Translation of Balanced Air Mass Models of Fronts and Associated Surface Pressure Changes

ROGER K. SMITH

Meteorologisches Institut, Universität München, München, Federal Republic of Germany

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ABSTRACT

The problem of explaining the surface pressure rise in simple balanced models of fronts, discussed at length by Sutcliffe, is reexamined. It is shown that air mass models for steadily translating fronts (including the Margules' front) are dynamically consistent, except along a vertical line above the surface front, only if there is vertical motion (subsidence for a cold front, ascent for a warm front) in the warm air that overlies the cold air. In this case, the local post-frontal pressure rise in a model cold front and the pre-frontal pressure fall in a model warm front can be attributed to advection. However, the presence of the vertical motion is a limiting factor in the applicability of such models.

The analysis resolves an apparent inconsistency between the surface pressure changes computed in Boussinesq models and the prediction of a theorem of Brunt.

Irrespective of the Boussinesq approximation, it is shown that, in the model, the surface pressure change at any fixed location bears no relation to the variation of surface pressure normal to the front at any given instant. This would imply that it is inappropriate to infer space cross-sections of pressure from observed time series at a single station, even for a steadily translating front. The result highlights a further limitation of balanced air mass models when applied to fronts in the atmosphere.

1. Introduction

The earliest and perhaps the simplest model for a front is that of Margules (1906) in which the front is considered to be a stationary sloping discontinuity separating two air masses of uniform, but different temperatures. The motion in these air masses is assumed to be geostrophic and parallel with the surface front, the vertical motion being everywhere zero. Frictional and diffusive processes are excluded. With these assumptions one obtains a diagnostic equation relating the slope of the front to the difference in mass flux and difference in density, or temperature, between the two air masses [see Eq. (5) below]. In essence, this relationship is an expression of thermal wind balance across the front.

In the author's experience, there is a widespread belief that Margules' solution may be extended to moving fronts, cold or warm fronts, simply by incorporating a uniform geostrophic wind component normal to the surface front. However, there are immediate difficulties with this approach. Basically, the presence of rotation precludes the introduction of a Galilean coordinate transformation in which the dynamics represented by Margules' solution is preserved, while the front translates at uniform speed c normal to its line of intersection with the ground. This translation velocity would lead to an unbalanced Coriolis torque, fc, in the along-front direction, where f is the Coriolis parameter. Even if we postulate the existence of an along-front pressure gradient in geostrophic balance with this torque, in which case the surface isobars no longer remain parallel with the front, it is still not possible to explain the translation of the front (Sawyer 1952, p. 170).

The foregoing problem was elucidated by Sutcliffe (1938) who noted that, in a geostrophic flow $\mathbf{u}_g$ over level ground with f = const, the surface pressure field p_s cannot change locally. This result, which actually dates back to Jeffreys (1919), follows from the pressure tendency equation

$$\frac{\partial p_s}{\partial t} = -g \int_0^\infty \nabla_h \cdot (\rho \mathbf{u}_h) dz + g\rho w_\infty, \qquad (1)$$

where ρ is the air density, $\mathbf{u}_h$ is the horizontal wind vector, w_∞ is the vertical velocity component at large heights and g is the acceleration due to gravity. Under the stated conditions, $\nabla_h \cdot (\rho \mathbf{u}_h) = \nabla_h \cdot (\rho \mathbf{u}_g) = 0$ and $w_\infty = 0$, in which case $\partial p_s/\partial t = 0$. Brunt (1939, pp. 308–309) showed that this is true even when there is an air mass discontinuity and it would appear that a geostrophically balanced translating front could in no way explain the surface pressure changes normally observed with frontal passages. Significantly, Brunt's analysis assumes that there is zero vertical motion on both sides of the discontinuity, but it applies, neverthe-

Corresponding author address: Prof. Roger K. Smith, Meteorologisches Institut, Uiversität München, Theresienshr. 37, 8000 München 2, FRG.

less, to the basic Margules' model. We show below that by relaxing this constraint it is possible to construct a more dynamically consistent extension of Margules' model to a translating front. We consider also the consequences of making the Boussinesq approximation wherein the foregoing difficulties are swept under the carpet.

The solutions for translating fronts, while no more realistic than those for stationary fronts, are of interest from a theoretical and historical standpoint and highlight the limitations of balanced air mass models[1] as a whole in describing the behavior of fronts in the atmosphere.

2. The translating Margules' model

The inconsistency in the naive extension of Margules' solution to a translating front described above is clearly exposed by considering the surface pressure distribution. Let us choose a coordinate system (x, y, z) with z vertical and let the surface front lie along the y-axis at time $t = 0$ with the cold air occupying the region $x < 0$, $0 < z < -x \tan\theta$, θ being the slope of the frontal interface. Let ρ_1, p_1, $\mathbf{u}_1$ and ρ_2, p_2, $\mathbf{u}_2$ denote the density, pressure and total air velocity in the warm and cold air masses, respectively, ρ_1 and ρ_2 being constant scalars and $\mathbf{u}_1$, $\mathbf{u}_2$ constant vectors; and let $\mathbf{n}$, $\mathbf{k}$ denote unit vectors normal to the sloping frontal discontinuity and to the earth's surface, respectively.

The assumption of geostrophy implies that

$$\mathbf{u}_{ih} = (\rho_i f)^{-1}\mathbf{k} \times \nabla p_i, \tag{2}$$

where $\mathbf{u}_{ih} = (u_i, v_i, 0)$ is the horizontal component of $\mathbf{u}_i = (u_i, v_i, w_i)$, and

$$\frac{\partial w_i}{\partial z} = 0. \tag{3}$$

Equation (3) follows immediately from the full continuity equation $\nabla \cdot \mathbf{u}_{ih} = 0$.

In the stationary front, the cross-front velocity components u_i are both zero, implying that $\partial p_i/\partial y = 0$ ($i = 1, 2$), and hence the isobars are parallel to the front in both air masses. The vertical velocities w_i are zero also, consistent with (3) together with the surface boundary condition that $w_i = 0$ at $z = 0$. It follows that the pressure is hydrostatic whereupon

$$\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} = \frac{1}{\rho_2}\frac{\partial p_1}{\partial x} + g'\frac{\partial h}{\partial x}, \tag{4}$$

where $h(x)$ is the height of the frontal surface and $g' = g(\rho_2 - \rho_1)/\rho_2$ is the reduced gravity. Margules' celebrated formula follows immediately from (2), namely

$$f(v_2 - \alpha v_1) = g'\frac{\partial h}{\partial x}, \tag{5}$$

where $\alpha = \rho_1/\rho_2$.

Suppose now we were to add a uniform geostrophic wind $c = u_1 = u_2$ to both air masses. Then the surface isobars would cross the front with a cyclonic change of direction at the front, itself. Consider the case of a cold front $c > 0$, moving towards some point A. Until the front arrives at A the pressure could not change because the flow in the warm air is geostrophic; nor could it change after the front had passed because the motion in the cold air is geostrophic also. Clearly, such a model is unable to explain either the observed pressure rise following the passage of a cold front, or the tendency for it to fall as the front approaches.

This moving cold front model has a further deficiency. The kinematic condition that the normal velocity is continuous across the frontal discontinuity is

$$\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}. \tag{6}$$

When there is no vertical motion, this implies that $u_1 = u_2$ and it follows then from Eq. (2) that $\partial p_2/\partial y = (1/\alpha)\partial p_1/\partial y$; i.e. the along-front pressure gradient is greater in the cold air. This means that at most one isobar can join across the front and that, at other positions along the front, the pressure is discontinuous (since of course $\alpha < 1$). This would imply an infinite horizontal pressure gradient force which, in turn, would drive an infinite acceleration. This is clearly unphysical! If we insist that the pressure is continuous across the front for all values of y, it follows that $\partial p/\partial y$ must be continuous also and hence from (2) that

$$u_2 = \alpha u_1 (< u_1). \tag{7}$$

Then the kinematic condition (5) implies that

$$w_1 = u_1(1 - \alpha)\partial h/\partial x, \tag{8}$$

in warm air, behind the surface front position.

In the warm air ahead of the front, $w_1 = 0$ as before in order to satisfy the surface boundary condition. Since $\partial h/\partial x < 0$ for a cold front, Eq. (8) implies that, in the model, there must be subsiding motion in the warm air overlying the cold air, while the opposite is true for a warm front. Thus it would appear that a more dynamically consistent, balanced model for a translating front has a cross-front flow pattern as shown in Fig. 1. Typical vertical motions implied by Eq. (8) are rather small, characteristic more of the broader-scale subsidence behind a front, rather than motions on the frontal scale, itself.

The moving front model is not entirely consistent immediately above the surface front where w_1 is discontinuous. This is manifest in a sharp bend in the streamlines (Fig. 1a). Note, that the continuity equation is not violated along this line although the balance approximation would be invalid in its neighborhood.

[1] A brief review and references can be found in Smith and Reeder (1988).

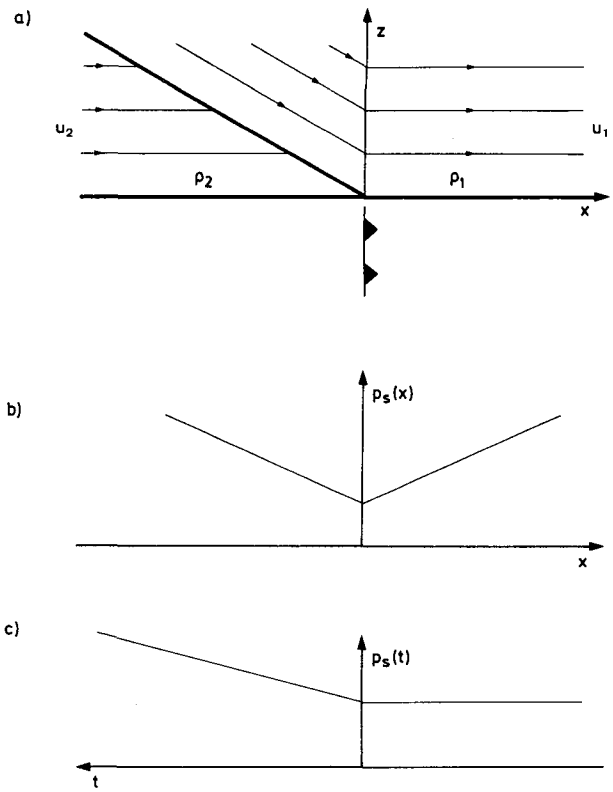

FIG. 1. (a) Vertical cross section of the moving Margules' cold front model when there is subsiding motion in the warm air overlying the frontal discontinuity; (b) corresponding spatial cross section of surface pressure at a fixed time; (c) time series of surface pressure at a fixed location, initially ahead of the front.

Accepting the need for subsiding motion in the warm air overlying the cold air, we are able to account for an increase in surface pressure following the passage of the model front, at least mathematically. Thus, if $w_\infty = -w_1 > 0$ in Eq. (1), the second term on the right is positive, implying an increase in the surface pressure with time. Using (8) it follows that

$$\frac{\partial p_s}{\partial t} = -g(\rho_2 - \rho_1)u_2 \frac{\partial h}{\partial x}. \tag{9}$$

This shows that the surface pressure rise can be associated with the horizontal advection of cooler and therefore denser air. In essence, by taking subsidence into account, the cross-front component of flow behind the front does not have to be so large as it otherwise would be to ensure a positive mass flux *into* any vertical column intersecting the cold air, as a part of the outgoing mass flux from the column is compensated for by the subsidence (cf. Brunt 1938, §185, Fig. 74). Thus, recognition of the vertical motion is essential to explain the post-frontal pressure change, but the existence of such motion limits the applicability of the model in practice (see below).

An important result highlighted by this model is the absence of any relationship between the cross-front pressure distribution at a particular *time and the* time series of surface pressure at a given location. While the spatial pressure distribution at any fixed time shows the surface front coincident with a pressure trough (Fig. 2b), the time series of surface pressure at any place shows a quite different behavior (Fig. 2c). In the latter case, the surface pressure at a fixed station remains uniform until the front arrives because the flow ahead of the front is geostrophic. Following the passage of the front, the pressure rises steadily according to Eq. (9) as the cold air depth increases.

If the surface front lies along the line $x = u_2 t$ at time t, the surface pressure distribution follows directly by partial integration of (2) and (9), using (4) to eliminate $\partial h/\partial x$, and can be written

$$p_s = p_0 + \begin{cases} f\rho_1(v_1 x - u_1 y), & \text{for} \quad x > u_2 t \\ f[\rho_2 v_2(x - u_2 t) - u_2(\rho_2 y - \rho_1 v_1 t)], & \\ & \text{for} \quad x > u_2 t, \end{cases} \tag{10}$$

where p_0 is a constant. Note that $\partial p_s/\partial t$ is a constant so that, following the passage of the front, any two isobars move at the same rate towards positive y, maintaining their spacing and orientation, consistent with a uniform geostrophic wind.

The lack of a relationship between the spatial and temporal variation of surface pressure would appear to have implications for the interpretation of pressure time series at a given station. It is commonly assumed that, to the extent that a front moves steadily, it is permissible to make a time-to-space conversion of the various frontal parameters (e.g. wind speed components, potential temperature etc . . .) by multiplying the time by the frontal speed, thereby obtaining a space cross-section of the front. The foregoing results show that, at least in the model, this is *not* possible with respect to surface pressure. In essence, this result is not new; indeed Sutcliffe (1938) emphasized the fact that a pressure field cannot be advected by a geostrophic flow and that extratropical cyclones move by a process of development in which ageostrophic circulations play an essential role. However, it seems worth emphasizing that the result is true for fronts. Thus, the commonly observed pressure fall before the arrival of a cold front must be attributed to a net divergence of the ageostrophic wind; it cannot be captured in a balanced air mass model. Such processes are, of course, contained in models that include frontogenetic processes (Hoskins 1982).

We may *not* infer that the time-to-space conversion of surface pressure is inappropriate to steadily-translating fronts in general (assuming that these exist!). The foregoing problems are a reflection of the assumption of strict geostrophy in the model. It would be possible, for example, to allow p_0 in Eq. (10) to vary

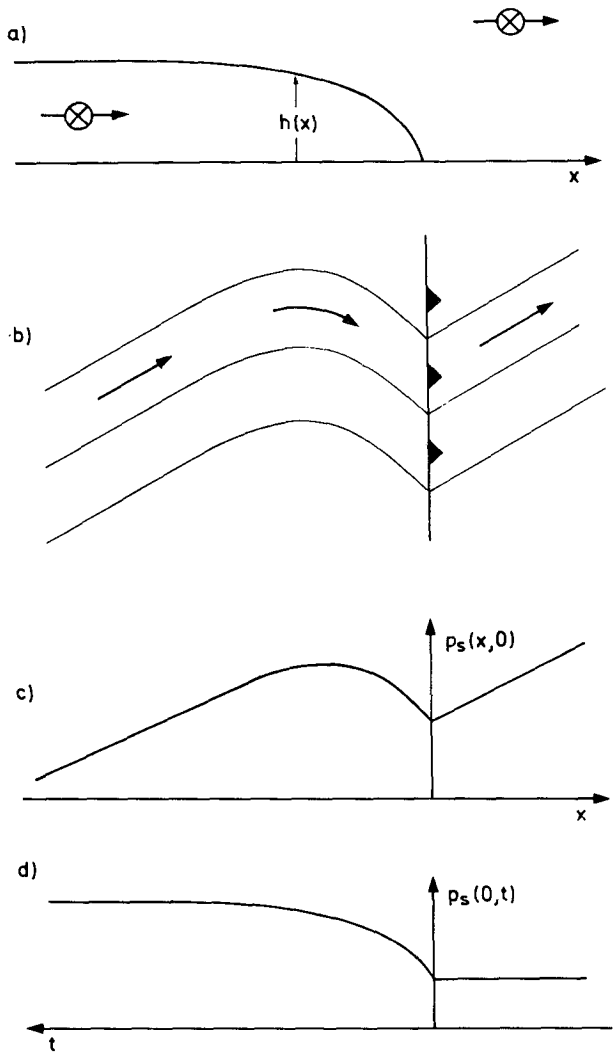

FIG. 2. Configuration of the uniformly translating cold front model of Davies (1984): (a) vertical cross section showing the wedge of cold air of variable $h(x)$ depicting the motion in each air mass. The arrows indicate the cross-front air flow; the vector tails, x, indicate the sense of the along-front motion in the warm air and in the cold air far from the front; (b) plan view of the surface isobars and flow direction corresponding with (a); (c) across-front variation of surface pressure at time $t = 0$; (d) time series of surface pressure at $x = 0$, taken to be the surface front position at $t = 0$.

linearly with time so that $\partial p_0/\partial t = -fu_2\rho_1 v_1$. Then in the warm air, $\partial p_s/\partial t + u_2 \partial p_s/\partial x = 0$, whereupon the surface pressure field *does* translate with the front.[2] We might argue that such an externally imposed pressure variation $p_0(t)$ models the ageostrophic redistribution of mass that necessarily accompanies a translating front in the atmosphere. However, there is no basis for accounting for this variation of $p_0(t)$ within a geostrophic model and this is evidently an important limitation of such models.

Notwithstanding the possibility of choosing $p_0(t)$ as described above, there is no guarantee that this is the "correct" choice for characterizing the ageostrophic redistribution of mass. Thus our results still signal caution in the interpretation of atmospheric data. Indeed, they highlight potential errors in inferring the spatial pressure variation across a front from its time variation at a given place as the pressure field is not advected by the cross-front geostrophic flow. Such time series contain only information about ageostrophic motions and/or of differential temperature advection (possibly geostrophic).

The above results are readily generalized to all steadily translating air mass models of fronts in which the slope of the frontal discontinuity, $\partial h/\partial x$, is nonuniform. One such example is discussed below.

3. Davies' Boussinesq models

Davies (1984) obtained a solution for a steadily moving cold front in which the cold air depth $h(x, t)$ is given by

$$h(x, t) = H[1 - \exp\{(x - ct)/L_R\}], \quad (11)$$

where $L_R = (g'H)^{1/2}/f$ is a Rossby radius of deformation for the flow and H is the depth of the cold air at large distances behind the surface front (see Fig. 4). Davies invokes the Boussinesq approximation (Spiegel and Veronis 1960) and ignores the difference in density between the two air masses when computing the pressure gradient force per unit mass. Hence the inconsistency in the along-front pressure gradient described earlier does not occur and, using our previous notation, $u_1 = u_2 = c$, $w_1 = 0$. Nevertheless it is pertinent to ask how the surface pressure can change with the passage of the front? According to Jeffreys' result and Brunt's extension thereof described in section 1, it should not change locally since u_1 and u_2 are both in geostrophic balance. The problem, which applies to all Boussinesq models, is resolved as follows. In the Boussinesq approximation one sets $\rho = \bar{\rho} + \rho'$, where $\bar{\rho}$ is an appropriately defined mean density and ρ' is the deviation therefrom. Basically, ρ' is then ignored unless multiplied by g, whereupon $(1/\rho)\nabla_h p$ is approximated as $(1/\bar{\rho})\nabla_h p$ and the continuity equation by

$$\nabla \cdot \mathbf{u} = 0. \quad (12)$$

Then in (7), α is unity (to zero order in $\rho'/\bar{\rho}$) and in (8), w_1 is zero. Using (1) and (12) the surface pressure tendency

[2] Note: the issue of time-to-space conversion is not resolved simply by rotating the x axis so that it points in the direction of flow in the warm air. Although the model prediction of zero prefrontal pressure change is then consistent with the (zero) spatial variation, a linear function of time $p_0(t) = |\mathbf{u}_2| t \sin\theta[-\rho_1|\mathbf{u}|\cos\alpha + \rho_2|\mathbf{u}_2| \cos(2\alpha - \theta)]$ must be included to allow for a time-to-space conversion of surface pressure behind the front. Here θ and α denote the wind directions in the warm and cold air, respectively, measured counterclockwise from the (original) positive x-axis.

$$\frac{\partial p_s}{\partial t} = -g \int_0^\infty \mathbf{u} \cdot \nabla \rho' dz. \qquad (13)$$

This equation states that surface pressure changes are associated with density advection and is analogous to Eq. (9), which was derived without approximation for the model with vertical motion in the warm air overlying the cold air. Clearly, in the extension of the Boussinesq model to $O(\rho'/\bar{\rho})$, such vertical motion must exist also. It follows that, in the extension of Davies' model to $O(\rho'/\bar{\rho})$, w_1 will be nonzero and given by Eq. (8). Furthermore, the local post-frontal pressure rise is given by Eq. (9) and there is no pre-frontal pressure change as before. The cross-front and temporal surface pressure variations are sketched in Figs. 2c and 2d. The complete surface pressure distribution is given by

$$\begin{aligned} p_s &= p_0 + f\bar{\rho}\,(v_1 x - u_1 y), \quad x > ct \\ &= p_0 + f\bar{\rho}\,[v_2 x - u_1 y + fL_R{}^2 \\ &\qquad \times \{1 - \exp[(x - ct)/L_R]\}, \quad x < ct. \qquad (14) \end{aligned}$$

As in section 2 it would be possible to choose $p_0(t)$ so that the surface pressure field translates with the front, without affecting other features of the solution.

4. Pressure coordinate formulations

The foregoing inconsistencies do not appear explicitly in the formulation of the analogous problems using pressure or some function of pressure as the vertical coordinate. However, it would be reasonable to assume that they occur implicitly, as in the Boussinesq models. This is difficult to prove, but it seems likely that they are concealed by the usual application of the surface boundary condition $\omega = 0$ at $p = p_s$; ω being the material derivative of pressure. As is well known, this is not identical with the condition that $w = 0$ at $z = 0$ in a height-coordinate model, even though it appears generally to provide a useful approximation thereto for many purposes.

5. Conclusion

A long known dynamical inconsistency in the formulation of balanced air mass models of fronts has been explored. The inconsistency can be resolved only at the expense of making physically unrealistic assumptions about the vertical motion in the warm air that overlies the cold air. The vertical motion must be invoked to account for the post frontal pressure rise associated with a cold front, or the prefrontal pressure fall associated with a warm front. Even for steadily translating models, there is, in general, no relationship between the pressure time series at a given station and the spatial variation of surface pressure at a given time.

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