Boundary Layer Meteorology

Chapter 3

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Equations for turbulent flow

- To quantitatively describe and forecast the state of the boundary layer, we turn to the governing equations of fluid dynamics.
- These include:
  - the continuity equation
  - the momentum equation (expressing Newton's 2^{nd} law)
  - the thermodynamic equation
  - the moisture equation
  - the equation of state

Special problems for turbulent flow

- In principle, the equations can be applied directly to turbulent flows, but this is generally too complicated.
- We would not be able to resolve all turbulent scales down to the smallest eddy to determine the initial condition.
- Instead, for simplicity, we pick some cut-off eddy size below which we include only the statistical effects of turbulence.
- In some mesoscale and synoptic scale models the cut off is on the order of 10 to 100 km, while for some boundary layer models known as large eddy simulation models, the cut off is on the order of 100 m.

Special problems for turbulent flow

- The complete set of equations as applied to the boundary layer are so complex that no analytic solution is known: we are forced to look for approximate solutions.
- We can seek
  - exact analytical solutions to simplified subsets of the equations, or
  - approximate numerical solutions to a more complete set of equations.
- We begin by formulating equations that are statistically averaged over the small eddy sizes.

Mean and turbulent parts of the flow

- There is a very easy way to isolate the large-scale variations from the turbulent ones: by averaging the wind speed measurements over a period of 30 min to one hour, we can eliminate or average out the positive and negative deviations of the turbulent velocities about the mean.
- Let

$$\underbrace{u}_{\text{instantaneous wind}} = \underbrace{\overline{U}}_{\text{mean wind}} + \underbrace{u'}_{\text{turbulent part}}$$

- The existence of a spectral gap allows us to partition the flow field in this manner.

Basic governing equations 1

- **Equation of state (ideal gas law for moist air)**

$$p = \rho R_d T_v$$

pressure **density** **specific gas constant** **virtual temperature**

$$T_v = T(1 + 0.61r)$$

Water vapour mixing ratio
or specific humidity

Basic governing equations 2

- **Mass conservation (continuity equation): two forms**

Flux form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0$$

Ordinary form

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$$

Boussinesq form

$$\frac{\partial u_j}{\partial x_j} = 0$$

Basic governing equations 3

- **Conservation of momentum (Newton's second law)**

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \delta_{i3} g - 2\varepsilon_{ijk}\Omega_j u_k + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

I II III IV V VI

Term I represents the rate-of-change of momentum (inertia)
Term II is the advection of momentum
Term III is the pressure gradient force
Term IV is the Coriolis force
Term V is the gravitational force
Term VI is the viscous stress term

Basic governing equations 4

Term VI ⇒

$$\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left\{\mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - \frac{2}{3}\mu\delta_{ij}\left[\frac{\partial u_k}{\partial x_k}\right]\right\}$$

Assuming that μ is not a function of position,

$$\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\mu}{\rho}\left\{\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\left[\frac{\partial u_j}{\partial x_j}\right] - \frac{2}{3}\frac{\partial}{\partial x_i}\left[\frac{\partial u_k}{\partial x_k}\right]\right\}$$

For an incompressible fluid

$$\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} = \nu\frac{\partial^2 u_i}{\partial x_j^2}$$

Basic governing equations 5

- Conservation of momentum ⇒

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \delta_{i3} g - 2\varepsilon_{ijk}\Omega_j u_k + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

- This is just the Navier-Stokes' equation

Basic governing equations 6

- Conservation of moisture (water substance)

Net water source

$$\frac{\partial q_T}{\partial t} + u_j \frac{\partial q_T}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_{q_T}}{\rho}$$

I II III IV

Molecular diffusivity for water vapour

q_T specific humidity for water substance

Term I represents the local rate-of-change of water substance
Term II is the advection of water substance
Term III is the diffusion of water vapour (specific humidity q)
Term IV is the net source of water substance

Basic governing equations 7

Put $q_T = q + q_L$ **and** $S_{q_T} = S_q + S_{q_L}$

$$\frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho} + \frac{E}{\rho}$$

$$\frac{\partial q_L}{\partial t} + u_j \frac{\partial q_L}{\partial x_j} = \frac{S_{q_L}}{\rho} - \frac{E}{\rho}$$

E **= rate-of-evaporation of liquid water**

Molecular diffusion of liquid water is neglected

Basic governing equations 8

- **Conservation of heat (First law of thermodynamics)**

$$\underset{\mathrm{I}}{\frac{\partial \theta}{\partial t}} + \underset{\mathrm{II}}{u_j \frac{\partial \theta}{\partial x_j}} = \underset{\mathrm{III}}{\nu_\theta \frac{\partial^2 \theta}{\partial x_j^2}} - \underset{\mathrm{IV}}{\frac{1}{\rho c_p} \frac{\partial Q_j^*}{\partial x_j}} - \underset{\mathrm{V}}{\frac{LE}{\rho c_p}}$$

Q_j^* **is the component of net radiation in the** j**-direction**

Term I **represents the rate-of-change of potential temperature**
Term II **is the advection of potential temperature**
Term III **is the effect of the molecular diffusion of heat**
Term IV **is the effect of the radiative flux divergence**
Term V **is the effect of the latent heat consumed by evaporation**

Basic governing equations 9

- **Conservation of a scalar quantity**

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \nu_c \frac{\partial^2 c}{\partial x_j^2} + S_c$$

I　　II　　III　　IV

c **is the scalar concentration (per unit mass of air)**

Term I represents the rate-of-change of the scalar
Term II is the advection of the scalar
Term III is the effect of the molecular diffusion of the scalar
Term IV is the source of the scalar quantity

Manipulation of the equation of state

Put $\rho = \overline{\rho} + \rho', \quad T_v = \overline{T}_v + T_v', \quad p = \overline{p} + p'$

$$p = \rho R T_v \Rightarrow \overline{p} + p' = R\left(\overline{\rho} + \rho'\right)\left(\overline{T}_v + T_v'\right)$$

$$\overline{p} + p' = R\left(\overline{\rho}\overline{T}_v + \rho'\overline{T}_v + \overline{\rho}T_v' + \rho_v' T_v'\right)$$

$$\Rightarrow \overline{p} = R\left(\overline{\rho}\overline{T}_v + \overline{\rho' T'}\right)$$

small

$$\Rightarrow \boxed{\overline{p} = R\overline{\rho}\overline{T}_v}$$

Manipulation of the equation of state

$$\overline{p} + p' = R\left(\overline{\rho}\overline{T}_v + \rho'\overline{T}_v + \overline{\rho}T_v' + \rho_v' T_v'\right)$$

$$\overline{p} = R\overline{\rho}\overline{T}_v$$

$$\Rightarrow \quad p' = R\left(\rho'\overline{T}_v + \overline{\rho}T_v' + \rho_v' T_v'\right)$$

$$\Rightarrow \quad \frac{p'}{\overline{p}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T}_v} + \frac{\rho_v' T_v'}{\overline{\rho}\overline{T}_v}$$

small

$$\Rightarrow \quad \boxed{\frac{p'}{\overline{p}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T}_v}}$$

$$\frac{p'}{\overline{p}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T}_v}$$

- **Static pressure fluctuations are associated with variations in the mass of air from column to column in the atmosphere.**
- **For the larger eddies and thermals in the BL, these fluctuations may be as large as 0.1 mb, while for smaller eddies the effect is smaller.**
- **Dynamic pressure fluctuations associated with wind speeds ≈ 10 m s^{-1} cause fluctuations of about 0.1 mb also.**
- **Thus for most BL situations**

$$\frac{p'}{\overline{p}} = \frac{10\ \text{Pa}}{10^5\ \text{Pa}} = 10^{-4} \qquad \textbf{cf} \qquad \frac{T_v'}{\overline{T}_v} = \frac{1\ \text{K}}{300\ \text{K}} = 3.3\times 10^{-3}$$

Shallow convection approximation

$$\cancel{\frac{p'}{\overline{p}}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T}_v} \quad \Rightarrow \quad \boxed{\frac{\rho'}{\overline{\rho}} \approx -\frac{T_v'}{\overline{T}_v}}$$

$$\Rightarrow \quad \boxed{\frac{\rho'}{\overline{\rho}} \approx -\frac{\theta_v'}{\overline{\theta}_v}}$$

Show as exercise

- Air that is warmer than average is less dense than average.
- These equations allow us to substitute temperature fluctuations, easily measurable quantities, in place of density fluctuations, which are not so easily measured.

Flux form of the advection terms

Advection term $= u_j \dfrac{\partial \xi}{\partial x_j}$ $\quad\xi$ **any dependent variable**

$$= u_j \frac{\partial \xi}{\partial x_j} + \xi \underline{\frac{\partial u_j}{\partial x_j}}$$

$$\boxed{\underline{\frac{\partial u_j}{\partial x_j} = 0}}$$

continuity eq.

Flux form $= \dfrac{\partial}{\partial x_j}\left(\underline{u_j \xi}\right)$

A kinematic flux

Conservation of momentum

Vertical component (put $x_3 = z$, $u_3 = w$)

Treat ν, μ as constants

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{\partial^2 w}{\partial x_j^2}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$$

$$(\bar{\rho} + \rho')\frac{D(\bar{w} + w')}{Dt} = -g(\bar{\rho} + \rho') - \frac{1}{\rho}\frac{\partial(\bar{p} + p')}{\partial z} + \mu \frac{\partial^2(\bar{w} + w')}{\partial x_j^2}$$

$$\left(1 + \frac{\rho'}{\bar{\rho}}\right)\frac{D(\bar{w} + w')}{Dt} = -g\frac{\rho'}{\bar{\rho}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\bar{w} + w')}{\partial x_j^2} - \frac{1}{\bar{\rho}}\left(\frac{\partial \bar{p}}{\partial z} + g\bar{\rho}\right)$$

$$\left(1 + \frac{\rho'}{\bar{\rho}}\right)\frac{D(\bar{w} + w')}{Dt} = -g\frac{\rho'}{\bar{\rho}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\bar{w} + w')}{\partial x_j^2} - \frac{1}{\bar{\rho}}\left(\frac{\partial \bar{p}}{\partial z} + g\bar{\rho}\right)$$

$$\frac{\rho'}{\bar{\rho}} = -\frac{T_v'}{\bar{T}_v} \approx 3.3 \times 10^{-3}$$

Assume hydrostatic equilibrium in the mean $\quad \dfrac{\partial \bar{p}}{\partial z} + g\bar{\rho} = 0$

$$\frac{D(\bar{w} + w')}{Dt} = -g\frac{\rho'}{\bar{\rho}} - \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \nu \frac{\partial^2(\bar{w} + w')}{\partial x_j^2}$$

Cannot neglect: Boussinesq approximation

$$-g\frac{\rho'}{\bar{\rho}} \approx g\frac{\theta_v'}{\bar{\theta}_v}$$

- **Although subsidence, $\overline{w}$, is important in mass conservation and in material advection from aloft, it is less important in the momentum equation.**
- **In fair weather BLs, it can vary from 0 to 0.1 m s^{-1}. This is small compared with w', which frequently varies over the range 0 to 5 m s^{-1}. In the momentum equation only, we can take:**

$$\overline{w} \equiv 0$$

$$\underline{\frac{Dw'}{Dt} = g\frac{\theta'_v}{\overline{\theta}_v}} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu\frac{\partial^2 w'}{\partial x_j^2}$$

- **Air that is warmer than average is accelerated upwards.**
- **The last two terms represent the effects of the vertical pressure gradient and viscous stress on the motion.**
- **Equation important in the evolution of convective thermals.**

Conservation of momentum

Horizontal component (put $x_1 = x$, $x_2 = y$, $u_1 = u$, $u_2 = v$)

Define the geostrophic wind by $f\left(u_g, v_g\right) = \left(-\frac{1}{\rho}\frac{\partial p}{\partial y}, \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$

Then

$$\frac{Du}{Dt} = -f(v_g - v) + \nu\frac{\partial^2 u}{\partial x_j^2}$$

$$\frac{Dv}{Dt} = \underline{f(u_g - u)} + \nu\frac{\partial^2 v}{\partial x_j^2}$$

Ageostrophic wind

Combined momentum equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\varepsilon_{ij3} f (u_{gj} - u_j) - \delta_{i3}\left[g \frac{\theta'_v}{\overline{\theta}_v} - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial z} \right] + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Here we have applied the shallow convection, incompressibility, hydrostatic and Boussinesq approximations and set

$$\mathbf{u_g} = (u_g, v_g, 0) .$$

Horizontal homogeneity

Expand the total derivative of any mean variable

$$\frac{D\overline{\xi}}{Dt} = \frac{\partial \overline{\xi}}{\partial t} + u \frac{\partial \overline{\xi}}{\partial x} + v \frac{\partial \overline{\xi}}{\partial y} + w \frac{\partial \overline{\xi}}{\partial z}$$

- **Averaged variables such as** θ **or turbulent** KE **exhibit large vertical variations over** 1 – 2 **km of the BL, but a much smaller horizontal variation over the same** 1 – 2 **scale.**
- **However,** $|w| << |(u,v)|$
- $\Rightarrow$ **the terms are comparable in many situations.**

Horizontal homogeneity

- Sometimes micrometeorologists wish to focus their attention on turbulence effects at the expense of neglecting mean advection.
- By assuming horizontal homogeneity, we can set

$$\frac{\partial \overline{\xi}}{\partial x} = 0, \quad \frac{\partial \overline{\xi}}{\partial y} = 0$$

- And neglecting subsidence gives $\overline{w} = 0$.
- While these assumptions are frequently made to simplify the derivations, they are rarely valid in the atmosphere.
- When made, the advection terms involving mean quantities disappear, leaving the important turbulent flux terms.

Reorienting and rotating the coordinate system

- We often use a Cartesian coordinate system aligned such that the (x,y,z) axes point (east, north, up).
- Sometimes it is convenient to rotate the axes about the vertical (z-) axis so that x and y point in other directions. Some examples include aligning the x-axis with:
  - The mean wind direction,
  - The geostrophic wind direction,
  - The direction of the surface stress, or
  - Perpendicular to shorelines or mountain ridges.
- The reason for doing this is to simplify some of the terms in the governing equations.

Equations for mean variables in a turbulent flow

The equation of state is assumed to hold in the mean: $\overline{p} = \overline{\rho} R \overline{T}$

Continuity equation:

$$\frac{\partial(\overline{u}_j + u'_j)}{\partial x_j} = 0 \xrightarrow{\text{Time average}} \overline{\frac{\partial \overline{u}_j}{\partial x_j} + \frac{\partial u'}{\partial x_j}} = 0$$

$$\frac{\partial \overline{u}_j}{\partial x_j} + \underbrace{\overline{\frac{\partial u'}{\partial x_j}}}_{=0} = 0 \Rightarrow \frac{\partial \overline{u}_j}{\partial x_j} = 0 \Rightarrow \frac{\partial u'_j}{\partial x_j} = 0$$

Can put turbulent advection terms in flux form

Conservation of momentum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \varepsilon_{ij3} f u_j - \frac{1}{\overline{\rho}} \frac{\partial(\overline{p} + p')}{\partial z} - \delta_{i3} g \frac{\theta'_v}{\overline{\theta}_v} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Expand the dependent variables into mean and turbulent parts, except where already done:

$$\frac{\partial(\overline{u}_i + u'_i)}{\partial t} + (\overline{u}_j + u'_j) \frac{\partial(\overline{u}_i + u'_i)}{\partial x_j} = \varepsilon_{ij3} f (\overline{u}_i + u') - \frac{1}{\overline{\rho}} \frac{\partial(\overline{p} + p')}{\partial x_i}$$

$$- \delta_{i3} \left[g - g \frac{\theta'_v}{\overline{\theta}_v} \right] + \nu \frac{\partial^2 (\overline{u}_i + u'_i)}{\partial x_j^2}$$

$$\frac{\partial(\bar{u}_i + u_i')}{\partial t} + (\bar{u}_j + u_j')\frac{\partial(\bar{u}_i + u_i')}{\partial x_j} = \varepsilon_{ij3} f(\bar{u}_j + u_j') - \frac{1}{\bar{\rho}}\frac{\partial(\bar{p} + p')}{\partial x_i}$$

$$- \delta_{i3}\left[g - g\frac{\theta_v'}{\bar{\theta}_v}\right] + \nu\frac{\partial^2(\bar{u}_i + u_i')}{\partial x_j^2}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_j\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \bar{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} = \varepsilon_{ij3} f\bar{u}_j + \varepsilon_{ij3} f u_j'$$

$$-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x_i} + -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x_i} - \delta_{i3}\left[g - g\frac{\theta_v'}{\bar{\theta}_v}\right] + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2}$$

Next average the whole equation

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_j\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \bar{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} = \varepsilon_{ij3} f\bar{u}_j + \varepsilon_{ij3} f u_j'$$

$$-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x_i} + -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x_i} - \delta_{i3}\left[g - g\frac{\theta_v'}{\bar{\theta}_v}\right] + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2}$$

$$\overline{\frac{\partial \bar{u}_i}{\partial t}} + \cancel{\overline{\frac{\partial u_i'}{\partial t}}} + \overline{\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}} + \cancel{\overline{\bar{u}_j\frac{\partial u_i'}{\partial x_j}}} + \cancel{\overline{u_j'\frac{\partial \bar{u}_i}{\partial x_j}}} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} = \overline{\varepsilon_{ij3} f\bar{u}_j} + \cancel{\overline{\varepsilon_{ij3} f u_j'}}$$

$$-\overline{\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x_i}} + -\cancel{\overline{\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x_i}}} - \delta_{i3} g - \delta_{i3} g\cancel{\overline{\frac{\theta_v'}{\bar{\theta}_v}}} + \nu\overline{\frac{\partial^2 \bar{u}_i}{\partial x_j^2}} + \nu\cancel{\overline{\frac{\partial^2 u_i'}{\partial x_j^2}}}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = \varepsilon_{ij3} f \bar{u}_j - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} - \delta_{i3} g + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2}$$

Add $u'_i \times$ the continuity equation and average to put the turbulent advection term in flux form and move this term to the right-hand-side:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \varepsilon_{ij3} f \bar{u}_j - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} - \delta_{i3} g + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \underline{\frac{\partial \overline{(u'_i u'_j)}}{\partial x_j}}$$

Note the prediction equation for the mean wind is very similar to the original conservation equation, except for the last term.

$$\underbrace{\frac{\partial \bar{u}_i}{\partial t}}_{\text{I}} + \underbrace{\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{II}} = \underbrace{\varepsilon_{ij3} f \bar{u}_j}_{\text{III}} \underbrace{- \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i}}_{\text{IV}} \underbrace{- \delta_{i3} g}_{\text{V}} + \underbrace{\nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2}}_{\text{VI}} \underbrace{- \frac{\partial \overline{(u'_i u'_j)}}{\partial x_j}}_{\text{VII}}$$

Term I represents the rate-of-change of mean momentum

Term II is the advection of mean momentum by the mean wind

Term III is the mean Coriolis force

Term IV is the mean pressure gradient force

Term V is the gravitational force

Term VI is the influence of the viscous stress on the mean motion

Term VII represents the influence of the Reynolds' stress on the mean motion. It can be interpreted also as the divergence of the turbulent momentum flux.

Turbulent momentum flux

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \varepsilon_{ij3} f \bar{u}_j - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} - \delta_{i3} g + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \frac{\partial \overline{(u_i' u_j')}}{\partial x_j}$$

Can write

$$-\frac{\partial \overline{(u_i' u_j')}}{\partial x_j} = \frac{1}{\bar{\rho}} \frac{\partial \tau_{ij}}{\partial x_j} \quad \textbf{where} \quad \tau_{ij} = -\bar{\rho}\, \overline{u_i' u_j'}$$

Implication: turbulence must be considered in predicting the turbulent BL, even if we are trying to predict only mean quantities. The last term can often be as large in magnitude, or larger, than many other terms in the equation.

Conservation of moisture

$$\frac{\partial \bar{q}_T}{\partial t} + \frac{\partial q_T'}{\partial t} + \bar{u}_j \frac{\partial \bar{q}_T}{\partial x_j} + \bar{u}_j \frac{\partial q_T'}{\partial x_j} + u_j' \frac{\partial \bar{q}_T}{\partial x_j} + u_j' \frac{\partial q_T'}{\partial x_j} =$$

$$\nu_q \frac{\partial^2 \bar{q}_T}{\partial x_j^2} + \nu_q \frac{\partial^2 q_T'}{\partial x_j^2} + \frac{S_{q_T}}{\bar{\rho}}$$

Mean source term

Proceeding as before

$$\frac{\partial \bar{q}_T}{\partial t} + \bar{u}_j \frac{\partial \bar{q}_T}{\partial x_j} = \nu_q \frac{\partial^2 \bar{q}_T}{\partial x_j^2} + \frac{S_{q_T}}{\bar{\rho}} - \frac{\partial \overline{(u_j' q_T')}}{\partial x_j}$$

$$\frac{\partial \overline{q}_T}{\partial t} + \overline{u}_j \frac{\partial \overline{q}_T}{\partial x_j} = \nu_q \frac{\partial^2 \overline{q}_T}{\partial x_j^2} + \frac{S_{q_T}}{\overline{\rho}} - \frac{\partial (\overline{u'_j q'_T})}{\partial x_j}$$

I **II** **III** **IV** **V**

Term I **represents the rate-of-change of mean total water**

Term II **is the advection of mean total water by the mean wind**

Term III **is the molecular diffusion of water vapour**

Term IV **is the mean source term for total water**

Term V **represents the divergence of the turbulent total water flux.**

Similar equations can be written down for the vapour and non-vapour parts of the specific humidity.

Conservation of heat

$$\frac{\partial \overline{\theta}}{\partial t} + \cancel{\frac{\partial \theta'}{\partial t}} + \overline{u}_j \frac{\partial \overline{\theta}}{\partial x_j} + \overline{u}_j \cancel{\frac{\partial \theta'_i}{\partial x_j}} + \cancel{u_j} \frac{\partial \overline{\theta}}{\partial x_j} + u'_j \cancel{\frac{\partial \theta'}{\partial x_j}} =$$

$$- \frac{1}{\overline{\rho} c_p} \frac{\partial \overline{Q}^*_j}{\partial x_i} - \frac{1}{\overline{\rho}} \cancel{\frac{\partial Q^*_j}{\partial x_i}} + \nu \frac{\partial^2 \overline{\theta}}{\partial x_j^2} + \nu \cancel{\frac{\partial^2 \theta'}{\partial x_j^2}} - \frac{LE}{\overline{\rho} c_p}$$

Proceeding as before

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{u}_j \frac{\partial \overline{\theta}}{\partial x_j} = - \frac{1}{\overline{\rho} c_p} \frac{\partial \overline{Q}^*_j}{\partial x_i} + \nu \frac{\partial^2 \overline{\theta}}{\partial x_j^2} - \frac{LE}{\overline{\rho} c_p} - \frac{\partial (\overline{u'_j \theta'})}{\partial x_j}$$

$$\frac{\partial\bar{\theta}}{\partial t}+\bar{u}_j\frac{\partial\bar{\theta}}{\partial x_j}=-\frac{1}{\bar{\rho}c_p}\frac{\partial\bar{Q}^*_j}{\partial x_i}+\nu\frac{\partial^2\bar{\theta}}{\partial x_j^2}-\frac{LE}{\bar{\rho}c_p}-\frac{\partial\overline{(u'_j\theta')}}{\partial x_j}$$

I II III IV V VI

Term I represents the rate-of-change of heat

Term II is the advection of heat by the mean wind

Term III is the molecular conduction of heat

Term IV is the mean radiative divergence source

Term V is the source associated with latent heat release

Term VI represents the divergence of the turbulent heat flux.

Conservation of heat

$$\frac{\partial\bar{c}}{\partial t}+\cancel{\frac{\partial c'}{\partial t}}+\bar{u}_j\frac{\partial\bar{c}}{\partial x_j}+\cancel{\bar{u}_j\frac{\partial c'}{\partial x_j}}+\cancel{u'_j}\frac{\partial\bar{c}}{\partial x_j}+u'_j\cancel{\frac{\partial c'}{\partial x_j}}=$$

$$\nu\frac{\partial^2\bar{c}}{\partial x_j^2}+\nu\cancel{\frac{\partial^2 c'}{\partial x_j^2}}+\underline{S_c}$$

Mean source term

Proceeding as before

$$\frac{\partial\bar{c}}{\partial t}+\bar{u}_j\frac{\partial\bar{c}}{\partial x_j}=\nu\frac{\partial^2\bar{c}}{\partial x_j^2}+\bar{S}_c-\frac{\partial\overline{(u'_j c')}}{\partial x_j}$$

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}_j \frac{\partial \overline{c}}{\partial x_j} = \nu \frac{\partial^2 \overline{c}}{\partial x_j^2} + \overline{S}_c - \frac{\partial \overline{(u_j' c')}}{\partial x_j}$$

I II III IV V

Term I **represents the rate-of-change of tracer concentration**

Term II **is the advection of tracer concentration by the mean wind**

Term III **is the molecular diffusion of tracer concentration**

Term IV **is the mean source of tracer concentration**

Term V **represents the divergence of the turbulent tracer concentration flux.**

Neglect of viscosity for mean motions

- **In each of the conservation equations except mass conservation, there are molecular diffusion/viscosity terms.**
- **Observations in the atmosphere indicate that the molecular diffusion terms are several orders of magnitude smaller than other terms and can be neglected.**

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = \varepsilon_{ij3} f \overline{u}_j - \frac{1}{\overline{\rho}} \frac{\partial \overline{p}}{\partial x_i} - \delta_{i3} g + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2} - \frac{\partial \overline{(u_i' u_j')}}{\partial x_j}$$

- **After making the hydrostatic approximation, all terms are of the same order of magnitude except the viscous term, which is** $O(1/Re) \approx 10^{-7}$ **time the others, except in the lowest few centimetres above the surface.**

Summary of mean flow equations 1

- Neglect molecular diffusion and viscosity and make the hydrostatic and Boussinesq approximations ⇒

$$\overline{p} = R\overline{\rho}\overline{T}_v$$

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}_j \frac{\partial \overline{u}}{\partial x_j} = -\varepsilon_{ij3} f(\overline{v}_g - \overline{v}_j) - \frac{\partial(\overline{u_j' u'})}{\partial x_j}$$

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}_j \frac{\partial \overline{v}}{\partial x_j} = +\varepsilon_{ij3} f(\overline{u}_g - \overline{u}) - \frac{\partial(\overline{u_i' v_j'})}{\partial x_j}$$

Summary of mean flow equations 2

$$\frac{\partial \overline{q}_T}{\partial t} + \overline{u}_j \frac{\partial \overline{q}_T}{\partial x_j} = \frac{\overline{S}_{q_T}}{\overline{\rho}} - \frac{\partial(\overline{u_j' q_T'})}{\partial x_j}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{u}_j \frac{\partial \overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho} c_p}\left[\frac{\partial \overline{Q}_j^*}{\partial x_i} + LE\right] - \frac{\partial(\overline{u_j' \theta'})}{\partial x_j}$$

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}_j \frac{\partial \overline{c}}{\partial x_j} = \overline{S}_c - \frac{\partial(\overline{u_j' c'})}{\partial x_j}$$

- Note the similarity in structure of the five prediction equations. The covariance terms that appear highlight the role of statistics in turbulent flow.

Summary of mean flow equations 3

- In the two momentum equations, the mean geostrophic wind components were defined using the mean horizontal pressure gradients:

$$\bar{u}_g = -\frac{1}{f\bar{\rho}}\frac{\partial \bar{p}}{\partial y}, \qquad \bar{v}_g = \frac{1}{f\bar{\rho}}\frac{\partial \bar{p}}{\partial x}$$

- We could write:

$$\frac{D(\;)}{Dt} \equiv \frac{\partial(\;)}{\partial t} + \bar{u}_j \frac{\partial(\;)}{\partial x_j}$$

where the total derivative D/Dt includes only the mean advection.

Examples

- Many applications must wait until more realistic PBL initial and boundary conditions have been covered.
- We examine here one or two artificial examples showing the use of the mean flow equations.

Problem 1

- Suppose that the turbulent flux decreases linearly with height according to $\overline{w'\theta'} = a - bz$, where $a = 0.3$ (K m s^{-1}) and $b = 3 \times 10^{-4}$ (K s^{-1}) .
- If the initial potential temperature profile is an arbitrary shape, then what will be the shape of the final profile one hour later? Neglect subsidence, radiation, latent heating, and assume horizontal homogeneity.

- **Solution:** **Neglecting subsidence, radiation, latent heating leaves:**

$$\frac{\partial\overline{\theta}}{\partial t}+\overline{u}\frac{\partial\overline{\theta}}{\partial x}+\overline{v}\frac{\partial\overline{\theta}}{\partial y}=-\frac{\partial(\overline{u'\theta'})}{\partial x}-\frac{\partial(\overline{v'\theta'})}{\partial y}-\frac{\partial(\overline{w'\theta'})}{\partial z}$$

- **Assuming horizontal homogeneity gives:**

$$\frac{\partial\overline{\theta}}{\partial t}=-\frac{\partial(\overline{w'\theta'})}{\partial z}$$

- **Substituting the expression for** $\overline{w'\theta'}$ **gives** $\partial\theta/\partial z = b$**.**
- **This is independent of** z**, so that air at each height in the sounding warms at the same rate. Integrating gives:**

$$\overline{\theta}(t)=\overline{\theta}(t_0)+b(t-t_0)$$

- **In one hour the warming is** 3×10^{-4} (K/s) $\times$ 3600 (s) =1.08 **K.**

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Problem 2

- **If a horizontal wind of** 10 **m s**$^{-1}$ **is advecting drier air into a region where the horizontal moisture gradient is** 5 **g water per kg of air per** 100 **km, then what vertical gradient of turbulent moisture flux in the BL is required to maintain a steady-state profile of specific humidity?**
- **Assume all the water is in vapour form, and that there is no body source of moisture. Be sure to state any additional assumptions you make.**

- **Solution:** **Steady-state** $\Rightarrow \partial(\;\;)/\partial t = 0 \Rightarrow$

$$\overline{u}\frac{\partial\overline{q}}{\partial x}+\overline{w}\frac{\partial\overline{q}}{\partial z}=-\frac{\partial(\overline{u'q'})}{\partial x}-\frac{\partial(\overline{v'q'})}{\partial y}-\frac{\partial(\overline{w'q'})}{\partial z}$$

Solution

- **No information was given about subsidence, or about horizontal flux gradients; therefore let's assume that they are zero for simplicity ⇒**

$$\bar{u}\frac{\partial \bar{q}}{\partial x} = -\frac{\partial(\overline{w'q'})}{\partial z}$$

$$\Rightarrow 10\ (\mathrm{ms^{-1}}) \times 5 \times 10^{-5}\ (\mathrm{g\ kg^{-1}\ m^{-1}}) = -\frac{\partial(\overline{w'q'})}{\partial z}$$

Thus $$\frac{\partial(\overline{w'q'})}{\partial z} = -5 \times 10^{-4}\ (\mathrm{g\ kg^{-1}\ s^{-1}})$$

- **A gradient of this magnitude corresponds to a** 0.5 ($\mathrm{g\ kg^{-1}\ m\ s^{-1}}$) **decrease of** $\overline{w'q'}$ **over a vertical distance of** 1 **km.**
- **Note a decrease of flux with height ⇒ a time increase of** $\bar{q}$**.**

Problem 3

- **Assume a turbulent BL at a latitude of** 44^{o}**N, where the mean wind is** 2 **m s^{-1} slower than geostrophic (i.e. the wind is subgeostrophic). Neglect subsidence and assume horizontal homogeneity and steady state conditions.**
- **(a) Find the Reynolds stress divergence necessary to support this velocity deficit.**
- **(b) If the stress divergence were related to molecular viscosity instead of turbulence, what curvature in the mean wind profile would be necessary?**

Solution: **(a) For simplicity, pick a coordinate system aligned with the stress ⇒**

Solution (a)

- Assuming horizontal homogeneity, steady state, and neglecting subsidence gives ⇒

$$0 = -f(\bar{v}_g - \bar{v}) - \frac{\partial \overline{(u'w')}}{\partial z}$$

$$-\frac{\partial \overline{(u'w')}}{\partial z} = f(\bar{v}_g - \bar{v}) \quad = 10^{-4}\,(s^{-1}) \times 2\,(ms^{-1})\,5 = 2 \times 10^{-4}\ m\ s^{-2}$$

Solution (b) The viscous stress term is expressed by $\nu\partial^2\bar{u}/\partial z^2$.

- Thus $\nu\partial^2 u/\partial z^2 = 2 \times 10^{-4}$ m s^{-2}.
- With $\nu = 1.5 \times 10^{-5}$ m^2 s^{-1} ⇒

$$\boxed{\frac{\partial^2 \bar{u}}{\partial z^2} = 13.33\ m^{-1} s^{-1}}$$

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Steady horizontally-homogeneous flow

Ignore temperature and moisture fluctuations

$$\cancel{\frac{\partial \bar{u}}{\partial t}} + \bar{u}_j \cancel{\frac{\partial \bar{u}}{\partial x_j}} = -f(\bar{v}_g - \bar{v}) - \cancel{\frac{\partial \overline{(u_j' u')}}{\partial x_j}} \rightarrow -\frac{\partial \overline{(u'w')}}{\partial z}$$

$$\cancel{\frac{\partial \bar{v}}{\partial t}} + \bar{u}_j \cancel{\frac{\partial \bar{v}}{\partial x_j}} = +f(\bar{u}_g - \bar{u}) - \cancel{\frac{\partial \overline{(u_j' v_j')}}{\partial x_j}} \rightarrow -\frac{\partial \overline{(v'w')}}{\partial z}$$

steady

horizontally-homogeneous

continuity $\quad \frac{\partial \bar{w}}{\partial z} = 0$

Steady horizontally-homogeneous flow

$$0 = -f(\overline{v}_g - \overline{v}) - \frac{\partial\overline{(u'w')}}{\partial z}$$

$$0 = +f(\overline{u}_g - \overline{u}) - \frac{\partial\overline{(v'w')}}{\partial z}$$

$$\frac{\partial\overline{w}}{\partial z} = 0$$

$$\mathbf{F}_3 = -\left(\frac{\partial\overline{(u'w')}}{\partial z}, \frac{\partial\overline{(v'w')}}{\partial z}\right)$$

Balance of forces