

# Buoyancy in tropical cyclones and other rapidly rotating atmospheric vortices

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Received 18 March 2004; accepted 8 March 2005

## Abstract

Motivated primarily by its application to understanding tropical-cyclone intensification and maintenance, we re-examine the concept of buoyancy in rapidly rotating vortices, distinguishing between the buoyancy of the symmetric balanced vortex or system buoyancy, and the local buoyancy associated with cloud dynamics. The conventional definition of buoyancy is contrasted with a generalized form applicable to a vortex, which has a radial as well as a vertical component. If, for the special case of axisymmetric motions, the balanced density and pressure distribution of a rapidly rotating vortex are used as the reference state, the buoyancy field then characterizes the unbalanced density perturbations, i.e. the local buoyancy. We show how to determine such a reference state without approximation.

The generation of the toroidal circulation of a vortex, which is necessary for vortex amplification, is characterized in the vorticity equation by the baroclinicity vector. This vector depends, inter-alia, on the horizontal (or radial) gradient of buoyancy evaluated along isobaric surfaces. We show that for a tropical-cyclone-scale vortex, the buoyancy so calculated is significantly different from that calculated at constant height or on surfaces of constant  $\sigma$  ( $\sigma = (p - p^*) / (p_s - p^*)$ , where  $p$  is the actual pressure,  $p^*$  some reference pressure and  $p_s$  is the surface pressure). Since many tropical-cyclone models are formulated using  $\sigma$ -coordinates, we examine the calculation of buoyancy on  $\sigma$ -surfaces and derive an expression for the baroclinicity vector in  $\sigma$ -coordinates. The baroclinic forcing term in the azimuthal vorticity equation for an axisymmetric vortex is shown to be approximately equal to the azimuthal component of the curl of the generalized buoyancy. A scale analysis indicates that the vertical gradient of the radial component of generalized buoyancy makes a comparatively small contribution to the

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generation of toroidal vorticity in a tropical cyclone, but may be important in tornadoes and possibly also in dust devils.

We derive also a form of the Sawyer–Eliassen equation from which the toroidal (or secondary) circulation of a balanced vortex may be determined. The equation is shown to be the time derivative of the toroidal vorticity equation in which the time rate-of-change of the material derivative of potential toroidal vorticity is set to zero. In analogy with the general case, the diabatic forcing term in the Sawyer–Eliassen equation is shown to be approximately equal to the time rate-of-change of the azimuthal component of the curl of generalized buoyancy.

Finally, we discuss the generation of buoyancy in tropical cyclones and contrast the definitions of buoyancy that have been used in recent studies of tropical cyclones. We emphasize the non-uniqueness of the buoyancy force, which depends on the choice of a reference density and pressure, and note that different, but equivalent interpretations of the flow dynamics may be expected to arise if different reference quantities are chosen.

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*Keywords:* Buoyancy force; Vortices; Tropical cyclones

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## 1. Introduction

Broadly speaking, tropical cyclones weaken when the secondary circulation induced by the surface boundary layer is strong enough to produce radial divergence above the boundary layer. The outward moving air parcels in this region tend to materially conserve their absolute angular momentum resulting in a decline in the tangential wind speed. It follows then that tropical-cyclone intensification requires a mechanism to produce horizontal convergence above the boundary layer, reversing the divergence there induced by the boundary layer. The only conceivable mechanism able to do this is the buoyancy force associated directly or indirectly with latent heat release in deep cumulus convection. In fact, the role of buoyancy on the axisymmetric dynamics of tropical-cyclone intensification was emphasized in a review of convective processes in hurricanes by the first author (Smith, 2000). Nevertheless, there is disagreement about the role of buoyancy in the recent tropical-cyclone literature. Zhang et al. (2000) examined the vertical forces in a numerical simulation of Hurricane Andrew (1992) and concluded that air in the eyewall was negatively buoyant and was forced upwards by perturbation pressure gradient forces. In contrast, Braun (2002) carried out a similar study in a simulation of Hurricane Bob (1991) and found that eyewall updrafts are positively buoyant with respect to an environment that includes the vortex-scale warm-core structure. Eastin (2002, 2003) examined the buoyancy of eyewall convective updrafts in hurricanes based on aircraft flight-level reconnaissance data and found that eyewall updraft cores were positively buoyant relative to a background meso-scale environment, defined using a running low-pass filter along the flight track. These updraft cores were found to occupy less than 5% of the total eyewall and rainband areas, but accomplish approximately 40% of the total upward mass, heat and momentum transport, consistent with the concept of “hot towers”.

One difficulty in rationalizing the results of the foregoing papers is that buoyancy can be defined in various ways and each of the papers employs a different definition. Nevertheless,

it seems timely to discuss further some of the issues concerning buoyancy in rapidly rotating vortices as well as its role in tropical-cyclone intensification. While some of the results to be presented are not new, it is our experience that many are not well known. We review first the conventional definition of buoyancy as well as its generalization to rapidly rotating vortices. In doing so, we contrast the various derivations of the buoyancy force and their underlying assumptions, including the definition for axisymmetric motions and for cloud-scale, asymmetric motions. We refer to these as system buoyancy and local buoyancy, respectively. In particular, we derive a form of the Sawyer–Eliassen equation for the diabatic forcing of the toroidal (or secondary) circulation of an axisymmetric balance vortex in terms of the rate-of-generation of generalized buoyancy. We examine also the more general (unbalanced) case and investigate the relationship between the azimuthal vorticity tendency and the generalized buoyancy. We show that the Sawyer–Eliassen equation is simply the time derivative of the toroidal vorticity equation in which the time rate-of-change of the material derivative of potential toroidal vorticity is set to zero.

Since many tropical-cyclone models are formulated in  $\sigma$ -coordinates, we show how to characterize the buoyancy distribution in this coordinate system. In particular, we derive an expression for the buoyancy gradient along  $\sigma$ -surfaces in terms of the baroclinicity vector. We compare also the calculation of buoyancy along height, pressure and  $\sigma$ -coordinates for a tropical-cyclone scale vortex. We contrast the definitions of buoyancy in the papers by Zhang et al. (2000), Braun (2002) and Eastin (2002). Finally, we discuss ways in which “buoyancy thinking” can be applied to elucidate certain types of fluid flow.

## 2. Buoyancy

### 2.1. The conventional definition

The buoyancy of an air parcel in a density-stratified air layer is defined as the difference between the weight of air displaced by the parcel (the upward thrust according to Archimedes principle) and the weight of the parcel itself (Emanuel, 1994, p. 6). This quantity is normally expressed per unit mass of the air parcel under consideration, i.e.

$$b = -g \frac{\rho - \rho_a}{\rho} \quad (1)$$

where  $\rho$  is the density of the parcel,  $\rho_a = \rho_a(z)$  is the density of the environment at the same height  $z$  as the parcel and  $g$  is the acceleration due to gravity. Here and elsewhere, the vertical coordinate  $z$  is defined to increase in the direction opposite to gravity. The calculation of the upward thrust assumes that the pressure within the air parcel is the same as that of its environment at the same level, an assumption that may be unjustified in a rapidly rotating vortex (see, e.g. Leslie and Smith, 1978). In the latter case, one can define a generalized buoyancy force, which acts normal to the isobaric surface intersecting the air parcel and which is proportional to the density difference between the parcel and its environment along that surface (see below).

The foregoing definition of buoyancy is used frequently in meteorology to determine the force acting on an air parcel that is displaced vertically in the atmosphere under some

assumption that enables the density in its new location to be calculated (e.g. an adiabatic or pseudo-adiabatic displacement). However, one can think also of a region of the atmosphere being ‘buoyant’ relative to some other region. For example, tropical cyclones are warm-core vortices in the sense that at each pressure level, the air in the central-core region is warmer (and therefore, less dense) than environmental air at large radii. Thus, we can regard the core as positively buoyant relative to the environment. The circumstances under which this positively buoyancy leads to vertical motion are explored in later sections, where it is shown, inter-alia, that the radial gradient of buoyancy is of consequence in generating vertical circulations.

A similar expression for the buoyancy force given in (1) may be obtained by starting from the vertical momentum equation and replacing the pressure  $p$  by the sum of a reference pressure  $p_{\text{ref}}$  and a perturbation pressure  $p'$ . The former is taken to be in hydrostatic balance with a prescribed reference density  $\rho_{\text{ref}}$ , which is often taken, for example, as the density profile in the environment. In real situations, the environmental density is not uniquely defined, but could be taken as the horizontally-averaged density over some large domain surrounding the air parcel. Neglecting frictional forces, the vertical acceleration per unit mass can be written alternatively as

$$\frac{Dw}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \text{or} \quad \frac{Dw}{Dt} = \frac{1}{\rho} \frac{\partial p'}{\partial z} + b \quad (2)$$

where  $w$  is the vertical velocity,  $D/Dt$  the material derivative and  $t$  is the time (see, e.g. Turner, 1973, Section 1.3).<sup>2</sup> Clearly, the sum of the vertical pressure gradient and gravitational force per unit mass acting on an air parcel is equal to the sum of the vertical gradient of perturbation pressure and the buoyancy force, where the latter is calculated from Eq. (1) by comparing densities at constant height. The expression for  $b$  in (2) has the same form as that in (1), but with  $\rho_{\text{ref}}$  in place of  $\rho_a$ . However, the derivation circumvents the need to assume that the local (parcel) pressure equals the environmental pressure when calculating  $b$ . The foregoing decomposition indicates that, in general, the buoyancy force is not uniquely defined, because it depends on the (arbitrary) choice of a reference density. However, the sum of the buoyancy force and the perturbation pressure gradient per unit mass is unique. If the motion is hydrostatic, the perturbation pressure gradient and the buoyancy force are equal and opposite, but they remain non-unique.

Using the gas law ( $p = \rho RT$ , where  $R$  is the specific gas constant) and the usual definition of virtual potential temperature, the buoyancy force per unit mass can be written approximately as

$$\mathbf{b} \approx g \left[ \frac{\theta - \theta_{\text{ref}}}{\theta_{\text{ref}}} - (1 - \kappa) \frac{p'}{p_{\text{ref}}} \right], \quad (3)$$

where  $\theta$  is the virtual potential temperature of the air parcel in K,  $\theta_{\text{ref}}$  is the corresponding reference value and  $\kappa = R/c_p$ , where  $c_p$  is the specific heat at constant pressure. The second term on the right-hand side of (3) is sometimes referred to as the ‘dynamic buoyancy’ (e.g. Zhang et al., 2000), but in some sense, this is a misnomer since buoyancy depends

<sup>2</sup> Section 1.2 (Emanuel, 1994) presents a similar derivation, but makes the inelastic approximation (Ogura and Phillips, 1962) in which the density of the denominator of (1) is approximated by that in the environment.

fundamentally on the density perturbation, and this term simply corrects the calculation of the density perturbation based on the virtual potential temperature perturbation. If the perturbation pressure gradient term in (2) is written in terms of the Exner function  $(p/p^*)^k$ , where  $p^* = 1000$  mb, the second term in (3) does not appear in the expression for buoyancy (see, e.g. Rotunno and Klemp, 1982, Section 2). The expression (3) is valid also in a rapidly rotating vortex, but as shown below, there exists then a radial component of buoyancy as well. When clouds are involved it may be advantageous to include the drag of hydrometeors in the definition of buoyancy, but we omit this additional effect here.

### 2.2. Buoyancy in rapidly rotating vortices

In a rapidly rotating, axisymmetric vortex, an air parcel experiences not only the gravitational force, but also the radial force  $C = v^2/r + fv$ , the sum of the centrifugal force  $v^2/r$  and Coriolis force  $fv$ , where  $v$  is the tangential wind component at radius  $r$  and  $f$  is the Coriolis parameter. If the vortex is in hydrostatic and gradient wind balance, the isobaric surfaces slope in the vertical and are normal to the effective gravity,  $\mathbf{g}_e = (C, 0, -g)$ , expressed in cylindrical coordinates  $(r, \lambda, z)$  (see Fig. 1). The Archimedes force acting on the parcel is then  $-\mathbf{g}_e \rho_{\text{ref}}$  and the effective weight of the parcel is  $\mathbf{g}_e \rho$ , where  $\rho_{\text{ref}}$  is now the far field (reference) density along the same isobaric surface as the parcel. Accordingly, we may define a generalized buoyancy force per unit mass:

$$\mathbf{b} = \mathbf{g}_e \frac{\rho - \rho_{\text{ref}}}{\rho} \tag{4}$$

analogous to the derivation of (1). Note that unless  $v(v + rf) < 0$ , fluid parcels that are lighter than their environment have an inward-directed component of generalized buoyancy force as well as an upward component, while heavier parcels have an outward component as well as a downward component. An estimate for the importance of the radial component of buoyancy force in various types of atmospheric vortices is given at the end of Section 3.

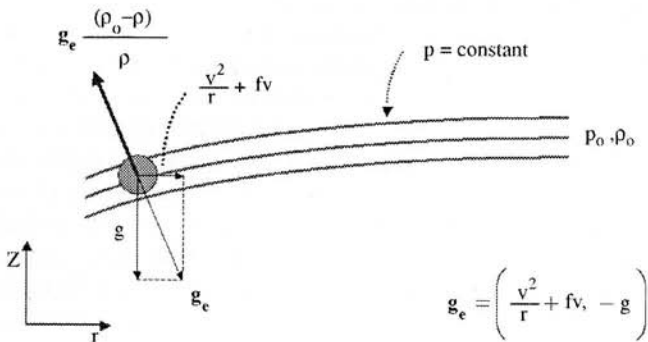


Fig. 1. Schematic radial-height cross-section of isobaric surfaces in a rapidly rotating vortex showing the forces on an air parcel including the gravitational force and the centrifugal and Coriolis forces. The Archimedes force  $-\mathbf{g}_e \rho_{\text{ref}}$  slopes upwards and inwards, while the weight  $\mathbf{g}_e \rho$  slopes downwards and outwards. Thus, the net buoyancy force acting on the parcel per unit mass is  $|\mathbf{g}_e|(\rho_{\text{ref}} - \rho)/\rho$  in the direction shown.

### 2.3. Buoyancy arising from parcel displacements: symmetric instability

The radial component of the generalized buoyancy as defined above is not the same as the centrifugal buoyancy that comes about if the air parcel has also a different tangential wind speed  $v^*$  to that of its local environment  $v$ . Such might be the case, for example, if the parcel were displaced horizontally to its current position, while conserving its absolute angular momentum. In this case, the net radial force that it experiences per unit mass is equal to  $[\rho(v^{*2}/r + fv^*) - \rho_{\text{ref}}(v^2/r + fv)]/\rho$  (Rayleigh, 1916). If, for example, the departures of the parcel values from the environmental values are small, this radial force is approximately equal to  $(\rho - \rho_{\text{ref}})(v^2/r + fv)/\rho + \rho_{\text{ref}}(2v/r + f)(v^* - v)/\rho$ . The first term is proportional to the density perturbation and is the radial component of the generalized buoyancy defined in Eq. (4), while the second term arises from the velocity perturbation. It is frequently assumed that, in contrast to vertical displacements, radial displacements do not produce any density perturbation so that only the second term is non-zero. Indeed, in his discussion of symmetric instability, Emanuel (1994, Chapter 12) regards this term as the analogous quantity to buoyancy in the radial direction in contrast to the definition in Eq. (4). Symmetric instability refers to the situation in which the flow is stable to either radial or vertical displacements, but unstable to certain slantwise displacements. The general case of axisymmetric and inviscid parcel displacements in any direction within a baroclinic circular vortex is discussed in Section 4 of Charney's (1973) lecture notes.

### 2.4. Buoyancy relative to a balanced vortex

Tropical cyclones are rapidly rotating warm-cored vortices and the warm core is therefore positively buoyant relative to the environment. On the cyclone-scale, however, hydrostatic and gradient-wind balance exist to a good approximation (Willoughby, 1990) and the radial density (or buoyancy) gradient is related by the thermal wind equation to the decay in the mean tangential circulation and density with height (see, e.g. Smith, 1980). Clearly much of the radial gradient of buoyancy force cannot be thought of as being "available" for driving a secondary (or toroidal) circulation of the vortex that is necessary for vortex amplification. Nevertheless, hydrostatic balance may be a poor approximation in individual convective clouds and a pertinent question is whether these clouds have significant local (unbalanced) buoyancy, which in turn might play an important role in the dynamics of storm intensification. This important question was addressed by Braun (2002), who answered it in the affirmative on the basis of his simulations of Hurricane Bob (1991). To address this question, it is necessary to define the perturbation pressure and perturbation density relative to some vortex-scale pressure and density distributions. The simplest case is when the primary vortex is approximately steady and axisymmetric. Then, we may take reference distributions  $p_0(r, z)$  and  $\rho_0(r, z)$ , respectively, that are in thermal wind balance with the tangential flow field  $v_0(r, z)$ . The thermal wind equation gives:

$$\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial r} + \left( \frac{v_0^2}{r} + fv_0 \right) \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} = - \left( \frac{2v_0}{r} + f \right) \frac{\partial v_0}{\partial z}. \quad (5)$$

This is a linear first-order partial differential equation for  $\ln \rho_0$ , the characteristics of which satisfy

$$\frac{dz}{dr} = \frac{1}{g} \left( \frac{v_0^2}{r} + fv_0 \right). \quad (6)$$

It is easy to show that these characteristics are just the isobaric surfaces<sup>3</sup> and the density variation along them is governed by the equation

$$\frac{d}{dr} \ln \rho_0 = -\frac{1}{g} \left( \frac{2v_0}{r} + f \right) \frac{\partial v_0}{\partial z}. \quad (7)$$

Given the vertical density profile,  $\rho_a(z)$ , at some radius  $R$ , these equations can be integrated inwards along the isobars to obtain the balanced axisymmetric density and pressure distributions (see Smith, 2005). We may use  $\rho_0(r, z)$  and  $p_0(r, z)$  as alternative reference quantities to define the buoyancy force in Eq. (2) (similar to Braun, 2002), without affecting the derivation of this equation. We denote the generalized buoyancy force so calculated by  $\mathbf{b}_B$ . It follows that  $\mathbf{b}_B = 0$  in the axisymmetric balanced state, whereas if the reference pressure and density at  $r = R$  are used,  $\mathbf{b}$  equals some non-zero function  $\mathbf{b}_0(r, z)$ . Clearly, the partition of force between perturbation pressure gradient and buoyancy will be different for the reference state characterized by  $\rho_0(r, z)$  and  $p_0(r, z)$  and interpretations of the dynamics will be different also, albeit equivalent to those using the more conventional reference quantities that depend on height only.

In the more general case, when the vortex structure has marked asymmetries and/or is evolving in time, it is necessary to allow for the azimuthal and/or time variations of the reference state as was done by Zhang et al. (2000) and Braun (2002) (see Section 6).

## 2.5. Buoyancy in axisymmetric balanced vortices

Axisymmetric balanced models of tropical-cyclone intensification (e.g. Ooyama, 1969) appear to capture many important observed features of tropical-cyclone behaviour. However, in an axisymmetric model that assumes exact thermal wind balance,  $\mathbf{b}_B(r, z, t) \equiv 0$  and the corresponding  $\partial p'/\partial z \equiv 0$ , even though there may be heat sources or sinks present that generate buoyancy  $\mathbf{b}$ . It is clear from the foregoing discussion that any diabatic heating or cooling in such models is incorporated directly into the balanced state, changing  $\mathbf{b}(r, z, t)$ , while  $\mathbf{b}_B(r, z, t)$  remains identically zero by definition. Obviously, non-zero values of  $\mathbf{b}_B$  relate to unbalanced motions provided that the appropriate reference state as defined above has been selected for the definition of buoyancy at any given time. It may be helpful to think of  $\mathbf{b}$  as characterizing the system buoyancy and  $\mathbf{b}_B$  as characterizing the local buoyancy.

<sup>3</sup> A small displacement  $(dr, dz)$  along an isobaric surface satisfies  $(\partial p_0/\partial r)dr + (\partial p_0/\partial z)dz = 0$ . Using the equations for hydrostatic balance,  $\partial p_0/\partial z = g\rho_0$  and gradient wind balance,  $\partial \rho_0/\partial r = \rho_0(v^2/r + fv)$ , gives the equation for the characteristics.

The axisymmetric balanced response of tropical cyclones to local sources of heating and momentum is discussed by Willoughby (1979) and Shapiro and Willoughby (1982). The streamfunction  $\psi$  for the toroidal circulation forced by diabatic heating, analogous to Shapiro and Willoughby's Eq. (5), but consistent with the thermal wind equation in Eq. (5) is

$$\frac{\partial}{\partial r} \left[ -g \frac{\partial \chi}{\partial z} \frac{1}{\rho r} \frac{\partial \psi}{\partial r} - \frac{\partial}{\partial z} (\chi C) \frac{1}{\rho r} \frac{\partial \psi}{\partial z} \right] + \frac{\partial}{\partial z} \left[ \left( \xi \chi (\zeta + f) + C \frac{\partial \chi}{\partial r} \right) \frac{1}{\rho r} \frac{\partial \psi}{\partial z} - \frac{\partial}{\partial z} (\chi C) \frac{1}{\rho r} \frac{\partial \psi}{\partial r} \right] = g \frac{\partial}{\partial r} (\chi^2 \dot{\theta}) + \frac{\partial}{\partial z} (C \chi^2 \dot{\theta}) \quad (8)$$

where  $\chi = 1/\theta$ ,  $\xi = 2v/r + f$ ,  $\zeta = (1/r)(\partial(rv)/\partial r)$  is the vertical component of relative vorticity and  $\dot{\theta} = d\theta/dt$  is the diabatic heating rate (see Appendices A and B). The equation shows that the buoyant generation of a toroidal circulation is related approximately to the curl of the rate-of-generation of generalized buoyancy defined in this case as  $\mathbf{b} = \mathbf{g}_e(\theta - \theta_a)/\theta_a$ . The approximation is based on replacing  $1/\theta$  and  $1/\theta_a$  by some global average value,  $1/\Theta$ , as in the anelastic approximation (Ogura and Phillips, 1962). Following Shapiro and Willoughby (1982), we note that Eq. (8) is an elliptic partial differential equation provided that the discriminant

$$D = -g \frac{\partial \chi}{\partial z} \left( \xi \chi (\zeta + f) + C \frac{\partial \chi}{\partial r} \right) - \left[ \frac{\partial}{\partial z} (\chi C) \right]^2 \quad (9)$$

is positive. With a few lines of algebra, one can show that  $D = g\rho\xi\chi^3 P$ , where  $P = (1/\rho\chi^3)(\partial v/\partial z)(\partial\chi/\partial r) - (\zeta + f)(\partial\chi/\partial z)$  is the Ertel potential vorticity (Shapiro and Montgomery, 1993).

We show in Appendix A that the Sawyer–Eliassen equation is the time derivative of the toroidal vorticity equation in which the time rate-of-change of the material derivative of potential toroidal vorticity, defined as  $\eta/(r\rho)$ , where  $\eta = \partial u/\partial z - \partial w/\partial r$ , is set to zero.

### 3. Buoyancy, the baroclinicity vector and toroidal vorticity generation

Broadly speaking, when the rotation is weak, it is the horizontal gradient of conventional buoyancy  $\mathbf{b}$ , rather than the buoyancy force per se, that is involved in generating horizontal vorticity, and hence toroidal circulations in vortices. What then is the role of the radial component of generalized buoyancy in rapidly rotating vortices, or more specifically its vertical gradient? To explore this question, we examine the vorticity equation, which may be written as:

$$\frac{D\omega}{Dt} = (\omega + \mathbf{f}) \cdot \nabla \mathbf{u} - (\omega + \mathbf{f}) \cdot \mathbf{u} + \mathbf{B} \quad (10)$$

where  $\omega$  is the three-dimensional vorticity vector,  $f$  is the Coriolis parameter, assumed here to be a constant vector, and  $\mathbf{B} = (1/\rho^2)\nabla\rho \wedge \nabla p$  is the baroclinicity vector (see, e.g. Gill, 1982, p. 238). The horizontal vorticity tendency associated with horizontal density variations



is characterized by the horizontal part of  $\mathbf{B}$ . In height coordinates, this component, denoted by a subscript “ $h$ ” is

$$\mathbf{B}_h = g\mathbf{k} \wedge \nabla_h \ln \rho + \frac{1}{\rho} \frac{\partial \rho}{\partial z} \mathbf{k} \wedge \left( \frac{1}{\rho} \nabla_h p \right). \quad (11)$$

If the difference between the density and the reference density is sufficiently small, the first term on the right-hand side of (11) is approximately equal to the horizontal gradient of (conventional) buoyancy,  $\nabla_h b$ , while the second term represents the generation of horizontal vorticity by a uniform horizontal pressure gradient force if the density decreases with height.

The analogue expressions to  $\mathbf{B}_h$  in pressure coordinates and  $\sigma$ -coordinates are

$$\mathbf{B}_p = g\mathbf{k} \wedge \nabla_p \ln \rho, \quad (12)$$

and

$$\mathbf{B}_\sigma = g\mathbf{k} \wedge \nabla_\sigma \ln \rho - \frac{g}{(\rho p^*)} \frac{\partial \rho}{\partial \sigma} \mathbf{k} \wedge \left[ \frac{1}{\rho} \nabla_\sigma p \right], \quad (13)$$

respectively, where the subscript on  $\nabla$  denotes differentiation keeping the relevant quantity constant. Note that in pressure coordinates there is only one term, highlighting the fact that horizontal circulation is produced by horizontal density gradients, and hence horizontal gradients of (conventional) buoyancy along isobaric surfaces,<sup>4</sup> as indicated in a different way in formulating the generalized buoyancy in Section 2.2. Clearly, the effective “buoyancy forcing” of the azimuthally-averaged secondary circulation (i.e. the circulation in the  $r$ - $z$ -plane) in a tropical cyclone may be determined from Eq. (11) or either of its equivalent forms in  $p$ - or  $\sigma$ -coordinates.

The role of buoyancy gradients in forcing a toroidal (or secondary) circulation of a rapidly rotating vortex may be conveniently illustrated for the case of an axisymmetric vortex initially in gradient wind balance on an  $f$ -plane. For simplicity, let us assume that the secondary circulation of the vortex is initially zero and that the initial wind field is  $v(r, z)\hat{\lambda}$ , where  $\hat{\lambda}$  is a unit vector in the azimuthal direction. The vorticity tendency obtained from (10) is then

$$\frac{\partial \omega_\lambda}{\partial t} = \frac{\partial C}{\partial z} + B_\lambda, \quad (14)$$

where  $\omega_\lambda = \hat{\lambda} \omega$  denotes the toroidal component of vorticity and  $B_\lambda = \hat{\lambda} \cdot \mathbf{B} = -\hat{\lambda} \cdot \mathbf{g}_e \wedge \nabla \ln \rho$ . If the relative density differences  $(\rho - \rho_{\text{ref}})/\rho$  are small compared with unity (i.e.<sup>5</sup>  $|b_r| \ll C$  and  $|b_z| \ll g$ ),  $B_\lambda$  may be expressed approximately as

$$B_\lambda \approx \frac{\partial b_r}{\partial z} - \frac{\partial b_z}{\partial r} + C \frac{\partial}{\partial z} \ln \rho_{\text{ref}} + g \frac{\partial}{\partial r} \ln \rho_{\text{ref}} \quad (15)$$

where  $b_r$  and  $b_z$  are the radial and vertical components of  $\mathbf{b}$ .

<sup>4</sup> This result is essentially equivalent to Bjerknes’ circulation theorem (see, e.g. Gill, 1982, Section 7.11).

<sup>5</sup> Recall that  $b_r$  is defined so that it is negative (i.e., radially-inward) if  $\rho < \rho_{\text{ref}}$ .

When  $\rho_{\text{ref}}$  is taken as a function of  $z$  only, for example, when the buoyancy is defined relative to the environmental density distribution, then

$$B_\lambda \approx \frac{\partial b_r}{\partial z} - \frac{\partial b_z}{\partial r} + C \frac{\partial}{\partial z} \ln \rho_{\text{ref}}, \quad (16)$$

and the tendency of azimuthal vorticity is

$$\frac{\partial \omega_\lambda}{\partial t} \approx \hat{\boldsymbol{\lambda}} \cdot \nabla \wedge \mathbf{b} + \left[ \frac{\partial C}{\partial z} + C \frac{\partial}{\partial z} \ln \rho_{\text{ref}} \right]. \quad (17)$$

In this case,  $\hat{\boldsymbol{\lambda}} \cdot \nabla \wedge \mathbf{b}$  is only one of three terms that contributes to the azimuthal vorticity tendency. However, when  $\rho_{\text{ref}}$  satisfies Eq. (5) for a balanced vortex, then

$$\frac{\partial \omega_\lambda}{\partial t} \approx \hat{\boldsymbol{\lambda}} \cdot \nabla \wedge \mathbf{b} \quad (18)$$

and  $\hat{\boldsymbol{\lambda}} \cdot \nabla \wedge \mathbf{b}$  is the sole term contributing to the vorticity tendency. If the full density field is in thermal wind balance in this case, then  $b_r$  and  $b_z$  are just the components of  $\mathbf{b}_B$  and are both identically zero as discussed in Section 2.2. In this case, we obtain the familiar result that the toroidal vorticity tendency for a steady vortex is zero and there is no toroidal circulation.

In the eyewall region of a tropical cyclone, where there is approximate cyclostrophic balance,  $|b_r|/|b_z| \approx v^2/rg$ , which for a maximum tangential wind speed of  $40 \text{ m s}^{-1}$  at a radius of 40 km is approximately  $4 \times 10^{-3}$ . However, it is the spatial derivatives of these buoyancy components that are of consequence for the estimating their relative contribution to the generation of toroidal vorticity and this will depend on the structure of the primary vortex. Let  $R$  and  $Z$  denote scales for the radial and vertical variation of  $b_z$  and  $b_r$ , respectively. Then, the ratio of the two terms  $\partial b_r/\partial z$  and  $\partial b_z/\partial r$  in  $\hat{\boldsymbol{\lambda}} \cdot \nabla \wedge \mathbf{b}$  are typically  $(R/Z)(v^2/rg)$  times this value in magnitude. Since tropical cyclones are broad-shallow vortices,  $R$  is likely to be larger than  $Z$ , so that the ratio  $|\partial b_z/\partial r|/|\partial b_r/\partial z|$  should be larger than  $4 \times 10^{-3}$ , but maybe not by more than an order of magnitude. Thus, the terms involving the vertical gradient of radial buoyancy in the foregoing equations should be comparatively small, at least on the scale of the vortex. However, in other rapidly rotating atmospheric vortices,  $\partial b_z/\partial r$  may not be negligible in the generation of toroidal vorticity. For example, in a tornado,  $v^2/rg$  is typically  $(100 \text{ m s}^{-1})^2/(100 \text{ m} \times 10 \text{ m s}^{-2}) = 10$  and in a dust devil  $(10 \text{ m s}^{-1})^2/(10 \text{ m} \times 10 \text{ m s}^{-2}) = 1$ . However, for these tall-thin vortices one might expect typical values for  $R/Z$  to be less than unity. Even if  $R/Z$  is an order of magnitude less than unity, then the term  $\partial b_z/\partial r$  may not be negligible for a tornado and a careful estimate of  $R/Z$  would be necessary to justify its neglect in a dust devil.

#### 4. Comparison of buoyancy calculations

As noted above, in strong vortical flows such as tropical cyclones, the pressure surfaces may have an appreciable slope in a vertical plane. Therefore, we may ask whether there is a significant difference in the buoyancy force calculated by comparing parcel and environmental temperatures (or potential temperatures) at constant height, constant pressure or

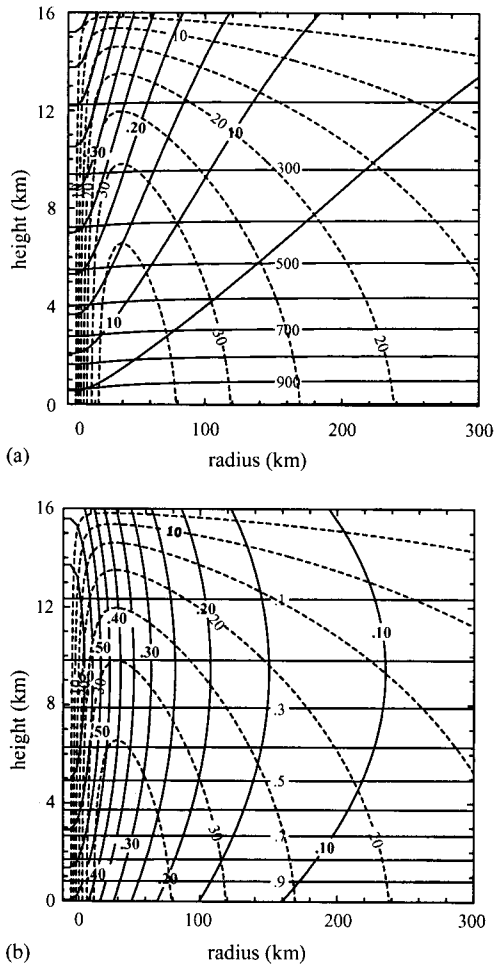


Fig. 2. Radial-height cross-section of the buoyancy distributions in a balanced tropical-cyclone strength vortex calculated by comparing parcel and environmental temperature at constant pressure (panel (a)) and constant height (panel (b)) (thick solid contours, contour interval  $0.05 \text{ m s}^{-2}$ ). Superimposed in each panel are the isotachs of tangential velocity (dashed lines, contour interval  $10 \text{ m s}^{-1}$ ). The near horizontal lines in panel (a) are the isobaric surfaces (contour interval 100 mb) and those in panel (b) are the surfaces of constant  $\sigma$  (contour interval 0.1).

constant  $\sigma$ ? To examine this question, we compare the various buoyancy distributions for a balanced tangential flow field typical in strength and structure to that of the inner-core region of a tropical cyclone with a maximum wind speed of about  $40 \text{ m s}^{-1}$  at a radius of 50 km near the surface and the maximum wind speed decreasing with height (see Fig. 2). The flow is determined by a slightly modified version of the analytically-prescribed pressure perturbation used by Smith (1968), detailed in Appendix B. The environmental sounding assumes a linear decrease in temperature with height ( $T = T_s(1 - \gamma z)$ , where  $T_s = 303 \text{ K}$  and

$\gamma = 2.12 \times 10^{-5} \text{ m}^{-1}$ ) so that the pressure and density distributions may be calculated analytically. This structure approximates closely that of the tropical atmosphere represented by the mean Jordan mean sounding for the Carribean region (Jordan, 1957).

The radial-height cross-sections in Fig. 2 show also the buoyancy distribution calculated by comparing parcel and environmental densities at constant pressure (i.e. the vertical component of  $\mathbf{g}_c(\rho_0 - \rho)/\rho$  in Fig. 1) (panel a), and by comparing densities at constant height (panel b). Superimposed in panel (a) are the isobars and in panel (b) the surfaces of constant  $\sigma$ . Note that the  $\sigma$ -surfaces are almost exactly horizontal. As a result the buoyancy distribution calculated by comparing densities at constant  $\sigma$  is almost identical to that calculated in height coordinates, and is therefore not shown. As would be anticipated from Fig. 1, the buoyancy force is larger when the density difference is calculated at constant height or constant  $\sigma$  instead of at constant pressure. The difference arises because for a warm-cored vortex, the pressure surfaces are higher in the environment than at the level of the air parcel, in which case the environmental density, and hence the density difference, is less than when the difference is calculated at constant height or constant  $\sigma$ .

In an axisymmetric configuration, the two terms on right-hand side of (11) may be written as  $(g/\rho_0)(\partial\rho_0/\partial r)_z$  and  $(v_0^2/r + fv_0)(1/\rho_0)(\partial\rho_0/\partial z)$  (these are just the terms on the left-hand side of (5)). The first is approximately equal to  $-\partial b_z/\partial r$  in Eq. (17), while the second is approximately equal to the term  $g_r\partial(\ln \rho_{\text{ref}})/\partial z$  in this equation. Fig. 3 shows radial-height cross-sections of these two terms for the vortex in Fig. 2. The second term is opposite in sign to the first, while the sum of these terms (not shown) is identical with the corresponding vorticity tendency calculated at constant pressure (Eq. (12)). For this reason, the latter quantity is a more appropriate measure of the horizontal vorticity tendency than that based on the gradient of buoyancy calculated at constant height as it is effectively the only term involved.

## 5. Definition of buoyancy in recent studies

As noted in the introduction, the papers by Zhang et al. (2000), Braun (2002) and Eastin (2002) make conflicting statements about the role of buoyancy in hurricanes and it seems appropriate to examine these in a little detail. Each of these papers uses a different definition for the reference state used to define buoyancy. Zhang et al. define a reference state that varies spatially and temporally by performing a running average of the numerical model output over four neighbouring grid points on constant  $\sigma$ -surfaces. In contrast, Braun used a Fourier decomposition into different azimuthal wavenumbers and selected wavenumbers 0 (the axisymmetric component) and 1 to define the reference state in an attempt to reduce the amount of horizontal variability resulting from such a procedure. Both of these methods lead to a definition of buoyancy force that is more akin to the local buoyancy,  $\mathbf{b}_B$ , defined in Section 2.2 than to system buoyancy,  $\mathbf{b}$ , defined by Eq. (4), although the foregoing authors consider only the vertical components of buoyancy. In contrast, because of the limitations of the flight-level data, Eastin chose to define the background meso-scale environment by applying a running Bartlett filter<sup>6</sup> to the thermodynamic data of each radial flight leg.

<sup>6</sup> See Eastin (2002).

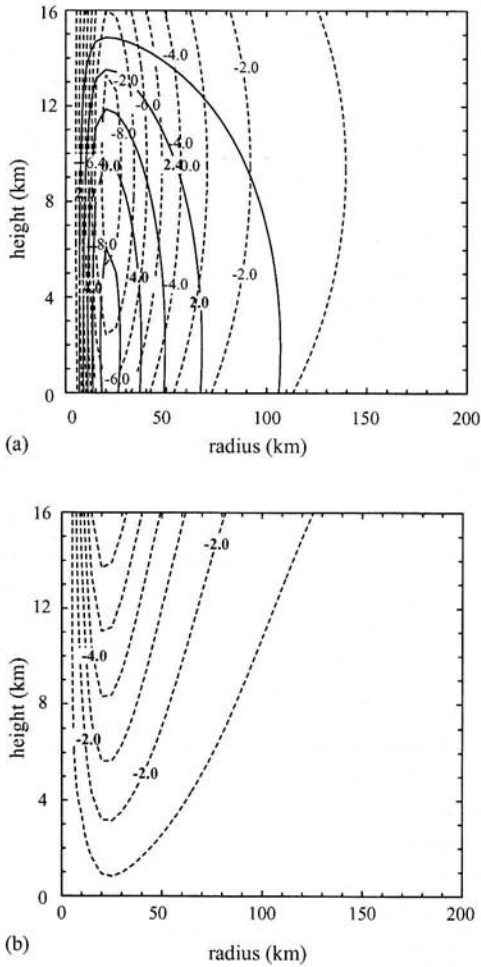


Fig. 3. Radial-height cross-sections of the azimuthal vorticity tendencies corresponding in the axisymmetric case with the terms: (a)  $(\partial \mathbf{b} / \partial r)_z$  (dashed contours),  $-(v_0^2/r + fv_0)(1/\rho)(\partial \mathbf{b} / \partial r)$  (solid contours) in the axisymmetric form of Eq. (11), and (b) their sum. Contour interval  $10^{-6} \text{ s}^{-2}$ . Dashed contours correspond with negative values.

This method corresponds more with the definition  $\mathbf{b}_B$ . It is immediately evident from our discussion in Section 2 that these different definitions may lead to different interpretations of the relative importance of the buoyancy and perturbation pressure gradient forces in the flow dynamics. Moreover, it is clear that such distinctions are artificial as Newton's second law knows only about the sum of the two forces, which is independent of the definition of the reference state. Of course, in a tropical cyclone, the effects of water loading will make an important contribution to the vertical force field, an effect considered by all the above authors. Both Zhang et al. and Eastin subdivide the buoyancy force into thermal and dynamic contributions, the former proportional to the virtual temperature perturbation and the latter to the pressure perturbation. However, these contributions arise only because

it is density that is the fundamental quantity in the definition of buoyancy (Eq. (1)) and because the virtual temperature is only a proxy for density at constant pressure (Eq. (2)). The so-called dynamic buoyancy has no dynamical significance, but merely corrects the calculation of density. In Section 4 of their paper, Zhang et al. following standard practice in studies of supercell thunderstorm dynamics (e.g. Klemp, 1987 and references), even go so far as to subdivide the perturbation pressure field into “dynamic” and “buoyancy” parts, which seems to unnecessarily obscure rather than clarify their interpretations of the dynamics! None of the foregoing authors consider the radial component of buoyancy, but fortunately, the scale analysis presented at the end of Section 3 indicates that the effects of this component should be small in tropical cyclones, at least on the scale of the vortex.

## 6. Origins of buoyancy in tropical cyclones

Tropical cyclones intensify when, as a direct or indirect result of latent heat release, the buoyancy  $\mathbf{b}$  in the core increases, even though a substantial fraction of the buoyancy distribution may be in thermal wind balance. Here we are referring to the core-scale buoyancy distribution relative to the environment, or system buoyancy, rather than the local buoyancy of individual clouds.<sup>7</sup> To a first approximation, the direct effect of latent heat release in saturated ascending air, such as in the eyewall clouds, or in the cores of individual convective clouds, is to maintain the air close to the moist adiabat from which the updraught originates. The indirect effect of latent heat release is to produce subsidence (or at least reduce the rate-of-ascent) in clear-air regions adjacent to (i.e. within a local Rossby radius of deformation of) deep convection. There is observational evidence (e.g. Betts, 1986; Xu and Emanuel, 1989) and evidence from model studies (Bretherton and Smolarkiewicz, 1989) that, again to a first approximation, the clear air properties are adjusted towards the same saturation moist adiabat as in the neighbouring convective cores, albeit in this case to one calculated reversibly. In either case, the thermal structure of the troposphere in a mature tropical cyclone, and thereby the radial distribution of buoyancy, would be determined largely by the radial distribution of moist entropy at the top of the subcloud layer, at least in regions of ascent (see, e.g. Emanuel, 1991). The inward-radial gradient of moist entropy, itself, is a result of the increase in surface entropy flux with increasing wind speed. We reiterate that the foregoing picture relates essentially to the generation of system buoyancy.

The extent to which local (unbalanced) buoyancy is produced will depend amongst other things on the rate at which the buoyancy is generated and the scale on which it is generated. For example, the simulations by Braun (2002) indicate that much of the eyewall updraft mass flux occurs within small-scale updrafts that are locally buoyant relative to the broad-scale thermal field of the vortex. A recent examination of the high resolution cloud resolving numerical simulation of the formation of Hurricane Diana (1984) has shown how buoyant cores growing in the rotation-rich environment of an incipient storm produce intense cyclonic vorticity anomalies in the lower troposphere by vortex tube stretching (Hendricks et al., 2004). These intense vorticity anomalies subsequently merge and axisymmetrize

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<sup>7</sup> In fact, the local buoyancy of individual clouds would be expected to decrease as the warm core of the vortex intensifies and spreads radially outwards.

to intensify the balanced circulation of the incipient meso-scale vortex (Montgomery and Enagonio, 1998; Möller and Montgomery, 2000; Montgomery and Brunet, 2002). In this case, subsidence warming is not the primary means for generating the cyclone's warm core. Rather, the warm-core temperature that materializes within the developing meso-scale vortex results from the tendency of the high vorticity cores of the buoyant plumes to 'trap' the heat released by the condensation process, as one might anticipate from local Rossby adjustment considerations (Schubert et al., 1980, Section 9) and quasi-balanced dynamics within enhanced vortical regions (Schubert and Hack, 1982, Montgomery et al., 2005).

## 7. The utility of buoyancy "thinking"

In light of the foregoing discussions, we might step back and ask why introduce the buoyancy in the first place? The concept is certainly useful in understanding why an air parcel rises or sinks when it is lighter or heavier than its environment, provided that the pressure and density of the environment are used as reference profiles. If the fluid is at rest, then a hydrostatic calculation (i.e. the calculation leading to (1)) tells us that the buoyancy force is the only force acting and the perturbation pressure gradient is zero. In most other situations, the perturbation pressure field may be important.<sup>8</sup>

It is perhaps worth remarking that, in most situations in the atmospheric sciences, interpretations of fluid motion in terms of force fields are not analogous to interpretations of rigid body motion so that caution is required in constructing "cause and effect" arguments based on local force fields in fluid flows. The typical problem in rigid body dynamics is to calculate the motion of a body subject to a prescribed set of forces. But if compressible waves are excluded as is commonly the case, the net force distribution in a fluid flow cannot be arbitrarily prescribed: it must be consistent with the continuity equation and the boundary constraints on the flow. Mathematically, the perturbation pressure field is then determined diagnostically and globally by the solution of a Poisson-type (elliptic) partial differential equation at each instant of time. With the foregoing caution, secure arguments may be constructed to "explain" at least the instantaneous pressure field, and hence the force distribution.

In most situations, the existence of buoyancy in a spatially local region of flow will "induce" a perturbation pressure field that permits the flow to evolve in a way that is consistent with mass continuity and the conditions at flow boundaries. Consider, for example, the role of local buoyancy in an isolated thermal rising through a non-rotating homogeneous fluid contained between two rigid horizontal boundaries (Fig. 4a). Our intuition tells us that as the thermal rises it will push away the fluid ahead of it and draw fluid upwards to follow it. This intuition is based on the fact the fluid is a continuous medium in which mass is conserved. However, for the motion to be possible, there must be forces at work. Outside the thermal, the only force is the perturbation pressure gradient. If air is drawn inwards and upwards behind the thermal, there must be an upward-directed perturbation pressure gradient force there, with relatively low pressure just behind the thermal. Likewise, above the thermal, and

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<sup>8</sup> The consequences and limitations of neglecting the perturbation pressure gradient when using parcel arguments in more general situations are discussed by Thorpe et al. (1989).

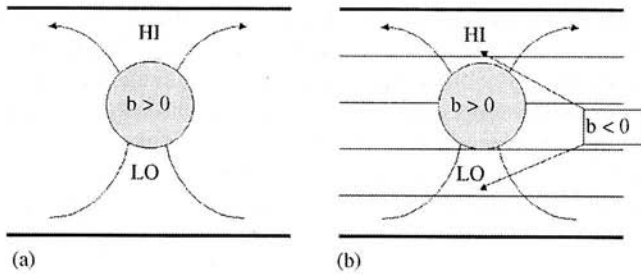


Fig. 4. Schematic diagram of the force field induced by a rising thermal in (a) a homogeneous fluid and (b) a stably-stratified fluid. The horizontal lines in (b) represent the mean isentropes.

especially just below the upper boundary, the pressure must be locally high as the air must decelerate vertically and accelerate horizontally. If the ambient air is stably stratified, the air pushed ahead of the thermal and that drawn up behind will be negatively buoyant with respect to the far field stratification (Fig. 4b). The negative buoyancy ahead will act with the adverse vertical pressure gradient to decelerate the air, while the negative buoyancy behind the thermal will inhibit the upward flow there. Clearly, this upward flow will be possible only if and where the perturbation pressure gradient exceeds the negative buoyancy. We can feel safe in asserting that if the air behind the thermal is accelerating upwards, then it is being driven by the vertical pressure gradient force. The interpretations presented in Braun (2002), although in the context of a more realistic three-dimensional hurricane flow, are consistent with the foregoing picture. Of course, if we were feeling perverse, we could substantially alter the picture by choosing a reference pressure and density different from those in the distant environment, which are the obvious choices in this case.

If the thermal element were located in a rapidly rotating vortex of homogeneous fluid, the buoyancy field would have a radially-inward component in addition to the upward component, and there would be a consequent effect on the induced perturbation pressure field, which in turn depends on how the reference pressure is defined and whether or not it includes the balanced pressure field associated with the rotating flow. In some situations, this radial component of “generalized” buoyancy, not considered by Zhang et al. (2000), Braun (2002) and Eastin (2002), may be of comparable importance for the vortex dynamics as the vertical component. Another aspect of the thermal model that should not be overlooked in rapidly rotating vortices is the potential impact of local rotation on the lateral entrainment into the thermal. For instance, when thermals are under “rotational control” the lateral entrainment can be significantly reduced relative to its non-rotating counterpart (Ayotte and Fernando, 1994; Helfrich, 1994; Julien et al., 1999). This potentially important effect warrants further theoretical and numerical examination in tropical cyclones and other intense convective vortices, but lies outside the scope of this paper.

## 8. Conclusions

In a rapidly rotating vortex, the effective buoyancy force has a radial component as well as a vertical component when calculated in height coordinates or sigma coordinates, but it



has only a single component when calculated in pressure coordinates, being normal to these surfaces. The buoyancy force is non-unique in the sense that it depends on the arbitrary choice of a reference density distribution; however, the sum of the buoyancy force and the perturbation pressure gradient is unique. The reference density may be taken as a function of radius, height, azimuth and time. If it is chosen to be that of an axisymmetric vortex in hydrostatic and gradient wind balance, the buoyancy force characterizes asymmetric buoyancy anomalies (that may or may not be unbalanced in a three-dimensional sense) and the buoyancy of the axisymmetric balanced vortex is zero by definition. In contrast, the same vortex has non-zero (system) buoyancy relative to a reference density that does not depend on radius.

In a tropical-cyclone scale vortex, the  $\sigma$ -surfaces are approximately horizontal and there is little difference between the buoyancy force calculated on sigma surfaces and that calculated at constant height.

In general, the generation of toroidal circulation can be characterized by the azimuthal vorticity tendency. When expressed in pressure coordinates, the radial density gradient is the only term contributing to the baroclinic production of azimuthal vorticity. In contrast, when expressed in height (or  $\sigma$ ) coordinates, there is an additional contribution associated with the decrease in density with height (or  $\sigma$ ) in the presence of centrifugal and Coriolis forces. If the buoyancy is defined relative to a balanced axisymmetric reference state, the azimuthal vorticity tendency is approximately equal to the curl of the generalized buoyancy. The radial component of buoyancy makes a negligibly small contribution to the toroidal vorticity generation in a tropical cyclone, at least on the scale of the vortex, but is significant in tornadoes and possibly important in dust devils.

The toroidal circulation of a balanced, symmetrically-stable, axisymmetric vortex can be determined by solving an elliptic partial differential equation, the Sawyer–Eliassen equation, for the streamfunction. The forcing term for this equation is approximately equal to the curl of the rate-of-generation of generalized buoyancy. The Sawyer–Eliassen equation is the time derivative of the toroidal vorticity equation in which the time rate-of-change of the material derivative of potential toroidal vorticity is set to zero.

The radial buoyancy gradient in the core of a mature tropical cyclone is determined to a large degree by the radial gradient of equivalent potential temperature in the subcloud layer, which in turn depends strongly on the radial gradient of surface moisture flux. Much of this buoyancy, which we have referred to as the system buoyancy, resides in the axisymmetric balanced part of the flow, but individual clouds may have positive (local) buoyancy relative to a reference density in thermal wind balance with the axisymmetric mean tangential circulation.

We suggest that apparent differences in interpretations of the relative role of buoyancy and perturbation pressure gradient forces in hurricanes in recent papers are most likely a result of the different reference states used to define these quantities.

## Acknowledgements

This work was stimulated by correspondence with Prof. Kerry Emanuel at the Massachusetts Institute of Technology. We thank John Molinari, Lloyd Shapiro and Michael

Reader for their thoughtful comments on earlier drafts of the paper. The work was supported in part by the US Office of Naval Research through Grant Nos. N00014-03-1-0185 and N00014-02-1-0474, and the National Aeronautics and Space Administration (NASA) Fourth Convection and Moisture Experiment (CAMEX) Grant NAG5-11140.

**Appendix A. Derivation of the Sawyer–Eliassen equation**

In the notation defined earlier, the azimuthal momentum equation and thermodynamic equation for an axisymmetric vortex may be written as

$$\frac{\partial v}{\partial t} + u(\zeta + f) + w \frac{\partial v}{\partial z} = 0$$

and

$$\frac{\partial \chi}{\partial t} + u \frac{\partial \chi}{\partial r} + w \frac{\partial \chi}{\partial z} = -\chi^2 \dot{\theta}$$

where  $\chi = 1/\theta$ ,  $Q = \chi^2 \dot{\theta}$  and  $\dot{\theta}$  is the diabatic heating rate, while the thermal wind Eq. (5) may be written as

$$g \frac{\partial \chi}{\partial r} + \frac{\partial(\chi C)}{\partial z} = 0.$$

Differentiating the latter equation with respect to time and eliminating the time derivatives of  $v$  and  $\chi$  using the first two gives

$$g \frac{\partial}{\partial r} \left( u \frac{\partial \chi}{\partial r} + w \frac{\partial \chi}{\partial z} - Q \right) + \frac{\partial}{\partial z} \left[ C \left( u \frac{\partial \chi}{\partial r} + w \frac{\partial \chi}{\partial z} - Q \right) + \chi \xi \left( u(\zeta + f) + w \frac{\partial v}{\partial z} \right) \right] = 0$$

Now the continuity equation

$$\frac{\partial}{\partial r}(\rho r u) + \frac{\partial}{\partial z}(\rho r w) = 0$$

implies the existence of a streamfunction  $\psi$  satisfying

$$u = -\frac{1}{r\rho} \frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r\rho} \frac{\partial \psi}{\partial r}$$

The Sawyer–Eliassen Eq. (8) follows by substitution for  $u$  and  $w$  after a little rearrangement using the thermal wind relationship and the definitions of  $C$ ,  $\xi$  and  $\zeta$ .

The Sawyer–Eliassen equation is an approximate form of the local time derivative of the equation for the toroidal vorticity  $\eta = \partial u/\partial z - \partial w/\partial r$ . Assuming the most general form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r\rho u) + \frac{\partial}{\partial z}(\rho w) = 0$$

the toroidal vorticity equation may be written as

$$r \frac{D}{Dt} \left( \frac{\eta}{r\rho} \right) = -\frac{1}{\rho} \frac{\partial C}{\partial z} + \frac{1}{\rho^2 \chi} \left( \frac{\partial \chi}{\partial r} \frac{\partial p}{\partial z} - \frac{\partial \chi}{\partial z} \frac{\partial p}{\partial r} \right) \quad (\text{A.1})$$

where  $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$  and  $\eta/(r\rho)$  is a ‘potential toroidal vorticity’, where the analogous ‘depth’ is ‘ $r$ ’, the radius of a toroidal vortex ring. If thermal wind balance exists, the right-hand-side of (A.1) may be written as

$$-\frac{1}{\rho \chi} \left( g \frac{\partial \chi}{\partial r} + \frac{\partial}{\partial z} (C\chi) \right).$$

Then the time derivative of (A.1) is

$$\frac{\partial}{\partial t} \left[ r \frac{D}{Dt} \left( \frac{\eta}{r\rho} \right) \right] = -\frac{\partial}{\partial t} \left[ \frac{1}{\rho \chi} \left( g \frac{\partial \chi}{\partial r} + \frac{\partial}{\partial z} (C\chi) \right) \right], \quad (\text{A.2})$$

The right-hand side of (A.2) gives the Sawyer–Eliassen equation when the thermal wind Eq. (5) is satisfied for all time. Then consistency requires that the left-hand-side is identically zero.

## Appendix B. Vortex profile used in Section 5

The tropical-cyclone scale vortex discussed in Section 5 is defined by the pressure perturbation:

$$p(s, z) = (p_c - p_\infty(0)) \left[ 1 - \exp\left(\frac{-x}{s}\right) \right] \exp\left(\frac{-z}{z^*}\right) \cos\left(\frac{1}{2}\pi \frac{z}{z_0}\right),$$

where  $p_c$  is the central pressure at the surface,  $p_\infty(0)$  is the surface pressure at large radial distance,  $s = r/r_m$ , and  $x$ ,  $r_m$ ,  $z_0$  and  $z^*$  are constants. We choose  $p_\infty(0) - p_c$  and  $x$  so that the maximum tangential wind speed is about  $40 \text{ m s}^{-1}$  at a radius of about 40 km and declines to zero at an altitude  $z_0 = 16 \text{ km}$ : specifically  $p_\infty(0) - p_c = 50 \text{ mb}$ ,  $z^* = 8 \text{ km}$  and  $x = 1.048$ . The exponential decay with height approximately matches the decrease in the environmental density with height and is necessary to produce a reasonably realistic tangential wind distribution that decreases in strength with height.

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