (A17) in Montgomery et al., 2019) predicted by the theory is inversely proportional to r_t^2, but r_t is not determined by the theory: it must be prescribed. In this sense alone, the theory is not a closed theory for intensification.

Based on the foregoing analysis, the premise that spin up in the E12 model is controlled by mixing in the upper tropospheric outflow layer may be a red herring. As shown by Montgomery et al. (2019) (see their Eq. (A18)), if one stops short of applying the mixing parameterization in the E12 theory, Eq. (1) would take the form

$$\frac{\partial V_m}{\partial \tau} = \frac{C_D V_m^2 M}{2h(T_b - T_t)}\frac{\partial T_o}{\partial M} - \frac{C_k}{2h}V_m^2, \qquad (3)$$

The symbols T_b and T_t are defined in footnote 4. Comparing this equation with (1), it would seem that *any* functional form for $\partial T_o/\partial M$ that leads to the M-surfaces being dragged inwards in the friction layer at a sufficient rate that the radial advection of M exceeds the azimuthal torque per unit depth would yield a model for vortex spin up! This is true, whether or not the functional form has any physical meaning.

The above considerations would explain why attempts by Montgomery et al. (2019) to test the E12 theory on the basis of idealized, three-dimensional numerical model experiments showed that vertical mixing in the upper tropospheric outflow layer has no appreciable effect on vortex intensification. These considerations suggest further that any attempt to find a physical interpretation of spin up in the friction layer brought about by vertical mixing in the upper troposphere may be a fruitless exercise.

Where then does the main limitation of the E12 theory arise? In our view, the most serious issue is the assumption that the M- and s^*-surfaces are everywhere congruent, even during the intensification stage. This assumption forces the upper level mixing to determine the relationship between the values of M and s^*, where the flow ascends out of the inflow layer. As a minimum to remove this constraint, one would require the value of M to be determined by a tendency equation for M_b, which, in turn, according to Newton's second law, would require also a tendency equation for the radial velocity in the layer of friction.

There appears to be a prevailing view among some of our previous reviewers that since the E12 theory makes predictions for vortex spin up and mature intensity, it is unnecessary to enquire into the basic mechanisms underlying the theory. This viewpoint would seem to be problematic since predictions must be always tested against observations to ascertain the veracity of the theory. As an example, the analytical solution derived by E12 (Eq. (2) above and Eq. (19) of E12[5]) predicts an intensifying vortex starting from *zero initial tangential velocity*. Although, as noted above, the theory is only strictly valid when the vortex has attained some degree of maturity, it is unclear how the vortex can intensify from a quiescent state as there would be no surface enthalpy flux at zero time and no turbulent mixing to "initiate" intensification at this time (the first term in Eq. (1)). Indeed, with this initial condition, both terms on the right-hand-side of Eq. (1) would be zero implying that there would be no intensification at zero time, in contradiction to the analytical solution (2).

Another aspect of the analytical solution (2) is that the intensification rate of the vortex is independent of the initial inner-core size of the vortex. In other words, small-scale vortices would spin up at the same rate as large-scale vortices. Surely, this prediction is inconsistent with forecaster experience and on this ground alone should raise serious concerns. Further, it is not consistent with three-dimensional model predictions (e.g. Kilroy and Smith 2017: see Fig. 2 and the physical interpretations of the behaviour shown).

Yet another curious feature of the analytical solution (2) is that the crucial effects of turbulent mixing represented by the first term on the right-hand-side of Eq. (1), without which intensification cannot occur, have totally disappeared as a result of the further approximations deriving (2) from Eq. (1).

The foregoing issues, together with the recent tests of the premise of the E12 theory by Montgomery et al. (2019) would seem to be sufficient motivation to examine the theory more deeply. This has been the primary aim of this paper.

[5]This analytical solution was endorsed by Peng et al. (2018), pgs. 2125, 2126, 2135.

3 Conclusions

We have presented here our understanding of the mathematical formulation of Emanuel's 2012 axisymmetric theory for tropical-cyclone intensification and the constraints it incorporates. We have shown that, when viewed in physical coordinates, spin up in the model must originate in the layer of friction. This spin up occurs when the M-surfaces in this layer move inwards at a sufficient rate that the radial advection of M exceeds the azimuthal torque per unit depth.

Because the tendency equation for M is used to determine the radial flow in the layer of friction, it is no longer available to determine the tendency of M there. Rather, this tendency is determined by the tendency of s_b in conjunction with the assumption that the M- and s^*-surfaces are congruent. It is the latter constraint that drags the M-surfaces inwards in the layer of friction. The tendency of s^* (or s_b) is controlled, in part, by the assumed parameterization of turbulent mixing in the upper troposphere and is unconstrained by a radial momentum equation in the friction layer.

Despite the foregoing mathematical formulation, the physics of spin up in the friction layer brought about by the mixing in the upper troposphere remain to be explained, although as we have argued, this may be a fruitless exercise.

Acknowledgements

This paper arose out of stimulating discussions we had with Rich Rotunno of the National Center for Atmospheric Research during our participation in the 33rd American Meteorological Society Conference on Hurricanes and Tropical Meteorology in Ponte Vedra, Florida in 2018. We are grateful also to Kerry Emanuel, Dave Raymond and and two anonymous reviewers for their thoughtful comments on earlier versions of the paper. MTM acknowledge the support of NSF grant AGS-1313948, ONR grant N0001417WX00336, and the U. S. Naval Postgraduate School. The views expressed herein are those of the authors and do not represent sponsoring agencies or institutions.

References

Bryan, G. H. and R. Rotunno, 2009: Evaluation of an analytical model for the maximum intensity of tropical cyclones. *J. Atmos. Sci.*, **66**, 3042–3060.

Emanuel, K., C. DesAutels, C. Holloway, and R. Korty, 2004: Environmental control of tropical cyclone intensity. *J. Atmos. Sci.*, **61**, 843–858.

Emanuel, K. A., 1986: An air-sea interaction theory for tropical cyclones. Part I: Steady state maintenance. *J. Atmos. Sci.*, **43**, 585–604.

— 1997: Some aspects of hurricane inner-core dynamics and energetics. *J. Atmos. Sci.*, **54**, 1014–1026.

— 2012: Self-stratification of tropical cyclone outflow. Part II: Implications for storm intensification. *J. Atmos. Sci.*, **69**, 988–996.

— 2017: Will global warming make hurricane forecasting more difficult? *Bull. Amer. Meteor. Soc.*, **98**, 495–501.

Emanuel, K. A. and R. Rotunno, 2011: Self-stratification of tropical cyclone outflow. Part I: Implications for storm structure. *J. Atmos. Sci.*, **68**, 2236–2249.

Kilroy, G. and R. K. Smith, 2017: The effects of initial vortex size on tropical cyclogenesis and intensification. *Quart. Journ. Roy. Meteor. Soc.*, **143**, 2832–2845.

Montgomery, M. T., J. Persing, and R. K. Smith, 2019: On the hypothesized outflow control of tropical cyclone intensification. *Quart. Journ. Roy. Meteor. Soc. (in press)*, **145**, xxx–yyy.

Montgomery, M. T. and R. K. Smith, 2014: Paradigms for tropical cyclone intensification. *Aust. Met. Ocean. Soc. Journl.*, **64**, 37–66.

— 2017: Recent developments in the fluid dynamics of tropical cyclones. *Annu. Rev. Fluid Mech.*, **49**, 1–33, doi:10.1146/annurev-fluid-010816-060022.

Ooyama, K. V., 1969: Numerical simulation of the life cycle of tropical cyclones. *J. Atmos. Sci.*, **26**, 3–40.

Peng, K., R. Rotunno, and G. H. Bryan, 2018: Evaluation of a time-dependent model for the intensification of tropical cyclones. *J. Atmos. Sci.*, **75**, 2125–2138.

Smith, R. K. and M. T. Montgomery, 2016: Understanding hurricanes. *Weather*, **71**, 219–223.

Smith, R. K., M. T. Montgomery, and H. Bui, 2018: Axisymmetric balance dynamics of tropical cyclone intensification and its breakdown revisited. *J. Atmos. Sci.*, **75**, 3169–3189.

Smith, R. K., M. T. Montgomery, and S. Vogl, 2008: A critique of Emanuel's hurricane model and potential intensity theory. *Quart. Journ. Roy. Meteor. Soc.*, **134**, 551–561.

