The efficiency of diabatic heating and tropical cyclone intensification

Roger K. Smith[a] and Michael T. Montgomery[b]
[a] *Meteorological Institute, Ludwig-Maximilians University of Munich, Munich*
[b] *Dept. of Meteorology, Naval Postgraduate School, Monterey, CA*
*Correspondence to: Roger K. Smith, Meteorological Institute, Ludwig-Maximilians University of Munich, Theresienstr. 37, 80333 Munich, Germany. Email: roger.smith@lmu.de

Widely-held arguments attributing the increasingly rapid intensification of tropical cyclones to the increasing "efficiency" of diabatic heating in the cyclone's inner core region associated with deep convection are examined. The efficiency, in essence the amount of temperature warming compared to the amount of latent heat released, is argued to increase as the vortex strengthens on account of the strengthening inertial stability. Another aspect of the efficiency ideas concerns the location of the heating in relation to the radius of maximum tangential wind speed, with heating inside this radius seen to be more efficient in rapidly developing a warm core thermal structure and, presumably, a rapid increase in the tangential wind.

A more direct interpretation of the increased spin up rate is offered when the diabatic heating is located inside the radius of maximum tangential wind speed. Further, we draw attention to the limitations of assuming a fixed diabatic heating rate as the vortex intensifies and offer reasons, on these grounds alone, why it is questionable to apply the efficiency argument to interpret the results of observations or numerical model simulations of tropical cyclones. Moreover, since the spin up of the maximum tangential winds in a tropical cyclone takes place in the boundary layer and the spin up of the eyewall is a result of the vertical advection of high angular momentum from the boundary layer, it is questionable also whether deductions about *efficiency* in theories that neglect the boundary layer dynamics and thermodynamics are relevant to reality.

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1. Introduction

The recent unanticipated rapid intensification of Hurricane Patricia (2015) off the Pacific coast of Mexico by approximately 50 m s^{-1} in 24 hours time is a reminder of the challenges involved in forecasting such events, which is in part a reflection of deficiencies in understanding the rapid intensification of these storms. A widely-held explanation for the rapid intensification of tropical cyclones is that, as the vortex intensifies, the accompanying increase in inertial stability makes inner-core heating by deep convection "more efficient" in warming the vortex core (Schubert and Hack 1982, Hack and Schubert 1986, Vigh and Schubert 2009).

In brief, the explanation goes as follows. It is well known that the rate of latent heat release in deep convection is much larger than that needed to account for the local temperature differences between a cloud updraught and its environment. It turns out that much of the heat release is offset by adiabatic cooling as rising air parcels expand to lower pressure (Holton 2004, p393). Schubert and Hack *op. cit.* consider the overturning circulation induced by a diabatic heating rate with *fixed* magnitude and spatial structure in different locations within a vortex, where the inertial stability is locally different. They note on p1692 that an increase in the local inertial stability acts to impede the strength of the secondary circulation produced by a given heating rate. As a result, there is less adiabatic cooling as air parcels rise and more of the prescribed heating is available to increase the temperature of the rising air: i.e. the heating is "more efficient"* in raising the temperature of the cloud updraught.

In their conclusions, Vigh and Schubert *op. cit.* state: "It has been known for several decades that one of the necessary conditions for hurricane development is that diabatic heating occur in the region of high inertial stability". As an illustration of the wide acceptance of the efficiency idea, the foregoing explanation has been invoked recently to help explain secondary eyewall formation in an idealized hurricane simulation using the full physics Weather Research and Forecasting (WRF) model (Rozoff *et al.* 2013) and discussed (but not endorsed) in a multiscale analysis of the rapid intensification of Hurricane Earl (2010) (Rogers *et al.* 2015).

In this paper, we review some of the key results and interpretations of vortex behaviour in the foregoing studies by Schubert and coworkers (section 2) and go on in section 3 to articulate some reservations we have about the realism of their assumption of a fixed heating rate as the inertial stability is changed. We draw attention also to the importance of including a representation of boundary layer dynamics and thermodynamics in any theory applicable to interpreting the behaviour of numerical model simulations of tropical cyclones or observations of real storms.

2. Idealized symmetric models

We examine now briefly the models studied by Schubert and Hack (1982), Hack and Schubert (1986) and Vigh and Schubert (2009). These studies all focus on the efficiency of diabatic heating in producing a temperature warming of the vortex core, the assumption being that, combined with the assumption of gradient and hydrostatic balance, the warming will be accompanied by a spin up of the tangential wind field. Because these studies do not consider friction or non-axisymmetric processes, the classical axisymmetric mechanism for vortex intensification (Ooyama 1969, Montgomery *et al.* 2014) provides a useful framework for discussion. In this mechanism, the radial gradient of latent heating rate in the inner-core region of a pre-existing weak vortex induces a secondary circulation that draws surfaces of absolute angular momentum (M) inwards in the lower troposphere above the frictional boundary layer, where M is approximately materially conserved. Here M is defined as $rv + \frac{1}{2}fr^2$, where r is the radius, v is the tangential velocity component and f is the Coriolis parameter (assumed constant). A local increase of M implies a local increase in the tangential wind speed because $v = M/r - \frac{1}{2}fr$.

2.1. *The Schubert and Hack 1982 model*

The equations used by Schubert and Hack (1982) are a simplified form of those used by Hack and Schubert (1986). In the 1982 paper, the conservation form of the tangential momentum equation (using M) is employed instead of the standard form (using v). Further, the Boussinesq approximation is made and gradient wind balance is assumed. Here we write down the more general form used by Hack and Schubert:

$$\frac{\partial u}{\partial t} + \underline{u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} + \frac{v^2}{r}} + fv + \frac{\partial \phi}{\partial r} = 0 \qquad (1)$$

$$\frac{\partial v}{\partial t} + \underline{u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}} + fu = 0 \qquad (2)$$

$$\frac{\partial \phi}{\partial z} = g\frac{\theta}{\theta_0} \qquad (3)$$

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{1}{\rho}\frac{\partial \rho w}{\partial z} = 0 \qquad (4)$$

$$\frac{\partial \log\theta}{\partial t} + \underline{u\frac{\partial \log\theta}{\partial r}} + w\frac{\partial \log\theta}{\partial z} = \frac{\tilde{Q}}{c_pT} \qquad (5)$$

where $z = [1 - (p/p_0)^\kappa](c_p\theta_0/g)$ is the pseudo-height, r is the radius, $\rho(z)$ is a known pseudo-density, u and, w are the radial and vertical components of velocity, θ the potential temperature, p is the pressure, p_0 is a reference pressure (taken as 1,000 mb), ϕ the geopotential and $\tilde{Q}$ is the diabatic heating rate in units of J kg^{-1} s^{-1}. The balance system used by Schubert and Hack approximates Eq. (1) by the gradient wind equation and combines this with Eq. (3) to form the thermal wind equation. Further it uses simply the material derivative of θ to define the heating rate, Q, where $Q = \theta\tilde{Q}/(c_pT)$.

Schubert and Hack (1982) specify a functional form for $Q(r,z)$, intended to represent a localized source of heating with maximum heating in the middle troposphere. They solve the Sawyer-Eliassen equation (their Eq. (2.6)) for the streamfunction of the secondary circulation for a prescribed radial distribution of M (their Eq. (2.7)). The Sawyer-Eliassen equation is derived from the balance system of equations in the standard way. The focus is on the magnitude of the local warming of the air, characterised by $\partial\theta/\partial t$, in relation to the heating rate Q as a function of the location of the heating within the vortex. Schubert and Hack define the efficiency of the heating as the ratio:

$$\frac{\int_0^{b_2} \frac{\partial\theta}{\partial t} r dr}{\int_0^{b_2} Q r dr} \qquad (6)$$

where b_2 is outer radius beyond which Q is set to zero. They show, *inter alia*, that this efficiency is larger when the heating is located within the central (high inertial stability) region of the vortex, a result that led them suggest that "local warming by cumulus convection is considerably greater if the convection is confined to a region of relatively high inertial stability".

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[*] This efficiency concept for characterizing the time-dependent spin up of a tropical cyclone vortex (defined precisely below) is not to be confused with the different concept of efficiency in its classical thermodynamic sense of a heat engine and the amount of useful work that can be performed by the engine during a thermodynamic cycle of its working substance (Adkins 1985; Emanuel 1986).

2.2. *The Hack and Schubert 1986 models*

Hack and Schubert (1986) refer to Eqs, (1)-(5) as their *nonlinear model* and they use them to compare the axisymmetric, inviscid response of a nonlinear vortex on an f-plane to a specified heat source Q, with that of the corresponding linear model. The *linear model* is obtained by omitting all the underlined terms in these equations and replacing $\partial \log \theta / \partial z$ with a specified mean tropical profile $\partial \log \theta_0 / \partial z$. The linear model is essentially a generalization of the linear model presented in Gill (1982, section 9.15).

Choosing a rather simple analytic time-independent form for $Q(r, z)$, they compare, *inter alia*, solutions for the minimum central pressure, p_{min}, and maximum tangential wind speed, v_{max}, of the linear and nonlinear models, starting with a prescribed initial vortex and mean profile of θ_0. The solutions for the two models (their Figures 1 and 2) showed dramatic differences in the time evolution of p_{min} and v_{max}.

In the linear model, p_{min} decreases linearly with time and v_{max} increases linearly with time (consistent with Gill 1982, section 9.15), whereas, in the nonlinear model, the decrease in p_{min} and increase in v_{max} are appreciably more rapid. Hack and Schubert conclude that the " ... nonlinear terms in the governing equations begin to play a significant role in the development of a tropical vortex at a very early stage in its evolution" and go on to state "Unfortunately, it is very difficult to determine what dynamical processes are of most physical significance in primitive-equation results like those presented above". At this point they change their approach and "attempt to understand the nonlinear behaviour observed in the primitive-equation integration using the transformed Eliassen balanced vortex model introduced by Schubert and Hack (1983)" on the grounds that " ... the balanced system allows us to derive an analytic measure of the efficiency[†] of an axisymmetric vortex at converting total potential energy (e.g., generated by latent heat release) to the kinetic energy of the balanced flow".

While not disputing the results of their analysis of the balance model *with fixed heating* and the deductions about *efficiency* as defined by them, we offer here an alternative and more direct explanation for the more rapid intensification of the maximum tangential wind in the nonlinear model.

A key difference between the linear and nonlinear models relates to the conserved quantity arising from Eq. (2), namely the absolute angular momentum, M. In the nonlinear case, M is simply as defined above, while in the linear version of Eq. (2), the conserved quantity is the *absolute linear momentum* per unit mass $M_L = v + fr$. The latter follows by replacing u by the material derivative of r and linearizing to give $u = \partial r / \partial t$, where it is understood that r in this definition of radial velocity represents the Lagrangian radius coordinate of a particle. As an air parcel converges in the nonlinear model conserving M, v increases inversely with r as r decreases (see end of section 1), while the linear approximation, $v = M_L - fr$ increases only linearly with decreasing r. It follows that much larger tangential wind speeds might be achieved by convectively-induced inward radial displacements of air parcels in the nonlinear case for the same diabatic heating rate. The caveat "might be" is necessary because the larger inertial stability[‡] in the nonlinear model ($(f + \zeta)(2v/r + f)$ compared with f^2) acts to limit radial displacements relative to the linear model. One has to do a calculation to see which effect will "win".

The foregoing effects are obscured by the transformation to potential radius coordinates carried out by Hack and Schubert since then the radial motion leading to the motion of the transformed coordinates (effectively the M-surfaces) is implicit in the balance theory.

2.3. *An intermediate model*

The foregoing differences between the conserved quantity in the linear and nonlinear models discussed above were invoked by Ulrich *et al.* (2002) to explain the differences in intensification between a hurricane-like vortex in an axisymmetric model and that of an inter-tropical convergence-zone-like disturbance in a slab-symmetric model, starting from an initial disturbance with the same lateral structure. The slab-symmetric model is obtained by dropping all terms proportional to $1/r$ in the partial differential equations defined above (specifically in Eqs. (1), (2) and (4)), but not all the nonlinear terms. Ulrich *et al.* showed that, although the two flow configurations have many similarities, the slab-symmetric model does not provide a dynamical surrogate for the hurricane. The main difference was attributed to a geometrical factor in the formula for the conservation of absolute angular momentum in the axisymmetric model, which for an inward-moving air parcel permits much larger tangential wind speeds to be attained than in the slab-symmetric model. As a result, the wind-speed dependent latent heat flux at the sea surface is much larger in the axisymmetric model, providing a larger energy supply to the growing disturbance per unit area than in the slab-symmetric case. In the Ulrich *et al*. model, the effects of deep convection were parameterized and not held fixed. A further geometrical effect is that, for the same inflow velocity profile in the boundary layer, there is larger mass convergence in the axisymmetric model.

2.4. *The Vigh and Schubert 2009 model*

Vigh and Schubert (2009) focus their analysis on the rapid development of the warm core and base their analysis on a partial differential equation for the geopotential tendency. The use of the geopotential tendency equation for describing the balanced evolution of a vortex has certain advantages over the use of the Sawyer-Eliassen equation for the meridional (overturning) required to keep the vortex in thermal wind balance. Unlike the Sawyer-Eliassen equation, the derivation of the geopotential tendency equation is not degenerate for a hypothetical steady state vortex (see Persing *et al.* 2013, Smith *et al.* 2014 for details). A mathematical advantage of using the geopotential tendency equation for studying the development of the mass field in the vortex is that it avoids the need to first invert for the overturning circulation, then advect the tangential wind component by the radial and vertical flow, and finally to link the changes in tangential wind to changes in the mass field by solving the thermal wind equation. In fact, for the

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[†] We note that the term "efficiency" is not quite the same in the Hack and Schubert and Schubert and Hack papers

[‡] The inertial (centrifugal) stability I^2 on constant pseudo-height surfaces is given by $I^2 = 1/r^3 \partial M^2 / \partial r = \eta \xi$, where $\eta = f + v/r + \partial v / \partial r$ is the absolute vertical vorticity at radius r.

idealized vortex studied by Vigh and Schubert (2009), the geopotential tendency equation gives a direct link between the heat and momentum forcing and the changes in the mass field of the vortex.

Acknowledging the advantages of the geopotential tendency equation, one still has to establish a connection with the spin up of the tangential wind. After solving the balance equation for the geopotential tendency, the tangential wind tendency could be obtained via the local time derivative of the gradient wind equation (Eq. (1) without the time-tendency and nonlinear advection terms):

$$\frac{\partial v}{\partial t} = \frac{1}{\xi}\frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial t}\right), \tag{7}$$

where $\xi = f + 2v/r$ denotes twice the local absolute rotation rate of the fluid at radius r. Accepting Vigh and Schubert's finding that the geopotential tendency response will be largest in the core region where the vorticity is relatively high, it is not obvious that *its radial gradient* will be positive at the location of v_{max}. Thus, physical considerations alone do not allow a prediction of the outcome of increasing warming on v_{max}. One has to do an additional calculation to determine whether v_{max} will increase or decrease.

From a different perspective, assuming that the vortex intensity increases, the inertial stability will increase also. For a fixed diabatic heating rate, this increase in inertial stability will reduce the strength of the secondary circulation and, in particular, the strength of the lower tropospheric inflow. However, the increase in inertial stability is reflected in an increase in the radial gradient of M^2 and hence of M. Since the rate of change of tangential wind speed in the classical paradigm for intensification is proportional to the radial advection of M (i.e. $-(u/r)\partial M/\partial r$), the reduction of the magnitude of inflow (i.e of $-u$) is to some extent mitigated by the increase in the magnitude of $\partial M/\partial r$ and again one cannot anticipate the change in $\partial v/\partial t$ *a priori*: one has to do the calculation (e.g. Smith *et al.* 2015a).

The foregoing argument ignores, of course, the contribution to the tangential wind tendency $(1/r)(\partial M/\partial t)$ from vertical advection $-(w/r)(\partial M/\partial z)$, which is small in regions where the secondary circulation is primarily horizontal. The latter contribution must become increasingly important where the secondary circulation turns upwards in the eyewall region of the tropical cyclone. Indeed, numerical model calculations suggest that the spin up of the eyewall updraught is dominated by the vertical advection of angular momentum; in particular, the angular momentum of air parcels emanating from the frictional boundary layer (Smith *et al.* 2009, Persing *et al.* 2013, Kilroy *et al.* 2015, Schmidt and Smith 2015). In fact, the recent calculations of Persing *et al.* (2013). Kilroy *et al.* (2015) and Schmidt and Smith (2015) suggest that the classical mechanism of spin up accounts primarily for the spin up of the outer circulation, which then through boundary-layer dynamics leads to a spin up of the maximum tangential wind in the boundary layer, itself. The air with high tangential momentum generated in the boundary layer is lofted into the eyewall where the flow has an outward radial component, which, according to the classical theory would, by itself, lead to spin down. Thus, *ideas relating to the efficiency of diabatic heating in producing inner-core warming do not obviously apply to the inner core dynamics of tropical cyclones, where boundary layer dynamics and thermodynamics are of the utmost importance* (see section 3.2).

Vigh and Schubert (2009) present "a simple theoretical argument to isolate the conditions under which a tropical cyclone can rapidly develop a warm-core thermal structure and subsequently approach a steady state." The theory is supported by analytical solutions to the transverse circulation equation for a line source of diabatic heating located inside or outside of the radius of maximum tangential wind speed in a barotropic vortex. They cite observational studies indicating that significant diabatic heating normally occurs within the high-inertial-stability region of most storms (i.e. within the radius of maximum tangential wind speed), a structure that they note supports the intensification of v_{max}. They go on to say that " ... the more interesting question still remains: what controls how rapidly a storm will intensify?"

The insights gained from their analytical solutions are succinctly summarized on p3349 of their paper: "The solutions emphasize the fact that diabatic heating in the low-inertial-stability region outside the radius of maximum wind is inefficient at generating a warm core, no matter how large the current storm intensity. In contrast, diabatic heating in the high-inertial-stability region inside the radius of maximum wind is efficient at generating a localized temperature tendency, and this efficiency increases dramatically with storm intensity. In other words, the present results emphasize that the vortex intensification rate depends critically on how much of the heating is occurring inside the radius of maximum wind." Surprisingly, their central focus was on the intensification rate in terms of warm core development, rather than directly in terms of the maximum tangential wind speed. However, we would argue that a consideration of the tangential wind tendency provides a more direct interpretation of their results consistent with results of Shapiro and Willoughby (1982).

Invoking the material conservation of M, it is clear that diabatic heating outside the radius of maximum tangential wind, r_{vmax}, will lead to outflow at r_{vmax} so that the M-surfaces in the vicinity of[§] r_{vmax} will move outwards accompanied by spin down there. On the other hand, diabatic heating inside r_{vmax} will lead to inflow at r_{vmax} and thereby to spin up. These considerations concerning the sensitive dependence of the sign of the spin tendency are independent of the degree of inertial stability. In this way we can see immediately why heating located outside of the high vorticity region is locked out of the intensification process for the maximum tangential wind. A schematic summarizing the foregoing is presented in Figure 1 (see caption for further discussion).

3. Some concerns/remarks

3.1. *The assumption of a fixed heating rate*

We consider now the assumption of a fixed heating rate used in all the papers by Schubert and coworkers. While we appreciate the analytical simplicity of fixing the heating rate in idealized calculations, for scientific completeness one must consider the potential ramifications of this assumption.

[§] We should caution that r_{vmax} is not tied to a particular M-surface and our argument does not invoke the material conservation of r_{vmax}.

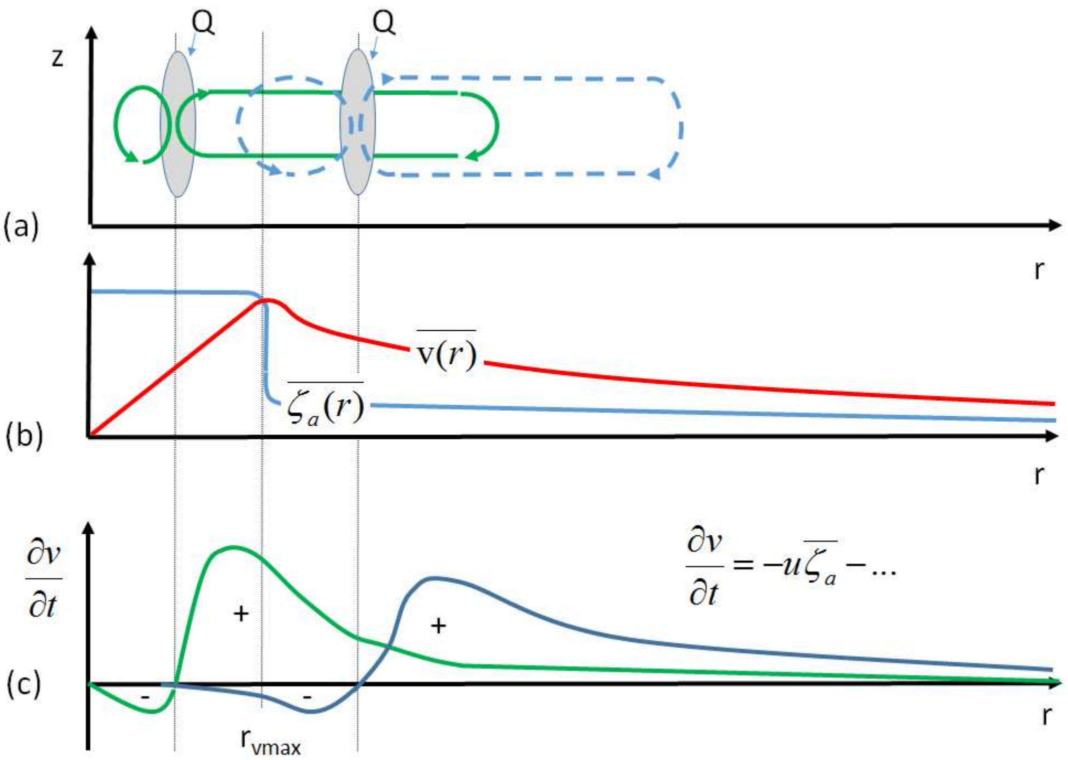

Figure 1. Idealized schematic illustrating the strong dependence of vortex spin up rate on the radial location of an imposed heating distribution $Q(r, z)$ in relation to the vorticity and wind distribution for a simple barotropic vortex where frictional effects are ignored. Panel (a) depicts two idealized positive heating distributions in radius-height coordinates, one located interior to the edge of high cyclonic vorticity, and the other located outside the edge of high cyclonic vorticity. Superimposed on this figure is the expected meridional overturning circulation for the imposed heating distributions from the Sawyer-Eliassen balance equation for the transverse streamfunction. Due to the much lower inertial stability of the region outside of the vortex core, the local Rossby length is much larger outside the core than inside. As a result, the radial scale of the streamfunction pattern is much larger outside the core than inside the core (Shapiro and Willoughby 1982, Figure 5). Panel (b) depicts the radial distribution of azimuthal mean cyclonic relative vorticity and tangential wind relative to the centre of the storm circulation. The vorticity and wind distribution resembles a modified Rankine vortex comprising a ‘high-vorticity core’ region of solid body rotation and an exterior ‘weak but nonzero vorticity skirt’ that decays slowly with radius outside the rapid transition region, consistent with observations (Mallen *et al.* 2005). Panel (c) depicts the radial distribution of the expected tangential wind tendency for the interior and exterior heating distributions, respectively. For the case of the heating distribution whose maximum is located interior to the radius of maximum tangential winds, the low-level meridional circulation outside of the heating maximum advects absolute angular momentum inwards, thereby increasing the tangential wind there and contributing to a contraction of the radius of the wind maximum (cf. Shapiro and Willoughby 1982). Inside of this heating distribution, the low-level meridional circulation is outwards. This low-level flow advects the absolute angular momentum surfaces outwards and leads to a weak spin down inside of the heating maximum. By similar reasoning, for the case of the heating distribution located outside the radius of maximum wind, the induced overturning circulation is such as to spin down the maximum tangential wind inside the heating maximum and increase the tangential wind outside the heating maximum.

To a first approximation, the diabatic heating rate for a rising air parcel is approximately related to the vertical velocity of the parcel, w, and its (saturation) equivalent potential temperature θ_e by the formula $Q = \mu w$, where $\mu = -L(\partial q_s/\partial z)_{\theta_e = constant}$, L is the latent heat of condensation and q_s is the saturation mixing ratio of water vapour. This formula follows from the definition of the heating rate in the form, $Q = -L(Dq_s/Dt)_{\theta_e = constant}$ when the material derivative Dq_s/Dt is approximated by the vertical advection term $w(\partial q_s \partial z)_{\theta_e = constant}$. For this reason, we would argue that the effect of inertial stability in reducing the radial inflow into the region of heating would be associated, in part, with a commensurate reduction of vertical velocity and, hence, heating rate.

Not only that, according to balance dynamics there are other effects that make the assumption of a fixed diabatic questionable when applied to real storms. Firstly, the increased inertial stability will reduce the radial scale of the updraught, which, in turn, will reduce the radial distribution of the diabatic heating (Schubert and Hack 1982, Shapiro and Willoughby 1982). Secondly, regions of high inertial stability inside the radius of maximum tangential wind will be less convectively unstable because of the balanced warm core in the interior of the vortex. If the vertical motion is suppressed in this way, there is no reason to suppose that the secondary circulation will extend through such a deep layer as when the vertical motion is not suppressed. Thus the vertical scale of the heating will be reduced. Thirdly, there is no reason to suppose that the factor μ relating Q to w will remain the same when the inertial stability is increased and especially if the heating is centred on a different radial location. For these reasons alone, we would argue that the gain in efficiency resulting from holding the magnitude and spatial structure of the heating rate fixed should not be applied to interpret the behaviour of real or model storms.

In their footnote 7, Schubert and Hack refer to a personal communication by K. Emanuel, who pointed out that, “while increased inertial stability suppresses the transverse circulation associated with the heating, it also implies larger transverse circulation associated with boundary layer pumping". The connection between increased inertial stability and increased boundary layer pumping is presumably because the increased inertial stability implies larger values of vertical vorticity above the boundary layer, but we would point out that the "boundary layer pumping" in a tropical cyclone is not a local effect: it depends on the radial profile of the gradient wind at the top of the boundary layer. This aspect is explored further in the following subsection.

3.2. *Boundary layer control*

The recent study by Kilroy *et al.* (2015) points strongly to the role of the boundary layer in controlling both the maximum tangential wind, which has been shown to occur within the boundary layer (Zhang *et al.* 2001, Smith *et al.* 2009, Zhang *et al.* 2011, Sanger *et al.* 2014, Montgomery *et al.* 2014), and the location of the eyewall updraught, at least in moderate strength and strong storms (e.g. tropical storm strength and above). Indeed, as shown by Kilroy *et al.*, calculations based on boundary layer theory[¶] indicate that the maximum ascent out of the boundary layer occurs inside the radius of maximum tangential wind speed, which would tend to initiate deep convection at these radii (cf. Rogers *et al.* 2015), i.e. in the high inertial stability region. This result is a feature of many earlier boundary layer calculations (e.g. Smith 1968, Carrier *et al.* 1971, Kepert 2001, Smith and Vogl 2008).

As argued above, the importance of having deep convection in this high inertial stability region is because it is the most favourable location for drawing M surfaces above the boundary layer closer to the axis of circulation, thereby amplifying v at these levels. If the convection is located outside the radius of maximum v, it will induce outflow at that radius and the maximum tangential wind above the boundary layer will tend to spin down as the M surfaces are drawn outwards. This argument is supported by the results of case studies of tropical lows in the Australian monsoon regime, including ones that intensified over the Australian continent (Smith *et al.* 2015c, Kilroy *et al.* 2015, Tang *et al.* 2015). These studies highlighted the importance, in general, for deep convection to occur close to the centre of an existing circulation for intensification. Further evidence for the importance of deep convection to occur inside r_{vmax} for intensification is provided by Rogers *et al.* (2015, p555).

Ultimately, as the vortex intensifies, the boundary layer control becomes paramount and frictional effects cannot be ignored in the spin up of the tangential circulation and the accompanying warm core. As noted above, because of the nonlinear nature of the boundary layer, the radial distribution of ascent out of the boundary layer into the eyewall is a non-local effect: it depends on the radial profile of tangential wind at radii well beyond the eyewall[‖]. Moreover, the radial distribution of diabatic heating rate in the eyewall depends on both the radial distribution of moist entropy of air leaving the boundary layer as well as the radial distribution of ascent at the top of the boundary layer. This result is seen in the formula for Q in the previous subsection (for a more in-depth discussion of this point see e.g. section 7.1 in Smith *et al.* 2015b).

Inertial stability is normally thought of as the resistance to a small radial displacement of an air parcel *above* the boundary layer, in an axisymmetric, balanced, rotating flow. The concept is not directly relevant to swirling boundary layer dynamics because there is a nonzero agradient force in that layer. This force is generally of one sign, changing from negative outside some radius near r_{vmax} to positive inside this radius. Thus, in general, a small radial displacement of an air parcel does not lead to a change in sign of the agradient force as it would above the boundary layer (Smith and Montgomery 2015). Of course, as noted in section 3.1 (see Fig. 1), inertial stability is a factor influencing the radial scale of the convectively-induced secondary circulation and affects the boundary layer flow indirectly by affecting the tangential wind profile at the top of the boundary layer.

4. Conclusions

We have examined widely-held arguments that attribute the increasingly rapid intensification of tropical cyclones to the increasing "efficiency" of diabatic heating in the cyclone's inner core associated with deep convection. In these arguments, the efficiency characterizes the amount of temperature warming compared to the amount of latent heat released and it is argued to increase as the vortex strengthens on account of the strengthening inertial stability, which, itself, has a weakening effect on the secondary circulation. Another aspect of the efficiency ideas concerns the location of the heating in relation to the radius of maximum tangential wind speed, with heating inside this radius seen to be more efficient.

We do not dispute the results of Schubert and coworkers' analyses of their axisymmetric balance models *with fixed heating* and the deductions about *efficiency* as defined by them. However, we do challenge the widespread use of these ideas when applied to interpret the results of numerical model simulations and observations of tropical cyclones in which the heating rate, itself, must depend also on the inertial stability. Here, we have bypassed the thermodynamic efficiency arguments and offered an alternative and more direct interpretation of the increased spin up rate when the diabatic heating is located inside the radius of maximum tangential wind speed.

While the efficiency ideas are focussed on the inflow above the frictional boundary layer and the effects of inertial stability thereon, the spin up of the maximum tangential winds in a tropical cyclone takes place in the boundary layer and the spin up of the eyewall is a result of the vertical advection of high angular momentum from the boundary layer. This being the case, it is unclear whether deductions about *efficiency* in theories that neglect the boundary layer dynamics and thermodynamics have any relevance to reality.

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[¶] Strictly speaking, boundary layer theory ultimately breaks down in the inertially-dominated, corner flow region where the boundary layer separates from the surface and the swirling wind erupts out of the boundary layer (Smith and Montgomery 2010).

[‖] This behaviour is in contrast to the linear boundary layer solution, which depends only on the local tangential wind speed above the layer (e.g. Kepert 2001, Vogl and Smith 2009).

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