

An extended analytic theory of tropical-cyclone motion in a barotropic shear flow

By ROGER K. SMITH* and HARRY C. WEBER

University of Munich, Germany

(Received 17 September 1992; revised 15 March 1993)

SUMMARY

An earlier analytic theory for the motion of a barotropic vortex in a zonally varying basic shear flow on a beta-plane is consolidated and extended. It is shown that the theory can be derived from a power-series expansion of variables representing the vortex asymmetry in terms of a single nondimensional parameter $\varepsilon = BL^2/U$, where B is the meridional gradient of basic state absolute vorticity and L and U are the length and velocity scales, respectively, for the *outer part of the vortex*. The derivation requires *inter alia* that $\varepsilon \ll 1$ and that the magnitude of the basic flow scales with ε . Insight is provided into the dynamics of vortex motion when the latter condition is not fulfilled, as in cases of moderate and large shear.

For vortices that decay appreciably more rapidly with radius than those studied in the earlier work it is necessary to include additional terms when calculating the asymmetric vorticity to obtain an accurate prediction of the vortex track. Analytic expressions have been obtained for these terms that lead to a theory that is valid for a broad range of vortex profiles.

Comparisons are made between the present theory and two recent analyses for the case of zero basic flow, contrasting the important differences in approach. The present theory gives results that are very close to one of these, a semi-spectral numerical method.

The paper concludes with a brief discussion of aspects of vortex stability.

1. INTRODUCTION

The last few years have seen a major upsurge in the study of tropical-cyclone motion with particular emphasis on the dynamical processes involved. The majority of studies have started from the assumption that the essential processes are captured in a barotropic model for some depth-averaged flow. Although it is not yet clear to what extent this assumption is valid, it does find support in the comparative skill of barotropic forecast models (see DeMaria 1987 and references). Whatever the case, the study of vortex motion in barotropic models is insightful and is a necessary prerequisite for understanding motion in a baroclinic atmosphere.

Two important papers on the subject of tropical-cyclone motion are those of Kasahara and Platzman (1963) and DeMaria (1985). Both papers considered barotropic vortex motion on a beta-plane in the presence of a basic zonal shear flow and found that vortices are not simply advected by this flow, but drift across it; the drift velocity being determined, *inter alia*, by the absolute-vorticity gradient of the vortex environment.

Through numerical integration of a nondivergent barotropic model, Chan and Williams (1987) showed that, in the absence of a basic flow, a tropical-cyclone-scale vortex on a northern hemisphere beta-plane drifts towards the north-west with a speed of a few metres per second. They attributed this so-called beta-drift to the development of an asymmetry to the initial symmetric vortex. The asymmetry is such that the stream-function centre becomes displaced with respect to the vorticity centre so that there is a finite component of flow across the latter.

Fiorino and Elsberry (1989) highlighted the vortex asymmetry by removing a symmetric vortex from the stream-function field, obtained by taking the azimuthal average of the total stream function about the stream-function centre. They showed that the remaining stream-function asymmetry had the form of a pair of 'gyres', the flow between them, across the vortex centre, being responsible for the north-westward drift of the

* Corresponding author: Meteorological Institute, University of Munich, Theresienstr. 37, 80333 Munich 2, Germany.

vortex. The problem was investigated in more depth by Smith *et al.* (1990, henceforth SUD) who considered the simultaneous evolution of the asymmetry in both the relative vorticity and stream-function fields. In contrast to Fiorino and Elsberry, these authors adopted a method of partitioning the flow between the *vortex* and the *vortex environment* first suggested by Kasahara and Platzman (1963), in which the *vortex* is taken to be the initial symmetric vortex, suitably relocated, and the *environment* is the residual flow which includes the vortex-induced asymmetries. SUD were able to provide an explanation for the growth in scale and intensity of the asymmetry with time, and for the development of a fine-scale structure thereto in the inner region of the vortex. Similar calculations were presented also by Shapiro and Ooyama (1990) who investigated the role of horizontal divergence on vortex motion, and showed this to be negligibly small in their shallow-water model.

The additional effects of horizontal shear have been studied numerically by Ulrich and Smith (1991) and analytically by Smith (1991, henceforth S91) who used a non-divergent barotropic model, and by Shapiro and Ooyama (1990) and Evans *et al.* (1991) who used a divergent, shallow-water model.

Smith and Ulrich (1990, henceforth SU) showed that the Kasahara–Platzman method of partitioning facilitates the construction of an approximate analytic theory for the prototype problem with zero basic flow studied by Chan and Williams (1987) and others. The theory hinges on a finding of SUD that, to a good first approximation, the asymmetry in the relative-vorticity field can be calculated by assuming that air-parcel trajectories follow circular paths around the vortex centre. With this approximation, the asymmetry has a simple analytic form that consists of a pair of vortex dipoles of different strengths, their axes oriented at right-angles to each another. This asymmetry may be inverted analytically to obtain the corresponding stream-function asymmetry, enabling the asymmetric flow across the vortex centre to be determined. A closure assumption was made that the vortex centre, defined as the relative-vorticity maximum, moves with the same velocity as the asymmetric stream-flow across it. The resulting analytic expression for this velocity may be integrated with respect to time to provide an analytic expression for the subsequent vortex track. This simple solution, which we refer to as the zero-order solution, bears remarkable similarity to that obtained from a numerical integration of the full nonlinear equations, but the analytically calculated track is too far westward. However, an iteration about the zero-order solution produces a first-order correction to it that considerably improves the agreement with the numerical solution. Indeed, for the standard tangential wind profile used by SUD, excellent agreement was obtained over a 36–48 hour period between the vortex tracks calculated analytically and numerically. In contrast, for a narrower vortex profile, the good agreement lasted only about 12 hours, after which the analytically calculated and numerically calculated vortex tracks diverged. The reasons for this breakdown have been investigated by Weber and Smith (1993) and Ross and Kurihara (1992) in complimentary studies that shed further light on the analytic theory in particular, and on vortex motion in general. We return to these studies later.

The analytic theory of SU was extended by S91 to the case of vortex motion in a horizontal shear flow by using the same iterative technique. Again, for the standard vortex profile of SUD, excellent agreement was obtained for a 36–48 hour period between the tracks calculated analytically and numerically. The availability of an analytic theory provides deeper insight into the precise effects of shear on vortex motion, enabling the numerical solutions to be better understood. Unfortunately, the *ad hoc* nature of the iterative approach makes it difficult to assess the range of validity of the theory. It is clear, for example, that the theory must break down for sufficiently strong unidirectional shear, or indeed, after a sufficiently long period of time (S91, section 7), even in the

absence of shear. In both situations the central assumption of the theory that air-parcel trajectories follow nearly circular paths around the vortex centre becomes progressively invalid.

The present paper seeks, *inter alia*, to establish the range of validity of the theory through a scale analysis of the equations of motion in a suitably partitioned form that govern 'the vortex' and its 'environment'. The appropriate choice of scales was made possible by insights gained from the analytic theory itself. The scale analysis enables us to relate the analytic theory to the recent studies by Peng and Williams (1990) and Ross and Kurihara (1992). Furthermore, it enables us to pinpoint important problems that remain to be addressed.

In brief the paper is organized as follows: the scale analysis of the barotropic vorticity equation for a vortex in a horizontal shear flow is carried out in section 2, where the sequence of equations developed by S91 for approximations to the asymmetric vorticity field are derived for a particular setting of the parameters. Refinements to the analytic theory are described in section 3 where comparisons are made between the predicted vortex tracks in the basic thought experiment at different levels of approximation and the track calculated from a numerical-model integration. Comparisons are made also in section 3 between the analytic theory and the two other recent theories. The effects of horizontal shear are considered in section 4, where limitations of the analytic theory as the magnitude of the shear increases are investigated. Section 5 addresses briefly the question of vortex stability and the conclusions are presented in section 6.

2. SCALE ANALYSIS FOR A MOVING VORTEX

We consider the basic thought experiment in the theory of tropical-cyclone motion. This refers to the initial-value problem in which a symmetric vortex with tangential velocity profile $V(r)$ and corresponding relative-vorticity profile $\xi_v(r)$ is prescribed at time $t = 0$ on a beta-plane. Here r measures radial distance from the vortex centre. The subsequent flow is determined by the barotropic vorticity equation which, when expressed in a coordinate system moving* with the vortex takes the form

$$\partial_t \xi + \mathbf{u} \cdot \nabla \xi + \mathbf{u} \cdot \nabla f = -\mathbf{c} \cdot \nabla f \quad (2.1)$$

where $\mathbf{u} = \mathbf{v} - \mathbf{c}$ is the wind vector relative to this coordinate frame, \mathbf{c} is the velocity of the frame, \mathbf{v} is the wind vector in a fixed frame, $\xi = \mathbf{k} \cdot \nabla \wedge \mathbf{v}$ is the vertical component of relative vorticity, f is the Coriolis parameter, \mathbf{k} is the unit vector in the vertical and t is the time. Note that, in the moving frame, f is a function of time as well as of the meridional coordinate. We partition the total flow into a *mean* flow, denoted by an overbar, the *symmetric* vortex, characterized by the vorticity ξ_v and tangential wind \mathbf{u}_v , and the *residual* flow denoted by a prime. Then Eq. (2.1) can be separated into a pair of equations for the mean and residual flow. These equations are:

$$\partial_t \bar{\xi} + \bar{\mathbf{u}} \cdot \nabla \bar{\xi} + \bar{\mathbf{u}} \cdot \nabla f = -\mathbf{c} \cdot \nabla f \quad (2.2)$$

and

$$\partial_t \xi' + \mathbf{u}_v \cdot \nabla \xi' + \mathbf{u}_v \cdot \nabla (f + \bar{\xi}) = -\mathbf{u}' \cdot \nabla \xi_v - \mathbf{u}' \cdot \nabla \xi' - \mathbf{u}' \cdot \nabla (\bar{\xi} + f) - \bar{\mathbf{u}} \cdot \nabla \xi_v - \bar{\mathbf{u}} \cdot \nabla \xi' \quad (2.3)$$

respectively. Note that in this method of partitioning, where the vortex is defined as the initial vortex suitably relocated, $\mathbf{u}_v \cdot \nabla \xi_v = 0$, and the equation for vortex vorticity $\partial_t \xi_v + \mathbf{u}_v \cdot \nabla \xi_v = 0$ is trivially satisfied. Note also that the term $-\mathbf{c} \cdot \nabla f$ can be removed

* Note that this coordinate system need not be uniformly translating.

from (2.2) by rewriting $\bar{\zeta} = \zeta_m + \zeta_*$, where $\partial_t \zeta_* = -\mathbf{c} \cdot \nabla f = \beta c_2$, $c_2 = dy_c/dt$ is the northward speed and y_c is the northward displacement of the vortex. It follows immediately that $\zeta_* = -\beta y_c$, which represents a spatially uniform vorticity associated with the translation of the coordinate frame. While such a decomposition is seemingly neat, in practice it would appear to be simpler to solve the mean-flow equation in a stationary reference frame. For example, in the case of a steady zonal mean flow, Eq. (2.2) is trivially satisfied when expressed in a stationary frame, but in the moving frame $\bar{\mathbf{u}}$ is a function of both y and t .

Our plan is to carry out a scale analysis of Eqs. (2.2) and (2.3). We begin with the case of zero basic flow ($\bar{\mathbf{u}} = 0$) and consider first the total-vorticity equation (2.1). Calculations are carried out for latitude 12.5° where $\beta = 2.23 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$. Taking length and velocity scales L and U for the asymmetric vortex, and assuming an advective time-scale L/U , the nondimensional form of (2.1) contains the single nondimensional parameter $\varepsilon = \beta L^2/U$, (see, for example, Peng and Williams 1990). For the tropical-cyclone-scale wind profiles used by Chan and Williams (1987) and SUD (the profiles used by SUD and the corresponding vorticity profiles are shown in Fig. 1), ε increases in magnitude from 5×10^{-3} , when U is taken to be the maximum tangential wind speed and L to be the radius at which it occurs, to about 10^{-1} if L is taken to be the radius at which the wind speed has reduced to gale force ($U = 15 \text{ m s}^{-1}$). Since it is the outer wind profile that is all important for vortex motion (Fiorino and Elsberry 1989) and since the outer region of the vortex is where the vorticity asymmetry has coherent structure (SUD, Fig. 2), the choice of scales characterizing the outer wind profile would seem to be more appropriate. Taking $L = 300 \text{ km}$ and $U = 15 \text{ m s}^{-1}$ gives $\varepsilon = 0.13$. A related complication in scaling Eq. (2.1) arises from the fact that $d\zeta_v/dr$ varies by more than two orders of magnitude across the vortex. The variation of $(d\zeta_v/dr)/\beta$ and $(d\zeta_v/dr)/(V/r^2)$ with radius for the two vortex profiles in Fig. 1(a) are shown also in Fig. 1. Considering first the broad vortex, we note that $d\zeta_v/dr$ is comparable in order of magnitude with β at radii where the vorticity asymmetry has coherent structure (typically beyond 300 km); in fact $d\zeta_v/dr = 3.6\beta$ at $r = 300 \text{ km}$ and 1.9β at $r = 400 \text{ km}$. Thus, for this vortex we assume that $d\zeta_v/dr$ scales with β , although similar arguments might have been used to scale it with U/L^2 in this region (see later). We consider the scaling for the narrow vortex later.

When a mean flow is present it is appropriate to redefine ε as BL^2/U , where B is a characteristic value for the basic-state absolute-vorticity gradient $\nabla(\bar{\zeta} + f)$, and to scale $d\zeta_v/dr$ with B instead of β . We assume that the spatial variation of $\nabla(\bar{\zeta} + f)$ is small over the vortex scale L . We choose length and velocity scales L_e and U_e for the mean flow, scaling $\bar{\zeta}$ with U_e/L_e . Guided by the results of SU and S91, we assume that $\mathbf{c} - \bar{\mathbf{u}}_{0c}$ scales with εU , where $\bar{\mathbf{u}}_{0c}$ is the mean flow speed at the vortex centre, that ζ_v and ζ' scale with U/L and $\varepsilon U/L$, respectively, and that \mathbf{u}_v and \mathbf{u}' scale with U and εU . Finally, we assume that $U_e = \varepsilon U$ and $L_e = L/\varepsilon$. Then, with the foregoing considerations, we take

$$\mathbf{u}_v = U\mathbf{u}_0 \quad (2.4a)$$

$$\mathbf{u}' = U(\varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \dots) \quad (2.4b)$$

$$\zeta' = (U/L)(\varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \dots) \quad (2.4c)$$

$$\nabla\zeta_v = B\nabla\zeta_0 \quad (2.4d)$$

$$\mathbf{c} = U(\varepsilon\mathbf{c}_1 + \varepsilon^2\mathbf{c}_2 + \dots) \quad (2.4e)$$

$$\bar{\mathbf{u}} = \varepsilon U\bar{\mathbf{u}}_0 \quad (2.4f)$$

$$\nabla\bar{\zeta} + \beta\mathbf{j} = B(\nabla\bar{\zeta}_0 + \mathbf{j}). \quad (2.4g)$$

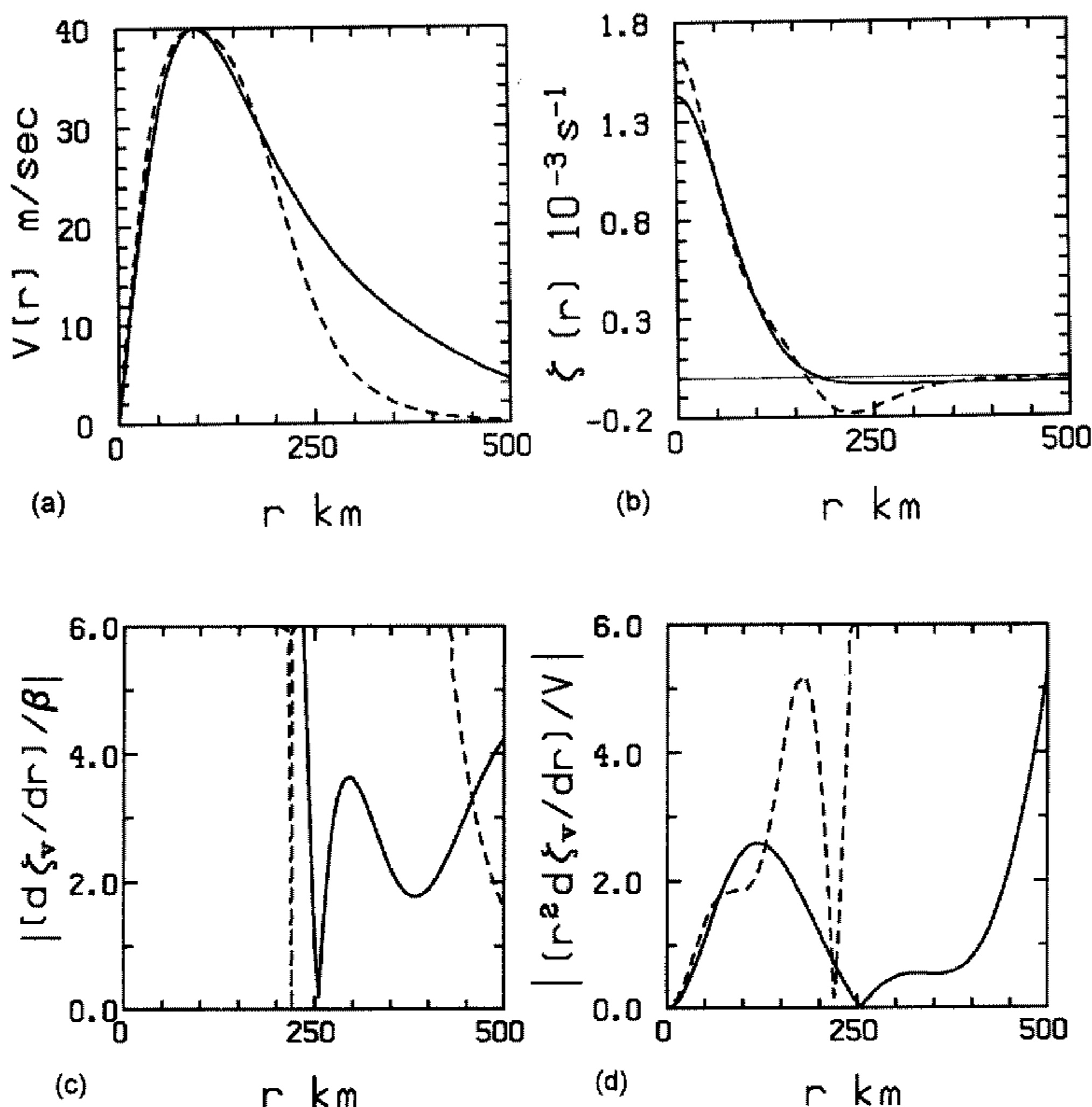


Figure 1. (a) Tangential wind profiles $V(r)$, and (b) vortex vorticity profiles $\zeta_v(r)$ for the broad vortex (solid lines) and the narrow vortex (dashed lines) used for the calculations herein. (c) and (d) Radial variation of the quantities $(d\zeta_v/dr)/\beta$ and $r^2(d\zeta_v/dr)/V(r)$, respectively, for the two profiles.

where \mathbf{j} is the unit vector pointing northwards. Substitution into (2.3) then gives to orders ε , ε^2 , ε^3 and ε^4 ,

$$\partial_t \zeta_1 + \mathbf{u}_0 \cdot \nabla \zeta_1 + \mathbf{u}_0 \cdot (\mathbf{j} + \nabla \bar{\zeta}_0) = 0 \quad (2.5)$$

$$\partial_t \zeta_2 + \mathbf{u}_0 \cdot \nabla \zeta_2 = -\mathbf{u}_1 \cdot \nabla \zeta_0 - \mathbf{u}_1 \cdot \nabla \zeta_1 - \mathbf{u}_1 \cdot (\mathbf{j} + \nabla \bar{\zeta}_0) - \bar{\mathbf{u}}_0 \cdot \nabla \zeta_0 - \bar{\mathbf{u}}_0 \cdot \nabla \zeta_1 \quad (2.6)$$

$$\partial_t \zeta_3 + \mathbf{u}_0 \cdot \nabla \zeta_3 = -\mathbf{u}_2 \cdot \nabla \zeta_0 - \mathbf{u}_2 \cdot \nabla \zeta_1 - \mathbf{u}_1 \cdot \nabla \zeta_2 - \mathbf{u}_2 \cdot (\mathbf{j} + \nabla \bar{\zeta}_0) - \bar{\mathbf{u}}_0 \cdot \nabla \zeta_2 \quad (2.7)$$

$$\partial_t \zeta_4 + \mathbf{u}_0 \cdot \nabla \zeta_4 = -\mathbf{u}_3 \cdot \nabla \zeta_0 - \dots \quad (2.8)$$

Equation (2.5) expresses mathematically the assumption made earlier that, to a first approximation, the vorticity asymmetry is determined by taking the absolute vorticity to be conserved following hypothetical air parcels moving in circular paths around the vortex; its solution has the simple mathematical form (S91, Eqs. (3.3)–(3.5))

$$\zeta_1 = \zeta_{1c}(r, t) \cos \bar{\theta} + \zeta_{1s}(r, t) \sin \bar{\theta} \quad (2.9)$$

where

$$\zeta_{1c}(r, t) = -Br \sin [\Omega(r)t] \quad (2.10a)$$

$$\zeta_{1s}(r, t) = -Br\{1 - \cos [\Omega(r)t]\} \quad (2.10b)$$

$\Omega(r) = V(r)/r$ and $\bar{\theta} = \theta + \theta_*$. The angle θ_* is that between the absolute-vorticity-gradient vector and north, measured clockwise from north.

The vortex velocity \mathbf{c} does not appear explicitly in (2.5). More generally, the contribution \mathbf{c}_n to \mathbf{c} at order ε^n does not appear in the equation for $\zeta_n (n > 1)$, but is determined by the requirement that, at this level of approximation, the relative flow across the vortex centre is zero. This is the second key assumption of the theory as it provides a closure of the equations at each order of ε , and allows the vortex track $\mathbf{x}_c(t)$ to be determined to any order of ε , since $d\mathbf{x}_c/dt = \mathbf{c}$.

The next-order correction is determined by calculating the rate of change of vorticity following similar hypothetical air parcels associated with:

- (i) the advection of the basic vortex vorticity by the lowest- or zero-order asymmetric flow (the term $-\mathbf{u}_1 \cdot \nabla \zeta_0$),
- (ii) the advection of the zero-order asymmetric vorticity by the zero-order asymmetric flow (the term $-\mathbf{u}_1 \cdot \nabla \zeta_1$),
- (iii) the advection of mean flow absolute vorticity by the zero-order asymmetric flow (the term $-\mathbf{u}_1 \cdot (\mathbf{j} + \nabla \bar{\zeta}_0)$),
- (iv) the advection of the zero-order vorticity field by the basic flow (the term $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_0$), and
- (v) the advection of the first-order vorticity field by the basic flow (the term $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_1$).

Note that the assumption that $\nabla \zeta_v$ is $O(B)$ brings terms involving $\nabla \zeta_0$ and $\nabla \zeta_1$ into the same equation, although higher-order terms $\nabla \zeta_n$ ($n = 2, 3 \dots$) appear in sequence. SU showed that the first of these terms contributes to the azimuthal wave-number 1 asymmetry and that, for the broader of the two vortex profiles they considered, it accounts for most of the difference between the numerically calculated vortex track and that obtained analytically based on the asymmetric stream function obtained by inverting (2.9). They showed that the second and third terms lead to corrections at azimuthal wave numbers 0 and 2 with no direct effect on the track, and hypothesized that their magnitude would be small. This was confirmed by S91; see section 5(b) therein. SU did not calculate any of the terms in Eqs. (2.7) and (2.8), assuming that these would be small also. As it turns out this assumption is valid for the broad vortex but, as shown by Ross and Kurihara (1992), the leading term on the right-hand side of Eq. (2.7) contributes to the wave-number 1 asymmetry also and, moreover, its inclusion accounts for most of the discrepancy between the analytically and numerically calculated tracks found by SU in the case of the narrow vortex. Ross and Kurihara used a numerical procedure to estimate the effect of this term. In section 3 and in the appendix we show how an analytic expression may be obtained, not only for the contribution of this term to the vorticity asymmetry, but also for its effect on the vortex track. S91 calculated the terms involving the basic flow in (iv) and (v) above and showed that their contribution to the vortex asymmetry at various wave numbers depends on the structure of the shear (see S91, Table 1).

Inspection of Fig. 1(d) suggests that it would be equally acceptable to scale $d\zeta_v/dr$ with U/L^2 , in which case the terms $-\mathbf{u}_n \cdot \nabla \zeta_0$ would need to be moved from the right-hand side of the equation for ζ_{n+1} in Eqs. (2.5)–(2.8) and placed on the left-hand

side of the equation for ζ_n . However, this would greatly complicate the solution of these equations, a complication that the calculations of SU and S91 have shown to be unnecessary.

The need to include additional terms in the theory for the case of the narrow vortex may be understood from the foregoing scale analysis. Choosing scales for U and L as before, it turns out that for this vortex $L = 237$ km when $U = 15$ m s⁻¹; then $\varepsilon = 0.08$, which is less than that for the broader vortex. However, in this case $|d\zeta_v/dr| = 1.3 \times 10^{-9}$ m⁻¹ s⁻¹ ($\sim 60\beta$) at $r = 300$ km, 2.1×10^{-10} m⁻¹ s⁻¹ ($\sim 9\beta$) at $r = 400$ km, and 1.2×10^{-9} m⁻¹ s⁻¹ ($\sim 54\beta$) at $r = L$. Clearly this places a strain on the assumption that $|\nabla\zeta_v|$ is $O(\beta)$ at radii where the beta-gyres have coherent structure. Indeed, it indicates that the terms involving $\nabla\zeta_0$ in Eqs. (2.6)–(2.8) should be elevated to equations of lower-order in ε , i.e. that (2.4d) should be rescaled. In fact, inspection of Figs. 1(c) and 1(d) suggests that U/L^2 might be a more appropriate scaling for $\nabla\zeta_v$ in this case, at least if $L = 237$ km! Despite this the results to be described show that the iterative procedure based on the sequential solution of (2.5)–(2.8) leads to an accurate solution for the vortex track in this case, provided that one includes the effects of higher-order terms of the type $-\mathbf{u}_n \cdot \nabla\zeta_0$. In fact we show later that it is necessary to calculate the effect of only one extra term, $-\mathbf{u}_2 \cdot \nabla\zeta_0$, in addition to $-\mathbf{u}_1 \cdot \nabla\zeta_0$, to achieve comparable accuracy in the vortex track with the calculation for the broad vortex. Evidently the iterative solution based on Eqs. (2.5)–(2.8) is more robust than the scale analysis would suggest.

3. REFINEMENTS TO THE ANALYTIC THEORY

The contributions to the vorticity asymmetry ζ_n arising from the terms $-\mathbf{u}_{n-1} \cdot \nabla\zeta_0$ ($n = 3, 4$) in Eqs. (2.7) and (2.8) have the form

$$\zeta_{n1} = \zeta_{nc}(r, t) \cos \bar{\theta} + \zeta_{ns}(r, t) \sin \bar{\theta} \quad (3.1)$$

where the functions ζ_{nc} , ζ_{ns} are given in the appendix. The remaining terms in Eq. (2.7) for the case of zero basic flow include wave-number 1 components also, but these are algebraically very complicated and have not been worked out. Their contribution would appear, however, to be small, as shown by the accuracy of the predicted vortex tracks when they are excluded (see below). Their effect when $\bar{\mathbf{u}}_0 = \mathbf{0}$ is included in the Fourier truncation wave-number expansion of Ross and Kurihara (1992).

The inclusion of higher-order terms in the analytic theory of SU can lead to a dramatic improvement in the extended-range calculation for vortex profiles that decay relatively rapidly with radius. This is illustrated in Fig. 2 which compares the analytically calculated tracks at various levels of approximation in ε with that from a full numerical calculation for the narrow and broad vortices studied by SU when there is no basic flow ($\bar{\mathbf{u}} = \mathbf{0}$). As in SU the numerical calculations were carried out in a 4000 km \times 4000 km domain with a grid resolution of 20 km \times 20 km and with the vortex initially at the centre of the domain. Figure 2(a) compares the calculated tracks for the narrow vortex for a 48-hour integration. As shown by SU, the lowest-order solution gives a track (A0) that from the initial instant is too far westward compared with the numerically calculated track (N), which is regarded as the control. The inclusion of the first-order correction gives track A1 which is a significant improvement, being almost identical with the control for the first 12 hours but deviating significantly therefrom at later times. At 48 hours the track error is over 80 km. As shown by Ross and Kurihara (1992) a considerable further improvement is obtained in this case by including the effects of the term $-\mathbf{u}_2 \cdot \nabla\zeta_0$ in Eq. (2.7). This leads to the track A2, whereby the 48-hour track error is reduced to only 25 km. Inclusion of the additional track correction arising from the

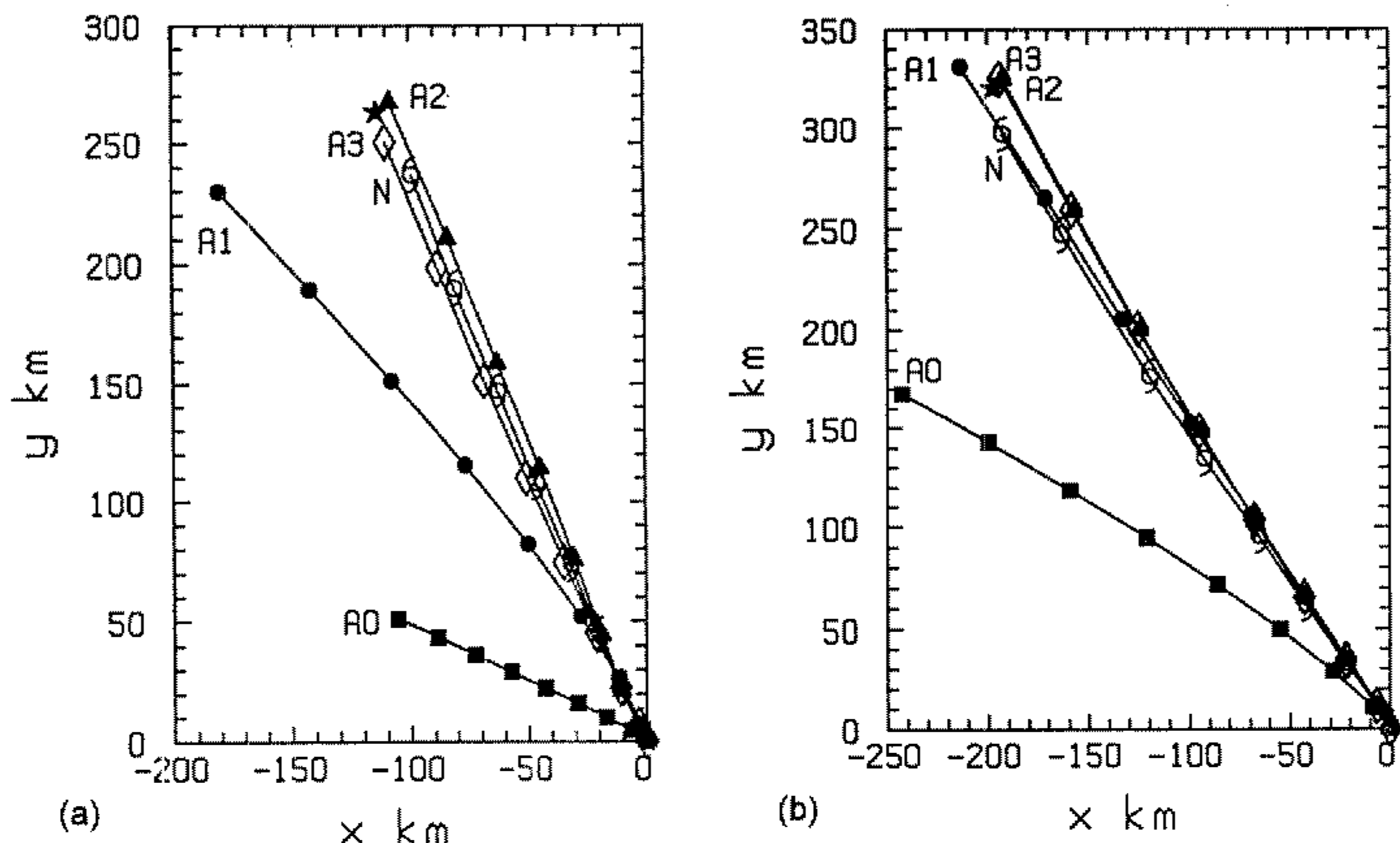


Figure 2. Analytically calculated vortex tracks at different levels of approximation in ε compared with the track of the corresponding numerical solution (denoted by N) in the case of zero basic flow; (a) is for the narrow vortex and (b) for the broad vortex in Fig. 1. Tracks A0 and A1 refer to the $O(\varepsilon)$ and $O(\varepsilon^2)$ theory, equivalent to that worked out by S91. Track A2 includes the effect of the first term on the right-hand side of Eq. (2.7), while track A3 includes this together with the first term on the right-hand side of Eq. (2.8). The 48-hour vortex position calculated by Ross and Kurihara (1992) is denoted by a star. The vortex positions are marked at 6-hour intervals by the following symbols: track N, cyclone symbol; track A0, solid squares; track A1, solid circles; track A2, solid triangles and track A3, diamonds.

term $-\mathbf{u}_3 \cdot \nabla \zeta_0$ in Eq. (2.8) gives the track A3 and halves the previous 48-hour track error to about 18 km, less than the grid size used in the control calculation. Details of the analytic calculations on which tracks A2 and A3 are based are given in the appendix. The numerically calculated vortex position obtained by Ross and Kurihara at 48 hours, shown in Fig. 2(a) by a solid star, agrees closely with our analytic calculation A2.

Figure 2(b) shows the corresponding track calculations for the broader vortex of SU. In this case the track A1 is a good approximation to the control for the full 48-hour period, the direction being almost identical but the displacement being 40 km too large. Track A2 has a 48-hour error of only 27 km, again close to the resolution of the control calculation, but in this case track A3 gives no further improvement. The relative accuracy of track A1 in this case compared with the case of the narrow vortex is consistent with the scale analysis presented in section 2. Note that, even for the broad vortex, the track correction at $O(\varepsilon)$ that incorporates the effects of the term and leads to the track A1 is not a small one—the difference between the tracks A0 and A1 is appreciable. This is consistent with the fact that β is numerically an underestimate for $|d\zeta_v/dr|$, and U/L^2 is an over-estimate in the outer part of the vortex (see Figs. 1(c) and 1(d)).

(a) Comparison with Ross and Kurihara's theory

At this point it is appropriate to note the difference in approach taken by Ross and Kurihara (1992) in comparison with our own. They begin by separating the symmetric and asymmetric contributions to the vorticity field in the barotropic vorticity equation,

equivalent to Eq. (2.3). They then expand the vorticity and velocity fields in a series of azimuthal wave-number components, the amplitude and phase of which are functions of radius and time. The radial dependence is discretized over a series of annular rings of radius r , and the time dependence of amplitude and phase in each ring is determined from equations derived from the symmetric and asymmetric components of the barotropic vorticity equation. The time integrations are carried out numerically. Different levels of approximation are obtained by truncating the equations at a certain azimuthal wave number. The lowest-order nontrivial calculation (denoted K1 by Ross and Kurihara) is obtained by truncating at wave number 1 while treating the symmetric vortex as time invariant. This corresponds with the numerical approximation to the zero-order solution (2.9) and the correction thereto, obtained by solving (2.6) and (2.7) with the wave-number 1 component of the forcing term $-\mathbf{u}_n \cdot \nabla \zeta_0$ ($n = 1, 2$) on the right-hand side. At the next order of truncation, denoted K01, the method picks up, *inter alia*, corrections to the symmetric velocity field arising from the term $-\mathbf{u}_1 \cdot \nabla \zeta_1$ in Eq. (2.6). This correction is taken into account when calculating, for example, the wave-number 1 asymmetry produced by the advection of absolute vorticity by the symmetric velocity component. As we have seen, they are important for an accurate calculation of the track in the case of the narrow vortex. Even so, at this level of truncation, Ross and Kurihara's scheme omits a contribution to the asymmetric vorticity from the term $\mathbf{u}_1 \cdot \nabla \zeta_0$ in Eq. (2.6) which, although of wave number 3, is comparable in magnitude with the one of wave number 1 that is retained (S91). It is evident from Ross and Kurihara's work that this term is of no consequence for the vortex track. Indeed, Ross and Kurihara's track K1 and that obtained from truncation at a wave number 2, denoted by K012, are both comparable with that of the refined analytic theory described earlier, and also with the numerically calculated track. Ross and Kurihara did not consider the effects of horizontal shear.

(b) *Comparison with Peng and Williams's theory*

Peng and Williams (1990) developed a similar power-series expansion to that in section 2, although again restricted to the case of zero basic flow. However, there are important differences between their approach and ours. While they transform the independent variables in the barotropic vorticity equation to a coordinate system moving with the vortex, they leave the dependent variables *in the stationary frame of reference*. As in section 2, they still expand the vorticity and tangential velocity as power series in ε in which, to zero order, the two quantities are dependent only on radius. This is consistent, however, only if the dependent variables are expressed in the moving coordinate system. In their case the zero-order vorticity should be a function also of time, satisfying the equation $\partial_t \zeta_0 + \mathbf{c} \cdot \nabla \zeta_0 = 0$, which unlike the situation in our case is not trivial (see section 2, below Eq. (2.3)). As it turns out this does not itself lead to an error in the calculation, but means that the vorticity tendency associated purely with the translation of the vortex, represented by term 3 in Eq. (2.8) of Peng and Williams, contributes to the calculated vortex asymmetry at wave number 1. In fact its structure is almost identical to the fields shown in SUD, Fig. 6. The presence of this contribution leads to complications in interpreting the vortex motion in terms of Peng and Williams's calculated stream-function asymmetries because, unlike the wave-number 1 contributions from other terms, this particular one must be regarded as a consequence of the motion rather than a cause thereof. Its presence would appear to account for much, if not all, of the inner gyre structures in many of Peng and Williams's figures.

Peng and Williams's theory differs from ours also in its lack of closure, whereupon the vortex speed and direction \mathbf{c} had to be determined from a separate nonlinear numerical calculation of the full initial-value problem. In addition, they adopted a scaling

in which the term $-\mathbf{u}_1 \cdot \nabla \zeta_0$ appeared at $O(\epsilon)$ in the power-series expansion of section 2 instead of at $O(\epsilon^2)$, thereby allowing for azimuthally propagating inertial wave modes in their analysis.

4. THE EFFECTS OF HORIZONTAL SHEAR

The effects of horizontal shear are basically twofold. Firstly, a non-uniform shear affects the zero-order vorticity asymmetry through its contribution to the absolute-vorticity gradient of the vortex environment, i.e. to the term $\mathbf{u}_0 \cdot (\mathbf{j} + \nabla \zeta_0)$ in Eq. (2.5). Secondly it produces distortion to the basic vortex vorticity and the vortex asymmetry, characterized in Eq. (2.6) by the terms $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_0$ and $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_1$, respectively. These terms were both calculated by S91 for uniform and linear zonal shear flows for the broader vortex in Fig. 1.

S91 compared both the analytically and numerically calculated vortex tracks in the foregoing cases and found very good agreement out to 48 hours. We present similar comparisons here for two cases of uniform shear (5 m s^{-1} per 1000 km: case US1, and 10 m s^{-1} per 1000 km: case US2) for both the broad and narrow vortex in Fig. 1, and investigate the effect of including one or both of the terms $-\mathbf{u}_2 \cdot \nabla \zeta_0$ in Eq. (2.7) and $-\mathbf{u}_3 \cdot \nabla \zeta_0$ in Eq. (2.8) in the track calculation. The results are shown in Fig. 3. The track A1 therein is that obtained from the $O(\epsilon^2)$ analysis (i.e. using only Eqs. (2.5) and (2.6) to calculate the vorticity asymmetry), equivalent to that worked out by S91. The track A2 includes the term $-\mathbf{u}_2 \cdot \nabla \zeta_0$ in Eq. (2.7), while in the track A3 the effect of the term $-\mathbf{u}_3 \cdot \nabla \zeta_0$ in Eq. (2.8) is included also.

In the case of the broad vortex (Fig. 3(a)), the inclusion of the additional terms has a minimal impact on the $O(\epsilon^2)$ analysis; in fact, in this case, the tracks A2 and A3 are slightly worse than A1. As in the case of zero basic flow, the analytically calculated tracks have a northward and eastward bias relative to the control, a difference that is not attributable to the finite domain size used to compute the numerical track*. In the presence of positive zonal shear, part of the eastward bias would be expected to be associated with the northward bias because a slightly larger northward component of vortex motion brings the vortex more rapidly into the increasing westerly flow.

The corresponding results for the narrow vortex are shown in Figs. 3(b) and 3(c). As in the case of zero basic flow, the inclusion of the additional term $-\mathbf{u}_2 \cdot \nabla \zeta_0$ in the analysis produces a significant improvement in the track compared with the $O(\epsilon^2)$ theory, whereas the inclusion also of the term $-\mathbf{u}_3 \cdot \nabla \zeta_0$ leads to only a minor additional improvement.

It is evident in both sets of calculations that the agreement between the analytically and numerically calculated tracks deteriorates as the magnitude of the shear increases. We explore now a possible reason for this deterioration.

In the scaling analysis of section 2 it was assumed that $U_e = \epsilon U$, i.e. that the shear is relatively weak. Under such circumstances the main effect of uniform shear is to produce a wave-number 2 contribution to the vorticity asymmetry (S91, Fig. 5) while a linear shear produces a wave-number 1 contribution (S91, Fig. 9). For a given magnitude of the shear, the validity of the assumption that, to a first approximation, air parcels move in circular paths about the vortex centre becomes restricted with increasing time to a progressively smaller region surrounding the vortex. It becomes similarly restricted in radius as the magnitude of the shear increases. In either case, with increasing shear

* It is known from calculations in the case of zero basic flow that the meridional displacement of the vortex slightly increases as the size of the computational domain increases (Fiorino and Elsberry 1989, p. 978).

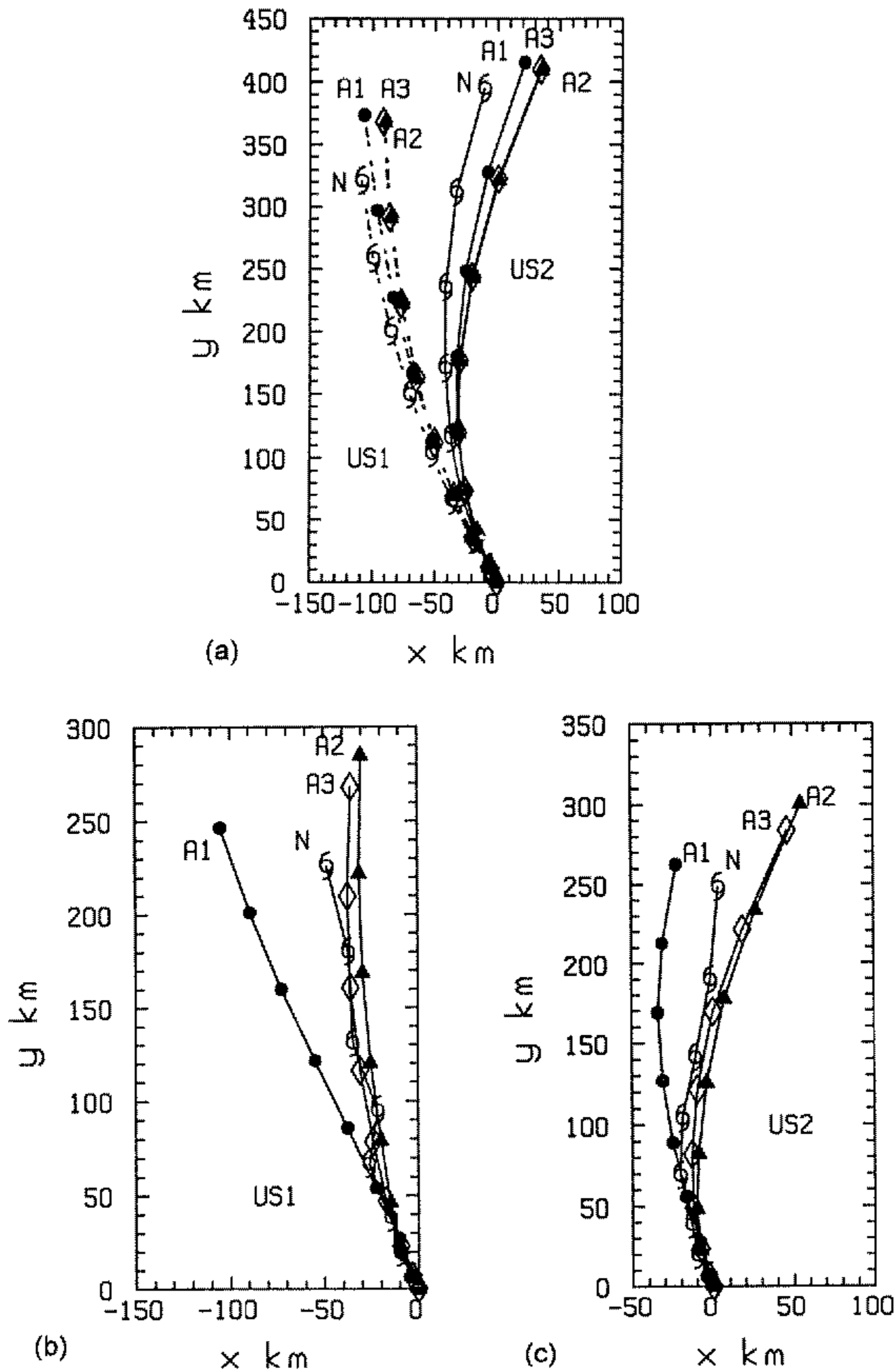


Figure 3. Equivalent track calculations to those in Fig. 2, but in the presence of a uniform zonal shear flow. (a) Tracks for the broad vortex for zonal shears of 5 m s^{-1} per 1000 km (leftmost tracks) and 10 m s^{-1} per 1000 km (rightmost tracks); (b) and (c) tracks for the narrow vortex for the smaller and larger shear, respectively.

or increasing time, the terms $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_0$ and $-\bar{\mathbf{u}}_0 \cdot \nabla \zeta_1$ in Eq. (2.6), which represent the distortion of the basic vortex and the zero-order vorticity asymmetry, respectively, by the environmental flow, become elevated in importance in the power-series expansion in section 2. In the case of uniform shear, a consequence is that the vorticity asymmetry is drawn out in the direction of the shear to a degree that it can be no longer represented

by a wave-number 2 asymmetry (see Smith and Ulrich 1993, Fig. 2). In essence, beyond a certain radius, the motion is closer to rectilinear than to circular.

In view of the success of the analytic theory in capturing the dynamics of the case of relatively weak shear, the question naturally arises: can it be extended to the case of moderate shear? In the former case, the zero-order vortex asymmetry was calculated on the assumption that air-parcel trajectories are circular about the vortex centre. For the case of a more strongly sheared flow, the relevant question seems to be: how accurately can the vortex asymmetry be calculated by ignoring the asymmetric flow that this induces? To explore this question we carried out a calculation for an initially symmetric vortex on an *f*-plane. In this problem, the vortex is simply carried along by the value of the shear flow at its centre and has no meridional component of motion. At a given time after the initial instant we calculated the back trajectories from a set of points on a uniform grid on the assumption that the total flow at any time is simply the sum of the imposed shear flow and the initial vortex. The trajectory equations were carried out using a standard fourth-order Runge–Kutta algorithm. Assuming that absolute vorticity is conserved along each trajectory, the vorticity field at time *t*, and therefore the vortex-induced asymmetry, can be calculated from the knowledge of the initial field.

Figure 4 compares the vorticity asymmetry at 24 and 48 hours calculated from the analytic theory, the back-trajectory method and from a full numerical solution of the barotropic vorticity equation for a uniform shear of 10 m s^{-1} per 1000 km (10^{-5} s^{-1}) on an *f*-plane. While there are still some small, but noticeable, differences in detail between the back-trajectory method and the full numerical solution at 24 hours, the former is generally more accurate than the analytic solution. Indeed the back-trajectory method retains considerable skill even at 48 hours. Evidently, it is reasonable to neglect the asymmetric flow to a first approximation when calculating the parcel trajectories. However, for moderate shear, it is inaccurate to ignore the effect of the mean flow when doing this. While the back-trajectory method is not an analytic theory, and is not proposed as a replacement for the numerical method, its use here helps us to identify the prime cause for the breakdown of the analytic theory as the shear increases.

Numerical experiments show that large shear leads to the rapid demise of a vortex as a coherent entity. An example of this effect was illustrated by SUD in a calculation in which a smaller and weaker vortex lies in close proximity to a larger and stronger one. In this simulation, the weaker vortex was rapidly destroyed by the large azimuthal shear of the stronger one. Presumably a strong horizontal deformation field would have a similar dramatic effect. As far as we are aware, the effect of a deformation field on vortex motion has not been studied at all. SUD investigated other cases of multiple vortex interaction numerically. Such problems pose a challenge to analytical treatment because the assumption that the absolute-vorticity gradient be slowly varying on the scale of each vortex is unlikely to be valid in cases of interest.

5. VORTEX STABILITY

The starting point of the analytic theory described in sections 2 and 3 is that, to a first approximation, the asymmetric vorticity field can be calculated on the assumption that air-parcel trajectories are circular about the moving vortex core. This assumption suppresses any azimuthally propagating wave modes, including unstable ones, that might otherwise exist. The linear theory of such modes would be described by Eq. (2.5) if the additional term $-\mathbf{u}_1 \cdot \nabla \zeta_0$ from Eq. (2.6) were included therein. Calculations by Weber and Smith (1993) show that no such waves exist for the broader vortex of Fig. 1, but an unstable wave of azimuthal wave number 2 exists for the narrow vortex. The e-folding

growth rate of the latter wave is approximately 7.4 hours. However, Weber and Smith show that its existence is of no consequence for the vortex motion in a 48-hour track forecast; being wave number 2, it would first have to generate a wave-number 1 asymmetry through nonlinear interaction with other modes to affect the vortex motion.

One particular result of the linear stability analysis is the existence of a non-rotating, neutrally stable mode of wave number 1. The stream-function perturbation associated with this mode has a radial structure proportional to* $V(r)$ and its radial scale is therefore the radius of the maximum tangential wind speed. The mode corresponds to the asymmetry that arises from a small horizontal displacement of the vortex, and its ubiquity is a consequence of the fact that the symmetric vortex is a solution of the equations of motion expressed in polar coordinates, even when it is not centred at the origin of these coordinates. It follows that the mode can always be removed by redefining the vortex centre, whereupon it might be regarded as a 'pseudo-mode' rather than a physical mode of the system.

A number of authors (e.g. Willoughby 1988, 1990; Peng and Williams 1990, 1991) have considered the possible influence of this mode on vortex motion. While its presence is usually evident in the numerical calculations (e.g. Fiorino and Elsberry 1989; Shapiro and Ooyama 1990; SUD), it is reasonable to presume, based on the foregoing discussion, that this is merely a result of inaccuracy in the algorithm used to determine the vortex centre (defined as the relative-vorticity maximum). To check this we calculated the residual stream-function field, obtained by subtracting the analytically calculated field from that calculated numerically for the broad and narrow vortex in the case of zero basic flow. The results are exemplified by the residual fields at 24 hours shown in Figs. 5(a) and 5(b) for each of these simulations. The inner-gyre structure is a prominent feature of the residual field in the case of the narrow vortex and has the structure of the pseudo-mode. As shown in Fig. 5(c) these gyres can be removed by displacing the vortex a small distance (746 m) to the south-west. We are led to conclude that the pseudo-mode is of no dynamical consequence for vortex motion.

6. CONCLUSIONS

We have extended the analytic theory of S91 so that it provides an accurate representation of the motion of tropical-cyclone-scale vortices with a broad range of sizes. We have shown also how it relates to other recent theories. The theory can be derived as a power-series expansion in the single small nondimensional parameter ε , provided that the environmental shear is sufficiently weak. As the magnitude of the shear increases, the analytic theory becomes progressively more inaccurate at any given time, at least to the level of approximation that the algebra is manageable. Despite this limitation, the availability of the theory provides much insight into the dynamics of vortex motion in a barotropic framework, and assists in the interpretation of numerical calculations.

The scale analysis indicates that for moderate shear, or for weak shear at longer times, it is necessary to include the advective effect of the ambient flow when calculating the zero-order vorticity asymmetry. We have illustrated the improvement in the structure

* This is true for any vortex profile $V(r)$; this 'pseudo-mode' appears to have been first discovered by Michalke and Timme (1967).

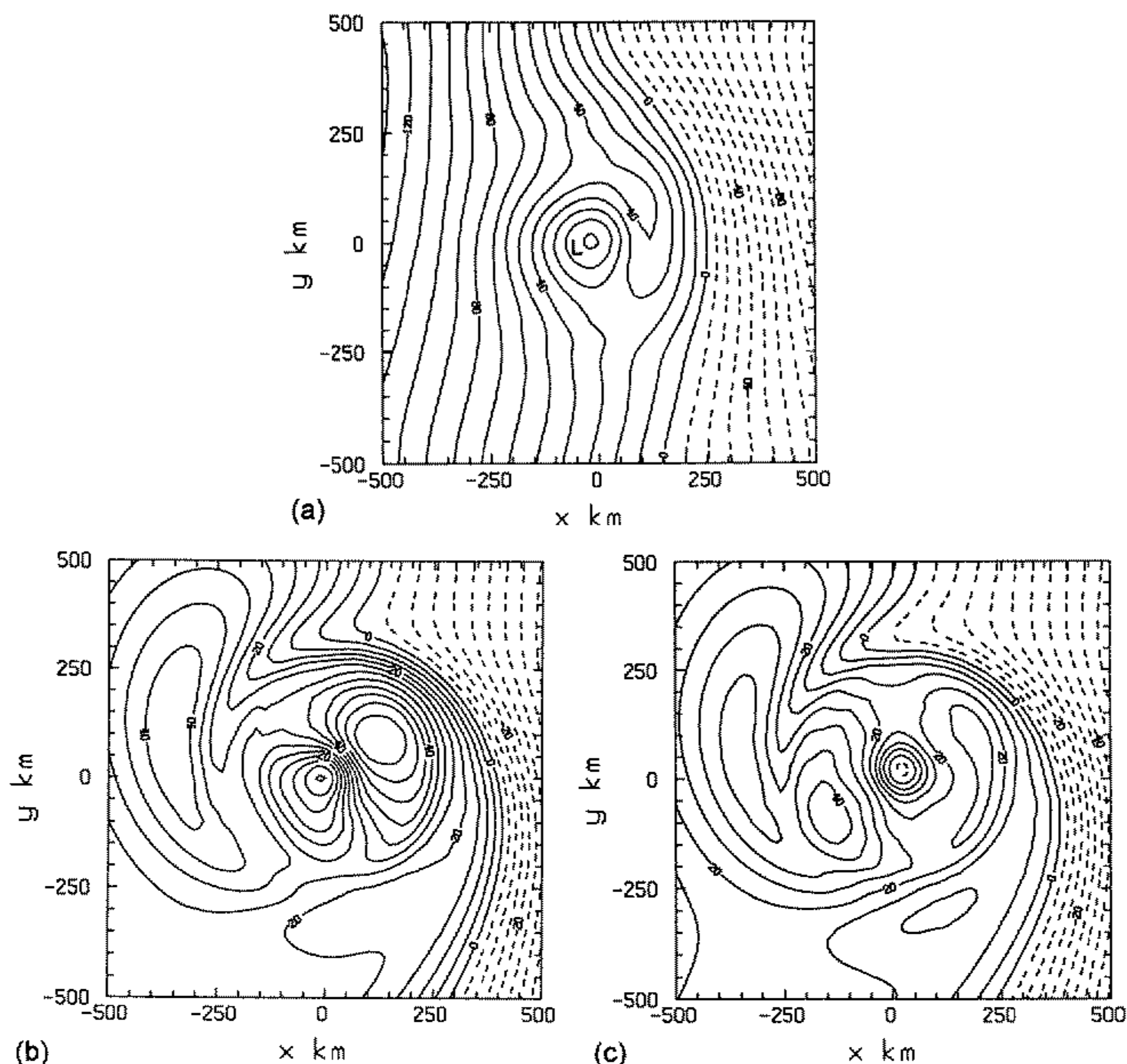


Figure 5. Residual stream-function fields after 24 hours obtained by subtracting the analytically calculated asymmetric stream-function fields from the corresponding numerically calculated fields. A constant value has been subtracted from the residual field so that the zero contour passes through the vortex centre. (a) Broad vortex, contour interval $10^4 \text{ m}^2 \text{ s}^{-1}$; (b) narrow vortex, contour interval $5 \times 10^3 \text{ m}^2 \text{ s}^{-1}$ (the inner gyres result from a relatively small inaccuracy in the centre location in the numerical model); and (c) as in (b) but with the accurate centre position 746 m south-west of the centre in (b). Note that the inner-gyre structure is no longer visible.

of this asymmetry that is obtained in the case of a vortex in uniform shear on an f -plane. Unfortunately, the calculations then become intractable to analytic development. In the extreme case of large ambient shear, the vortex is rapidly distorted by the shear and eventually loses its coherence as a quasi-symmetrical entity.

A notable difference between the analytically calculated and the numerically calculated vorticity and stream-function asymmetries is the existence of small-scale inner gyres in the latter. These have the structure of the neutrally stable, non-rotating pseudo-mode that arises in the solution of the linear stability analysis for the vortex and corresponds with a displacement of the vortex centre. We have shown that the amplitude of the inner gyres is consistent with a displacement of the vortex centre by less than one twentieth of the grid size used to obtain the numerical solution. We argue that the existence of these gyres in the numerical model is *purely* a result of inaccuracy of the vortex centre-finding algorithm and that the pseudo-mode is of no *dynamical* consequence in vortex motion, a view that is contrary to that expressed in some other recent studies.

ACKNOWLEDGEMENTS

We would like to express our sincere thanks to our colleague, Wolfgang Ulrich, for allowing us to use his barotropic model to calculate the numerical tracks in Figs. 2 and 3. We are grateful also to Drs Terry Williams and Lloyd Shapiro for their perceptive comments on an earlier version of the manuscript. Finally we acknowledge financial support from the German Research Council and the US Office of Naval Research through Contract No. N00014-90-J-1487.

APPENDIX

Calculations of $\xi_{nc}(r, t)$, $\xi_{ns}(r, t)$, in Eq. (3.1)

The calculation of $\xi_{nc}(r, t)$, $\xi_{ns}(r, t)$ uses the same procedure as SU and S91. Having calculated the asymmetric stream-function contribution $\psi_{21c} = \psi_{21c}(r, t) \cos \bar{\theta} + \psi_{21s}(r, t) \sin \bar{\theta}$ associated with the wave-number 1 component of ξ_2 , the eastward (x) and northward (y) components of \mathbf{u}_2 can be written as

$$u_2 = \left[\frac{\psi_{21c}}{r} - \frac{\partial \psi_{21c}}{\partial r} \right] \sin \bar{\theta} \cos \bar{\theta} - \frac{\partial \psi_{21s}}{\partial r} \sin^2 \bar{\theta} - \frac{\partial \psi_{21c}}{\partial r} \cos \bar{\theta} - c_{2x} \quad (\text{A.1})$$

and

$$v_2 = \frac{\partial \psi_{21c}}{\partial r} \cos \bar{\theta} + \frac{\psi_{21c}}{r} \sin^2 \bar{\theta} - \left[\frac{\psi_{21c}}{r} - \frac{\partial \psi_{21s}}{\partial r} \right] \sin \bar{\theta} \cos \bar{\theta} - c_{2y} \quad (\text{A.2})$$

respectively, where c_{2x} , c_{2y} are the x and y components of \mathbf{c}_2 . The drift-speed correction \mathbf{c}_2 is determined by

$$\mathbf{c}_2 = \left[- \frac{\partial \psi_{21s}}{\partial r} \Big|_{r=0}, \frac{\partial \psi_{21c}}{\partial r} \Big|_{r=0} \right] \quad (\text{A.3})$$

parallel to the calculation of \mathbf{c} in SU.

From (2.7) it follows that the contribution ξ_{31} from the first term on the right-hand side is given by

$$\frac{d\xi_{31}}{dt} = -\mathbf{u}_2 \cdot \nabla \xi_0 \quad (\text{A.4})$$

where $d/dt = \partial/\partial t + \mathbf{u}_0 \cdot \nabla$. As detailed in S91, this can be integrated with respect to time to give Eq. (3.1) for $n = 3$. For $\kappa = c, s$ we obtain

$$\xi_{31\kappa} = \frac{\beta t^2}{4} \frac{d\xi_0}{dr} \int_0^r \left[1 - \frac{p^2}{r^2} \right] \frac{d\xi_0}{dp} \int_0^p q \left[1 - \frac{q^2}{p^2} \right] \eta_\kappa(r, p, q, t) dq dp \quad (\text{A.5})$$

with

$$\eta_c = \frac{\sin \Omega_r - \sin \Omega_q}{(\Omega_r - \Omega_q)(\Omega_p - \Omega_q)} - \frac{\Omega_q (\sin \Omega_r - \sin \Omega_p)}{\Omega_p (\Omega_r - \Omega_p)(\Omega_p - \Omega_q)} - \frac{\sin \Omega_r}{\Omega_r \Omega_p} \quad (\text{A.6})$$

$$\eta_s = -\frac{\cos \Omega_r - \cos \Omega_q}{(\Omega_r - \Omega_q)(\Omega_p - \Omega_q)} + \frac{\Omega_q (\cos \Omega_r - \cos \Omega_p)}{\Omega_p (\Omega_r - \Omega_p)(\Omega_p - \Omega_q)} - \frac{1 - \cos \Omega_r}{\Omega_r \Omega_p} \quad (\text{A.7})$$

and $\Omega_r = \Omega(r)t$, $\Omega_p = \Omega(p)t$ and $\Omega_q = \Omega(q)t$. The integrals in Eq. (A.5) can be evaluated easily using quadrature. The corresponding stream function $\psi_{31} = \psi_{31c}(r, t) \cos \bar{\theta} + \psi_{31s}(r, t) \sin \bar{\theta}$ is calculated using

$$\psi_{31\kappa} = \frac{r}{2} \int_0^r \left(1 - \frac{p^2}{r^2}\right) \xi_{31\kappa}(p, t) dp - \frac{r}{2} \int_0^\infty \xi_{31\kappa}(r, t) dr, \quad (\kappa = c, s) \quad (\text{A.8})$$

from which the corrections of the drift speed and the track of the vortex are computed following SU.

The procedure to calculate ξ_{41} follows exactly the one described above, but now for given ψ_{31} , to yield Eq. (3.1) for $n = 4$ with

$$\begin{aligned} \xi_{41c} = & \frac{\beta t^3}{8} \frac{d\xi_0}{dr} \left\{ \int_0^r a(r, p) (\sin \Omega_r - \sin \Omega_p) \int_0^p b(p, q) \int_0^q e(q, s) f(r, p, q, s) ds dq dp - \right. \\ & - \int_0^r a(r, p) \int_0^p b(p, q) (\sin \Omega_r - \sin \Omega_q) \int_0^q e(q, s) g(r, p, q, s) ds dq dp + \\ & + \int_0^r a(r, p) \int_0^p b(p, q) \int_0^q e(q, s) h(r, p, q, s) (\sin \Omega_r - \sin \Omega_s) ds dq dp - \\ & \left. - \sin \Omega_r \int_0^r a(r, p) \int_0^p b(p, q) k(r, p, q) \int_0^q e(q, s) ds dq dp \right\} \quad (\text{A.9}) \end{aligned}$$

$$\begin{aligned} \xi_{41s} = & \frac{\beta t^3}{8} \frac{d\xi_0}{dr} \left\{ \int_0^r a(r, p) \int_0^p b(p, q) (\cos \Omega_r - \cos \Omega_q) \int_0^q e(q, s) g(r, p, q, s) ds dq dp - \right. \\ & - \int_0^r a(r, p) (\cos \Omega_r - \cos \Omega_p) \int_0^p b(p, q) \int_0^q e(q, s) f(r, p, q, s) ds dq dp - \\ & - \int_0^r a(r, p) \int_0^p b(p, q) \int_0^q e(q, s) h(r, p, q, s) (\cos \Omega_r - \cos \Omega_s) ds dq dp - \\ & \left. - (1 - \cos \Omega_r) \int_0^r a(r, p) \int_0^p b(p, q) k(r, p, q) \int_0^q e(q, s) ds dq dp \right\} \quad (\text{A.10}) \end{aligned}$$

and

$$a = \left(1 - \frac{p^2}{r^2}\right) \frac{d\xi_0}{dp}, \quad b = \left(1 - \frac{q^2}{p^2}\right) \frac{d\xi_0}{dq}, \quad e = s \left(1 - \frac{s^2}{q^2}\right),$$

$$f = \frac{1}{\Omega_{rp}} \left[\frac{\Omega_s}{\Omega_q \Omega_{pq} \Omega_{qs}} - \frac{\Omega_s}{\Omega_{ps} \Omega_{qs}} + \frac{1}{\Omega_p \Omega_q} \right],$$

$$g = \frac{\Omega_s}{\Omega_q \Omega_{rq} \Omega_{pq} \Omega_{qs}}, \quad h = \frac{1}{\Omega_{rs} \Omega_{ps} \Omega_{qs}}, \quad k = \frac{1}{\Omega_r \Omega_p \Omega_q},$$

with the same abbreviations as above and where $\Omega_{rp} = [\Omega(r) - \Omega(p)]t$, $\Omega_{rq} = [\Omega(r) - \Omega(q)]t$, etc. The resulting stream function $\psi_{41} = \psi_{41c}(r, t) \cos \bar{\theta} + \psi_{41s}(r, t) \sin \bar{\theta}$, the correction of drift speed and track are calculated in the same way as explained above.

REFERENCES

- Chan, J. C. and Williams, R. T. 1987 Analytical and numerical studies of the beta-effect on tropical cyclone motion. Part I: Zero mean flow. *J. Atmos. Sci.*, **44**, 1257–1265
- DeMaria, M. 1985 Tropical cyclone motion in a nondivergent barotropic model. *Mon. Weather Rev.*, **113**, 1199–1210
- 1987 Tropical cyclone track prediction with a barotropic spectral model. *Mon. Weather Rev.*, **115**, 2346–2357
- Evans, J. L., Holland, G. L. and Elsberry, R. L. 1991 Interactions between a barotropic vortex and an idealized subtropical ridge. Part I: Vortex motion. *J. Atmos. Sci.*, **48**, 301–314
- Fiorino, M. and Elsberry, R. L. 1989 Some aspects of vortex structure related to tropical cyclone motion. *J. Atmos. Sci.*, **46**, 975–990
- Kasahara, A. and Platzmann, G. W. 1963 Interaction of a hurricane with a steering field and its effect upon the hurricane trajectory. *Tellus*, **15**, 321–335
- Michalke, A. and Timme, A. 1967 On the inviscid instability of certain two-dimensional vortex-type flows. *J. Fluid Mech.*, **29**, 647–666
- Peng, M. S. and Williams, R. T. 1990 Dynamics of vortex asymmetries and their influence on vortex motion on a β -plane. *J. Atmos. Sci.*, **47**, 1987–2003
- 1991 Stability analysis of barotropic vortices. *Geo. Astro. Fluid Dyn.*, **58**, 263–283
- Ross, R. J. and Kurihara, Y. 1992 A simplified scheme to simulate asymmetries due to the beta effect in barotropic vortices. *J. Atmos. Sci.*, **49**, 1620–1628
- Shapiro, L. J. and Ooyama, K. V. 1990 Barotropic vortex evolution on a beta plane. *J. Atmos. Sci.*, **47**, 170–187
- Smith, R. K. 1991 An analytic theory of tropical-cyclone motion in a barotropic shear flow. *Q. J. R. Meteorol. Soc.*, **117**, 685–714
- Smith, R. K. and Ulrich, W. 1990 An analytic theory of tropical-cyclone motion using a barotropic model. *J. Atmos. Sci.*, **47**, 1973–1986
- 1993 Vortex motion in relation to the absolute vorticity gradient of the vortex environment. *Q. J. R. Meteorol. Soc.*, **119**, 207–215
- Smith, R. K., Ulrich, W. and Dietachmayer, G. 1990 A numerical study of tropical-cyclone motion using a barotropic model. I: The role of vortex asymmetries. *Q. J. R. Meteorol. Soc.*, **116**, 337–362
- Ulrich, W. and Smith, R. K. 1991 A numerical study of tropical-cyclone motion using a barotropic model. II: Spatially-varying large-scale flows. *Q. J. R. Meteorol. Soc.*, **117**, 107–124
- Weber, H. and Smith, R. K. 1993 The stability of barotropic vortices: implications for tropical cyclone motion. *Geo. Astro. Fluid. Dyn.* (in press)
- Willoughby, H. E. 1988 Linear motion of a shallow-water, barotropic vortex. *J. Atmos. Sci.*, **45**, 1906–1928
- 1990 Linear normal modes of a moving, shallow-water, barotropic vortex. *J. Atmos. Sci.*, **47**, 2141–2148